Section 3.4 Subsequences and the Bolzano-Weierstrass Theorem

In this section we will introduce the notion of a subsequence of a sequence of real numbers. Informally, a subsequence of a sequence is a selection of terms from the given sequence such that the selected terms form a new sequence. Usually the selection is made for a definite purpose. For example, subsequences are often useful in establishing the convergence or the divergence of the sequence. We will also prove the important existence theorem known as the Bolzano-Weierstrass Theorem, which will be used to establish a number of significant results.

3.4.1 Definition Let $X = (x_n)$ be a sequence of real numbers and let $n_1 < n_2 < \cdots < n_k < \cdots$ be a strictly increasing sequence of natural numbers. Then the sequence $X' = (x_{n_k})$ given by

$$(x_{n_1}, x_{n_2}, \ldots, x_{n_k}, \ldots)$$

is called a **subsequence** of X.

For example, if $X := (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots)$, then the selection of even indexed terms produces the subsequence

$$X' = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \ldots, \frac{1}{2k}, \ldots\right),$$

where $n_1 = 2$, $n_2 = 4, \ldots, n_k = 2k, \ldots$. Other subsequences of $X = (1/n)$ are the following:

$$\left(\frac{1}{1}, \frac{1}{3}, \frac{1}{5}, \ldots, \frac{1}{2k-1}, \ldots\right), \quad \left(\frac{1}{2!}, \frac{1}{4!}, \frac{1}{6!}, \ldots, \frac{1}{(2k)!}, \ldots\right).$$

The following sequences are *not* subsequences of $X = (1/n)$:

$$\left(\frac{1}{2}, \frac{1}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \ldots\right), \quad \left(\frac{1}{1}, 0, \frac{1}{3}, 0, \frac{1}{5}, 0, \ldots\right).$$

A tail of a sequence (see 3.1.8) is a special type of subsequence. In fact, the m-tail corresponds to the sequence of indices

$$n_1 = m + 1, n_2 = m + 2, \ldots, n_k = m + k, \ldots.$$

But, clearly, not every subsequence of a given sequence need be a tail of the sequence.

Subsequences of convergent sequences also converge to the same limit, as we now show.

3.4.2 Theorem *If a sequence $X = (x_n)$ of real numbers converges to a real number x, then any subsequence $X' = (x_{n_k})$ of X also converges to x.*

Proof. Let $\varepsilon > 0$ be given and let $K(\varepsilon)$ be such that if $n \geq K(\varepsilon)$, then $|x_n - x| < \varepsilon$. Since $n_1 < n_2 < \cdots < n_k < \cdots$ is an increasing sequence of natural numbers, it is easily proved (by Induction) that $n_k \geq k$. Hence, if $k \geq K(\varepsilon)$, we also have $n_k \geq k \geq K(\varepsilon)$ so that $|x_{n_k} - x| < \varepsilon$. Therefore the subsequence (x_{n_k}) also converges to x. Q.E.D.

3.4.3 Examples **(a)** $\lim(b^n) = 0$ if $0 < b < 1$.

We have already seen, in Example 3.1.11(b), that if $0 < b < 1$ and if $x_n := b^n$, then it follows from Bernoulli's Inequality that $\lim(x_n) = 0$. Alternatively, we see that since

$0 < b < 1$, then $x_{n+1} = b^{n+1} < b^n = x_n$ so that the sequence (x_n) is decreasing. It is also clear that $0 \leq x_n \leq 1$, so it follows from the Monotone Convergence Theorem 3.3.2 that the sequence is convergent. Let $x := \lim x_n$. Since (x_{2n}) is a subsequence of (x_n) it follows from Theorem 3.4.2 that $x = \lim(x_{2n})$. Moreover, it follows from the relation $x_{2n} = b^{2n} = (b^n)^2 = x_n^2$ and Theorem 3.2.3 that

$$x = \lim(x_{2n}) = \left(\lim(x_n)\right)^2 = x^2.$$

Therefore we must have either $x = 0$ or $x = 1$. Since the sequence (x_n) is decreasing and bounded above by $b < 1$, we deduce that $x = 0$.

(b) $\lim(c^{1/n}) = 1$ for $c > 1$.

This limit has been obtained in Example 3.1.11(c) for $c > 0$, using a rather ingenious argument. We give here an alternative approach for the case $c > 1$. Note that if $z_n := c^{1/n}$, then $z_n > 1$ and $z_{n+1} < z_n$ for all $n \in \mathbb{N}$. (Why?) Thus by the Monotone Convergence Theorem, the limit $z := \lim(z_n)$ exists. By Theorem 3.4.2, it follows that $z = \lim(z_{2n})$. In addition, it follows from the relation

$$z_{2n} = c^{1/2n} = (c^{1/n})^{1/2} = z_n^{1/2}$$

and Theorem 3.2.10 that

$$z = \lim(z_{2n}) = \left(\lim(z_n)\right)^{1/2} = z^{1/2}.$$

Therefore we have $z^2 = z$ whence it follows that either $z = 0$ or $z = 1$. Since $z_n > 1$ for all $n \in \mathbb{N}$, we deduce that $z = 1$.

We leave it as an exercise to the reader to consider the case $0 < c < 1$. □

The following result is based on a careful negation of the definition of $\lim(x_n) = x$. It leads to a convenient way to establish the divergence of a sequence.

3.4.4 Theorem *Let $X = (x_n)$ be a sequence of real numbers, Then the following are equivalent:*

(i) *The sequence $X = (x_n)$ does not converge to $x \in \mathbb{R}$.*
(ii) *There exists an $\varepsilon_0 > 0$ such that for any $k \in \mathbb{N}$, there exists $n_k \in \mathbb{N}$ such that $n_k \geq k$ and $|x_{n_k} - x| \geq \varepsilon_0$.*
(iii) *There exists an $\varepsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.*

Proof. (i) ⇒ (ii) If (x_n) does not converge to x, then for some $\varepsilon_0 > 0$ it is impossible to find a natural number k such that for all $n \geq k$ the terms x_n satisfy $|x_n - x| < \varepsilon_0$. That is, for each $k \in \mathbb{N}$ it is *not true* that for *all* $n \geq k$ the inequality $|x_n - x| < \varepsilon_0$ holds. In other words, for each $k \in \mathbb{N}$ there exists a natural number $n_k \geq k$ such that $|x_{n_k} - x| \geq \varepsilon_0$.

(ii) ⇒ (iii) Let ε_0 be as in (ii) and let $n_1 \in \mathbb{N}$ be such that $n_1 \geq 1$ and $|x_{n_1} - x| \geq \varepsilon_0$. Now let $n_2 \in \mathbb{N}$ be such that $n_2 > n_1$ and $|x_{n_2} - x| \geq \varepsilon_0$; let $n_3 \in \mathbb{N}$ be such that $n_3 > n_2$ and $|x_{n_3} - x| \geq \varepsilon_0$. Continue in this way to obtain a subsequence $X' = (x_{n_k})$ of X such that $|x_{n_k} - x| \geq \varepsilon_0$ for all $k \in \mathbb{N}$.

(iii) ⇒ (i) Suppose $X = (x_n)$ has a subsequence $X' = (x_{n_k})$ satisfying the condition in (iii). Then X cannot converge to x; for if it did, then, by Theorem 3.4.2, the subsequence X' would also converge to x. But this is impossible, since none of the terms of X' belongs to the ε_0-neighborhood of x. Q.E.D.

Since all subsequences of a convergent sequence must converge to the same limit, we have part (i) in the following result. Part (ii) follows from the fact that a convergent sequence is bounded.

3.4.5 Divergence Criteria *If a sequence $X = (x_n)$ of real numbers has either of the following properties, then X is divergent.*

(i) *X has two convergent subsequences $X' = (x_{n_k})$ and $X'' = (x_{r_k})$ whose limits are not equal.*
(ii) *X is unbounded.*

3.4.6 Examples **(a)** The sequence $X := ((-1)^n)$ is divergent.

The subsequence $X' := ((-1)^{2n}) = (1, 1, \ldots)$ converges to 1, and the subsequence $X'' := ((-1)^{2n-1}) = (-1, -1, \ldots)$ converges to -1. Therefore, we conclude from Theorem 3.4.5(i) that X is divergent.

(b) The sequence $(1, \frac{1}{2}, 3, \frac{1}{4}, \ldots)$ is divergent.

This is the sequence $Y = (y_n)$, where $y_n = n$ if n is odd, and $y_n = 1/n$ if n is even. It can easily be seen that Y is not bounded. Hence, by Theorem 3.4.5(ii), the sequence is divergent.

(c) The sequence $S := (\sin n)$ is divergent.

This sequence is not so easy to handle. In discussing it we must, of course, make use of elementary properties of the sine function. We recall that $\sin(\pi/6) = \frac{1}{2} = \sin(5\pi/6)$ and that $\sin x > \frac{1}{2}$ for x in the interval $I_1 := (\pi/6, 5\pi/6)$. Since the length of I_1 is $5\pi/6 - \pi/6 = 2\pi/3 > 2$, there are at least two natural numbers lying inside I_1; we let n_1 be the first such number. Similarly, for each $k \in \mathbb{N}$, $\sin x > \frac{1}{2}$ for x in the interval

$$I_k := \big(\pi/6 + 2\pi(k-1), 5\pi/6 + 2\pi(k-1)\big).$$

Since the length of I_k is greater than 2, there are at least two natural numbers lying inside I_k; we let n_k be the first one. The subsequence $S' := (\sin n_k)$ of S obtained in this way has the property that all of its values lie in the interval $[\frac{1}{2}, 1]$.

Similarly, if $k \in \mathbb{N}$ and J_k is the interval

$$J_k := \big(7\pi/6 + 2\pi(k-1),\ 11\pi/6 + 2\pi(k-1)\big),$$

then it is seen that $\sin x < -\frac{1}{2}$ for all $x \in J_k$ and the length of J_k is greater than 2. Let m_k be the first natural number lying in J_k. Then the subsequence $S'' := (\sin m_k)$ of S has the property that all of its values lie in the interval $[-1, -\frac{1}{2}]$.

Given any real number c, it is readily seen that at least one of the subsequences S' and S'' lies entirely outside of the $\frac{1}{2}$-neighborhood of c. Therefore c cannot be a limit of S. Since $c \in \mathbb{R}$ is arbitrary, we deduce that S is divergent. □

The Existence of Monotone Subsequences

While not every sequence is a monotone sequence, we will now show that every sequence has a monotone subsequence.

3.4.7 Monotone Subsequence Theorem *If $X = (x_n)$ is a sequence of real numbers, then there is a subsequence of X that is monotone.*

Proof. For the purpose of this proof, we will say that the mth term x_m is a "peak" if $x_m \geq x_n$ for all n such that $n \geq m$. (That is, x_m is never exceeded by any term that follows it

in the sequence.) Note that, in a decreasing sequence, every term is a peak, while in an increasing sequence, no term is a peak.

We will consider two cases, depending on whether X has infinitely many, or finitely many, peaks.

Case 1: X has infinitely many peaks. In this case, we list the peaks by increasing subscripts: $x_{m_1}, x_{m_2}, \ldots, x_{m_k}, \ldots$. Since each term is a peak, we have

$$x_{m_1} \geq x_{m_2} \geq \cdots \geq x_{m_k} \geq \cdots.$$

Therefore, the subsequence (x_{m_k}) of peaks is a decreasing subsequence of X.

Case 2: X has a finite number (possibly zero) of peaks. Let these peaks be listed by increasing subscripts: $x_{m_1}, x_{m_2}, \ldots, x_{m_r}$. Let $s_1 := m_r + 1$ be the first index beyond the last peak. Since x_{s_1} is not a peak, there exists $s_2 > s_1$ such that $x_{s_1} < x_{s_2}$. Since x_{s_2} is not a peak, there exists $s_3 > s_2$ such that $x_{s_2} < x_{s_3}$. Continuing in this way, we obtain an increasing subsequence (x_{s_k}) of X. Q.E.D.

It is not difficult to see that a given sequence may have one subsequence that is increasing, and another subsequence that is decreasing.

The Bolzano-Weierstrass Theorem

We will now use the Monotone Subsequence Theorem to prove the Bolzano-Weierstrass Theorem, which states that every bounded sequence has a convergent subsequence. Because of the importance of this theorem we will also give a second proof of it based on the Nested Interval Property.

3.4.8 The Bolzano-Weierstrass Theorem *A bounded sequence of real numbers has a convergent subsequence.*

First Proof. It follows from the Monotone Subsequence Theorem that if $X = (x_n)$ is a bounded sequence, then it has a subsequence $X' = (x_{n_k})$ that is monotone. Since this subsequence is also bounded, it follows from the Monotone Convergence Theorem 3.3.2 that the subsequence is convergent. Q.E.D.

Second Proof. Since the set of values $\{x_n : n \in \mathbb{N}\}$ is bounded, this set is contained in an interval $I_1 := [a, b]$. We take $n_1 := 1$.

We now bisect I_1 into two equal subintervals I_1' and I_1'', and divide the set of indices $\{n \in \mathbb{N} : n > 1\}$ into two parts:

$$A_1 := \{n \in \mathbb{N} : n > n_1, x_n \in I_1'\}, \quad B_1 = \{n \in \mathbb{N} : n > n_1, x_n \in I_1''\}.$$

If A_1 is infinite, we take $I_2 := I_1'$ and let n_2 be the smallest natural number in A_1. If A_1 is a finite set, then B_1 must be infinite, and we take $I_2 := I_1''$ and let n_2 be the smallest natural number in B_1.

We now bisect I_2 into two equal subintervals I_2' and I_2'', and divide the set $\{n \in \mathbb{N} : n > n_2\}$ into two parts:

$$A_2 = \{n \in \mathbb{N} : n > n_2, x_n, \in I_2'\}, \quad B_2 := \{n \in \mathbb{N} : n > n_2, x_n \in I_2''\}$$

If A_2 is infinite, we take $I_3 := I_2'$ and let n_3 be the smallest natural number in A_2. If A_2 is a finite set, then B_2 must be infinite, and we take $I_3 := I_2''$ and let n_3 be the smallest natural number in B_2.

We continue in this way to obtain a sequence of nested intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_k \supseteq \cdots$ and a subsequence (x_{n_k}) of X such that $x_{n_k} \in I_k$ for $k \in \mathbb{N}$. Since the length of I_k is equal to $(b-a)/2^{k-1}$, it follows from Theorem 2.5.3 that there is a (unique) common point $\xi \in I_k$ for all $k \in \mathbb{N}$. Moreover, since x_{n_k} and ξ both belong to I_k, we have

$$|x_{n_k} - \xi| \le (b-a)/2^{k-1},$$

whence it follows that the subsequence (x_{n_k}) of X converges to ξ. Q.E.D.

Theorem 3.4.8 is sometimes called the Bolzano-Weierstrass Theorem *for sequences*, because there is another version of it that deals with bounded sets in $\mathbb{R}$ (see Exercise 11.2.6).

It is readily seen that a bounded sequence can have various subsequences that converge to different limits or even diverge. For example, the sequence $((-1)^n)$ has subsequences that converge to -1, other subsequences that converge to $+1$, and it has subsequences that diverge.

Let X be a sequence of real numbers and let X' be a subsequence of X. Then X' is a sequence in its own right, and so it has subsequences. We note that if X'' is a subsequence of X', then it is also a subsequence of X.

3.4.9 Theorem *Let $X = (x_n)$ be a bounded sequence of real numbers and let $x \in \mathbb{R}$ have the property that every convergent subsequence of X converges to x. Then the sequence X converges to x.*

Proof. Suppose $M > 0$ is a bound for the sequence X so that $|x_n| \le M$ for all $n \in \mathbb{N}$. If X does not converge to x, then Theorem 3.4.4 implies that there exist $\varepsilon_0 > 0$ and a subsequence $X' = (x_{n_k})$ of X such that

$$(1) \qquad |x_{n_k} - x| \ge \varepsilon_0 \quad \text{for all} \quad k \in \mathbb{N}.$$

Since X' is a subsequence of X, the number M is also a bound for X'. Hence the Bolzano-Weierstrass Theorem implies that X' has a convergent subsequence X''. Since X'' is also a subsequence of X, it converges to x by hypothesis. Thus, its terms ultimately belong to the ε_0-neighborhood of x, contradicting (1). Q.E.D.

Limit Superior and Limit Inferior

A bounded sequence of real numbers (x_n) may or may not converge, but we know from the Bolzano-Weierstrass Theorem 3.4.8 that there will be a convergent subsequence and possibly many convergent subsequences. A real number that is the limit of a subsequence of (x_n) is called a *subsequential limit of* (x_n). We let S denote the set of all subsequential limits of the bounded sequence (x_n). The set S is bounded, because the sequence is bounded.

For example, if (x_n) is defined by $x_n := (-1)^n + 2/n$, then the subsequence (x_{2n}) converges to 1, and the subsequence (x_{2n-1}) converges to -1. It is easily seen that the set of subsequential limits is $S = \{-1, 1\}$. Observe that the largest member of the sequence itself is $x_2 = 2$, which provides no information concerning the limiting behavior of the sequence.

An extreme example is given by the set of all rational numbers in the interval $[0, 1]$. The set is denumerable (see Section 1.3) and therefore it can be written as a sequence (r_n). Then it follows from the Density Theorem 2.4.8 that every number in $[0, 1]$ is a subsequential limit of (r_n). Thus we have $S = [0, 1]$.

A bounded sequence (x_n) that diverges will display some form of oscillation. The activity is contained in decreasing intervals as follows. The interval $[t_1, u_1]$, where $t_1 := \inf\{x_n : n \in \mathbb{N}\}$ and $u_1 := \sup\{x_n : n \in \mathbb{N}\}$, contains the entire sequence. If for each $m = 1, 2, \ldots$, we define $t_m := \inf\{x_n : n \ge m\}$ and $u_m := \sup\{x_n : n \ge m\}$, the sequences (t_m) and (u_m) are monotone and we obtain a nested sequence of intervals $[t_m, u_m]$ where the mth interval contains the m-tail of the sequence.

The preceding discussion suggests different ways of describing limiting behavior of a bounded sequence. Another is to observe that if a real number v has the property that $x_n > v$ for *at most* a finite number of values of n, then no subsequence of (x_n) can converge to a limit larger than v because that would require infinitely many terms of the sequence be larger than v. In other words, if v has the property that there exists N_v such that $x_n \le v$ for all $n \ge N_v$, then no number larger than v can be a subsequential limit of (x_n).

This observation leads to the following definition of limit superior. The accompanying definition of limit inferior is similar.

3.4.10 Definition Let $X = (x_n)$ be a bounded sequence of real numbers.

(a) The **limit superior** of (x_n) is the infimum of the set V of $v \in \mathbb{R}$ such that $v < x_n$ for at most a finite number of $n \in \mathbb{N}$. It is denoted by

$$\limsup(x_n) \quad \text{or} \quad \limsup X \quad \text{or} \quad \overline{\lim}(x_n).$$

(b) The **limit inferior** of (x_n) is the supremum of the set of $w \in \mathbb{R}$ such that $x_m < w$ for at most a finite number of $m \in \mathbb{N}$. It is denoted by

$$\liminf(x_n) \quad \text{or} \quad \liminf X \quad \text{or} \quad \underline{\lim}(x_n).$$

For the concept of limit superior, we now show that the different approaches are equivalent.

3.4.11 Theorem *If (x_n) is a bounded sequence of real numbers, then the following statements for a real number x^* are equivalent.*

(a) $x^* = \limsup(x_n)$.

(b) *If $\varepsilon > 0$, there are at most a finite number of $n \in \mathbb{N}$ such that $x^* + \varepsilon < x_n$, but an infinite number of $n \in \mathbb{N}$ such that $x^* - \varepsilon < x_n$.*

(c) *If $u_m = \sup\{x_n : n \ge m\}$, then $x^* = \inf\{u_m : m \in \mathbb{N}\} = \lim(u_m)$.*

(d) *If S is the set of subsequential limits of (x_n), then $x^* = \sup S$.*

Proof. (a) implies (b). If $\varepsilon > 0$, then the fact that x^* is an infimum implies that there exists a v in V such that $x^* \le v < x^* + \varepsilon$. Therefore x^* also belongs to V, so there can be at most a finite number of $n \in \mathbb{N}$ such that $x^* + \varepsilon < x_n$. On the other hand, $x^* - \varepsilon$ is not in V so there are an infinite number of $n \in \mathbb{N}$ such that $x^* - \varepsilon < x_n$.

(b) implies (c). If (b) holds, given $\varepsilon > 0$, then for all sufficiently large m we have $u_m < x + \varepsilon$. Therefore, $\inf\{u_m : m \in \mathbb{N}\} \le x^* + \varepsilon$. Also, since there are an infinite number of $n \in \mathbb{N}$ such that $x^* - \varepsilon < x_n$, then $x^* - \varepsilon < u_m$ for all $m \in \mathbb{N}$ and hence $x^* - \varepsilon \le \inf\{u_m : m \in \mathbb{N}\}$. Since $\varepsilon > 0$ is arbitrary, we conclude that $x^* = \inf\{u_m : m \in \mathbb{N}\}$. Moreover, since the sequence (u_m) is monotone decreasing, we have $\inf(u_m) = \lim(u_m)$.

(c) implies (d). Suppose that $X' = (x_{n_k})$ is a convergent subsequence of $X = (x_n)$. Since $n_k \ge k$, we have $x_{n_k} \le u_k$ and hence $\lim X' \le \lim(u_k) = x^*$. Conversely, there exists n_1 such that $u_1 - 1 \le x_{n_1} \le u_1$. Inductively choose $n_{k+1} > n_k$ such that

$$u_k - \frac{1}{k+1} < x_{n_{k+1}} \le u_k.$$

Since $\lim(u_k) = x^*$, it follows that $x^* = \lim(x_{n_k})$, and hence $x^* \in S$.

(d) implies (a). Let $w = \sup S$. If $\varepsilon > 0$ is given, then there are at most finitely many n with $w + \varepsilon < x_n$. Therefore $w + \varepsilon$ belongs to V and $\lim\sup(x_n) \le w + \varepsilon$. On the other hand, there exists a subsequence of (x_n) converging to some number larger than $w - \varepsilon$, so that $w - \varepsilon$ is not in V, and hence $w - \varepsilon \le \lim\sup(x_n)$. Since $\varepsilon > 0$ is arbitrary, we conclude that $w = \lim\sup(x_n)$. Q.E.D.

As an instructive exercise, the reader should formulate the corresponding theorem for the limit inferior of a bounded sequence of real numbers.

3.4.12 Theorem A bounded sequence (x_n) is convergent if and only if $\lim\sup(x_n) = \lim\inf(x_n)$.

We leave the proof as an exercise. Other basic properties can also be found in the exercises.

Exercises for Section 3.4

1. Give an example of an unbounded sequence that has a convergent subsequence.
2. Use the method of Example 3.4.3(b) to show that if $0 < c < 1$, then $\lim(c^{1/n}) = 1$.
3. Let (f_n) be the Fibonacci sequence of Example 3.1.2(d), and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, determine the value of L.
4. Show that the following sequences are divergent.
   (a) $(1 - (-1)^n + 1/n)$, (b) $(\sin n\pi/4)$.
5. Let $X = (x_n)$ and $Y = (y_n)$ be given sequences, and let the "shuffled" sequence $Z = (z_n)$ be defined by $z_1 := x_1, z_2 := y_1, \ldots, z_{2n-1} := x_n, z_{2n} := y_n, \ldots$. Show that Z is convergent if and only if both X and Y are convergent and $\lim X = \lim Y$.
6. Let $x_n := n^{1/n}$ for $n \in \mathbb{N}$.
   (a) Show that $x_{n+1} < x_n$ if and only if $(1 + 1/n)^n < n$, and infer that the inequality is valid for $n \ge 3$. (See Example 3.3.6.) Conclude that (x_n) is ultimately decreasing and that $x := \lim(x_n)$ exists.
   (b) Use the fact that the subsequence (x_{2n}) also converges to x to conclude that $x = 1$.
7. Establish the convergence and find the limits of the following sequences:
   (a) $\left((1 + 1/n^2)^{n^2}\right)$, (b) $((1 + 1/2n)^n)$,
   (c) $\left((1 + 1/n^2)^{2n^2}\right)$, (d) $((1 + 2/n)^n)$.
8. Determine the limits of the following.
   (a) $((3n)^{1/2n})$, (b) $((1 + 1/2n)^{3n})$.
9. Suppose that every subsequence of $X = (x_n)$ has a subsequence that converges to 0. Show that $\lim X = 0$.
10. Let (x_n) be a bounded sequence and for each $n \in \mathbb{N}$ let $s_n := \sup\{x_k : k \ge n\}$ and $S := \inf\{s_n\}$. Show that there exists a subsequence of (x_n) that converges to S.
11. Suppose that $x_n \ge 0$ for all $n \in \mathbb{N}$ and that $\lim((-1)^n x_n)$ exists. Show that (x_n) converges.
12. Show that if (x_n) is unbounded, then there exists a subsequence (x_{n_k}) such that $\lim(1/x_{n_k}) = 0$.
13. If $x_n := (-1)^n/n$, find the subsequence of (x_n) that is constructed in the second proof of the Bolzano-Weierstrass Theorem 3.4.8, when we take $I_1 := [-1, 1]$.

14. Let (x_n) be a bounded sequence and let $s := \sup\{x_n : n \in \mathbb{N}\}$. Show that if $s \notin \{x_n : n \in \mathbb{N}\}$, then there is a subsequence of (x_n) that converges to s.
15. Let (I_n) be a nested sequence of closed bounded intervals. For each $n \in \mathbb{N}$, let $x_n \in I_n$. Use the Bolzano-Weierstrass Theorem to give a proof of the Nested Intervals Property 2.5.2.
16. Give an example to show that Theorem 3.4.9 fails if the hypothesis that X is a bounded sequence is dropped.
17. Alternate the terms of the sequences $(1 + 1/n)$ and $(-1/n)$ to obtain the sequence (x_n) given by
$$(2, -1, 3/2, -1/2, 4/3, -1/3, 5/4, -1/4, \ldots).$$
Determine the values of $\lim\sup(x_n)$ and $\lim\inf(x_n)$. Also find $\sup\{x_n\}$ and $\inf\{x_n\}$.
18. Show that if (x_n) is a bounded sequence, then (x_n) converges if and only if $\lim\sup(x_n) = \lim\inf(x_n)$.
19. Show that if (x_n) and (y_n) are bounded sequences, then
$$\lim\sup(x_n + y_n) \le \lim\sup(x_n) + \lim\sup(y_n).$$
Give an example in which the two sides are not equal.

Section 3.5 The Cauchy Criterion

The Monotone Convergence Theorem is extraordinarily useful and important, but it has the significant drawback that it applies only to sequences that are monotone. It is important for us to have a condition implying the convergence of a sequence that does not require us to know the value of the limit in advance, and is not restricted to monotone sequences. The Cauchy Criterion, which will be established in this section, is such a condition.

3.5.1 Definition A sequence $X = (x_n)$ of real numbers is said to be a **Cauchy sequence** if for every $\varepsilon > 0$ there exists a natural number $H(\varepsilon)$ such that for all natural numbers $n, m \ge H(\varepsilon)$, the terms x_n, x_m satisfy $|x_n - x_m| < \varepsilon$.

The significance of the concept of Cauchy sequence lies in the main theorem of this section, which asserts that a sequence of real numbers is convergent if and only if it is a Cauchy sequence. This will give us a method of proving a sequence converges without knowing the limit of the sequence.

However, we will first highlight the definition of Cauchy sequence in the following examples.

3.5.2 Examples **(a)** The sequence $(1/n)$ is a Cauchy sequence.

If $\varepsilon > 0$ is given, we choose a natural number $H = H(\varepsilon)$ such that $H > 2/\varepsilon$. Then if $m, n \ge H$, we have $1/n \le 1/H < \varepsilon/2$ and similarly $1/m < \varepsilon/2$. Therefore, it follows that if $m, n \ge H$, then
$$\left|\frac{1}{n} - \frac{1}{m}\right| \le \frac{1}{n} + \frac{1}{m} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$
Since $\varepsilon > 0$ is arbitrary, we conclude that $(1/n)$ is a Cauchy sequence.

(b) The sequence $(1 + (-1)^n)$ is *not* a Cauchy sequence.

The negation of the definition of Cauchy sequence is: There exists $\varepsilon_0 > 0$ such that for every H there exist at least one $n > H$ and at least one $m > H$ such that $|x_n - x_m| \ge \varepsilon_0$. For the terms $x_n := 1 + (-1)^n$, we observe that if n is even, then $x_n = 2$ and $x_{n+1} = 0$. If we take $\varepsilon_0 = 2$, then for any H we can choose an even number $n > H$ and let $m := n + 1$ to get

$$|x_n - x_{n+1}| = 2 = \varepsilon_0.$$

We conclude that (x_n) is not a Cauchy sequence. □

Remark We emphasize that to prove a sequence (x_n) is a Cauchy sequence, we may not assume a relationship between m and n, since the required inequality $|x_n - x_m| < \varepsilon$ must hold for *all* $n, m \geq H(\varepsilon)$. But to prove a sequence is *not* a Cauchy sequence, we may specify a relation between n and m as long as arbitrarily large values of n and m can be chosen so that $|x_n - x_m| \geq \varepsilon_0$.

Our goal is to show that the Cauchy sequences are precisely the convergent sequences. We first prove that a convergent sequence is a Cauchy sequence.

3.5.3 Lemma *If $X = (x_n)$ is a convergent sequence of real numbers, then X is a Cauchy sequence.*

Proof. If $x := \lim X$, then given $\varepsilon > 0$ there is a natural number $K(\varepsilon/2)$ such that if $n \geq K(\varepsilon/2)$ then $|x_n - x| < \varepsilon/2$. Thus, if $H(\varepsilon) := K(\varepsilon/2)$ and if $n, m \geq H(\varepsilon)$, then we have

$$\begin{aligned} |x_n - x_m| &= |(x_n - x) + (x - x_m)| \\ &\leq |x_n - x| + |x_m - x| < \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that (x_n) is a Cauchy sequence. Q.E.D.

In order to establish that a Cauchy sequence is convergent, we will need the following result. (See Theorem 3.2.2.)

3.5.4 Lemma *A Cauchy sequence of real numbers is bounded.*

Proof. Let $X := (x_n)$ be a Cauchy sequence and let $\varepsilon := 1$. If $H := H(1)$ and $n \geq H$, then $|x_n - x_H| < 1$. Hence, by the Triangle Inequality, we have $|x_n| \leq |x_H| + 1$ for all $n \geq H$. If we set

$$M := \sup\{|x_1|, |x_2|, \ldots, |x_{H-1}|, |x_H| + 1\},$$

then it follows that $|x_n| \leq M$ for all $n \in \mathbb{N}$. Q.E.D.

We now present the important Cauchy Convergence Criterion.

3.5.5 Cauchy Convergence Criterion *A sequence of real numbers is convergent if and only if it is a Cauchy sequence.*

Proof. We have seen, in Lemma 3.5.3, that a convergent sequence is a Cauchy sequence.

Conversely, let $X = (x_n)$ be a Cauchy sequence; we will show that X is convergent to some real number. First we observe from Lemma 3.5.4 that the sequence X is bounded. Therefore, by the Bolzano-Weierstrass Theorem 3.4.8, there is a subsequence $X' = (x_{n_k})$ of X that converges to some real number x^*. We shall complete the proof by showing that X converges to x^*.

Since $X = (x_n)$ is a Cauchy sequence, given $\varepsilon > 0$ there is a natural number $H(\varepsilon/2)$ such that if $n, m \geq H(\varepsilon/2)$ then

$$(1) \qquad |x_n - x_m| < \varepsilon/2.$$

Since the subsequence $X' = (x_{n_k})$ converges to x^*, there is a natural number $K \geq H(\varepsilon/2)$ belonging to the set $\{n_1, n_2, \ldots\}$ such that

$$|x_K - x^*| < \varepsilon/2.$$

Since $K \geq H(\varepsilon/2)$, it follows from (1) with $m = K$ that

$$|x_n - x_K| < \varepsilon/2 \quad \text{for} \quad n \geq H(\varepsilon/2).$$

Therefore, if $n \geq H(\varepsilon/2)$, we have

$$\begin{aligned} |x_n - x^*| &= |(x_n - x_K) + (x_K - x^*)| \\ &\leq |x_n - x_K| + |x_K - x^*| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Sincee $\varepsilon > 0$ is arbitrary, we infer that $\lim(x_n) = x^*$. Therefore the sequence X is convergent. Q.E.D.

We will now give some examples of applications of the Cauchy Criterion.

3.5.6 Examples **(a)** Let $X = (x_n)$ be defined by

$$x_1 := 1, \quad x_2 := 2, \quad \text{and} \quad x_n := \frac{1}{2}(x_{n-2} + x_{n-1}) \quad \text{for} \quad n > 2.$$

It can be shown by Induction that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$. (Do so.) Some calculation shows that the sequence X is not monotone. However, since the terms are formed by averaging, it is readily seen that

$$|x_n - x_{n+1}| = \frac{1}{2^{n-1}} \quad \text{for} \quad n \in \mathbb{N}.$$

(Prove this by Induction.) Thus, if $m > n$, we may employ the Triangle Inequality to obtain

$$\begin{aligned} |x_n - x_m| &\leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| \\ &= \frac{1}{2^{n-1}} + \frac{1}{2^n} + \cdots + \frac{1}{2^{m-2}} \\ &= \frac{1}{2^{n-1}}\left(1 + \frac{1}{2} + \cdots + \frac{1}{2^{m-n-1}}\right) < \frac{1}{2^{n-2}}. \end{aligned}$$

Therefore, given $\varepsilon > 0$, if n is chosen so large that $1/2^n < \varepsilon/4$ and if $m \geq n$, then it follows that $|x_n - x_m| < \varepsilon$. Therefore, X is a Cauchy sequence in $\mathbb{R}$. By the Cauchy Criterion 3.5.5 we infer that the sequence X converges to a number x.

To evaluate the limit x, we might first "pass to the limit" in the rule of definition $x_n = \frac{1}{2}(x_{n-1} + x_{n-2})$ to conclude that x must satisfy the relation $x = \frac{1}{2}(x + x)$, which is true, but not informative. Hence we must try something else.