Suprema and Infima

We now introduce the notions of upper bound and lower bound for a set of real numbers. These ideas will be of utmost importance in later sections.

2.3.1 Definition Let S be a nonempty subset of $\mathbb{R}$.

(a) The set S is said to be **bounded above** if there exists a number $u \in \mathbb{R}$ such that $s \leq u$ for all $s \in S$. Each such number u is called an **upper bound** of S.

(b) The set S is said to be bounded **below** if there exists a number $w \in \mathbb{R}$ such that $w \leq s$ for all $s \in S$. Each such number w is called a **lower bound** of S.

(c) A set is said to be **bounded** if it is both bounded above and bounded below. A set is said to be **unbounded** if it is not bounded.

For example, the set $S := \{x \in \mathbb{R} : x < 2\}$ is bounded above; the number 2 and any number larger than 2 is an upper bound of S. This set has no lower bounds, so that the set is not bounded below. Thus it is unbounded (even though it is bounded above).

If a set has one upper bound, then it has infinitely many upper bounds, because if u is an upper bound of S, then the numbers $u + 1, u + 2, \ldots$ are also upper bounds of S. (A similar observation is valid for lower bounds.)

In the set of upper bounds of S and the set of lower bounds of S, we single out their least and greatest elements, respectively, for special attention in the following definition. (See Figure 2.3.1.)

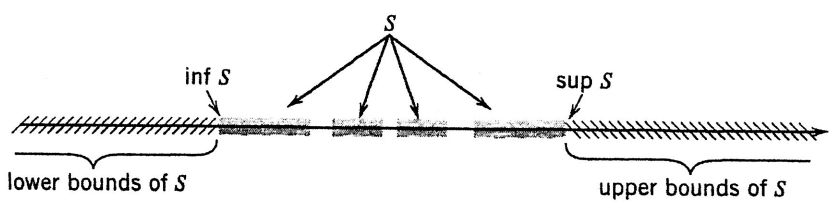

Figure 2.3.1 inf S and sup S

2.3.2 Definition Let S be a nonempty subset of $\mathbb{R}$.

(a) If S is bounded above, then a number u is said to be a **supremum** (or a **least upper bound**) of S if it satisfies the conditions:

(1) u is an upper bound of S, and
(2) if v is any upper bound of S, then $u \leq v$.

(b) If S is bounded below, then a number w is said to be an **infimum** (or a **greatest lower bound**) of S if it satisfies the conditions:

(1′) w is a lower bound of S, and
(2′) if t is any lower bound of S, then $t \leq w$.

It is not difficult to see that *there can be only one supremum of a given subset S of* $\mathbb{R}$. (Then we can refer to *the* supremum of a set instead of *a* supremum.) For, suppose that u_1 and u_2 are both suprema of S. If $u_1 < u_2$, then the hypothesis that u_2 is a supremum implies that u_1 cannot be an upper bound of S. Similarly, we see that $u_2 < u_1$ is not possible. Therefore, we must have $u_1 = u_2$. A similar argument can be given to show that the infimum of a set is uniquely determined.

If the supremum or the infimum of a set S exists, we will denote them by

$$\sup S \quad \text{and} \quad \inf S.$$

We also observe that if u' is an arbitrary upper bound of a nonempty set S, then $\sup S \leq u'$. This is because $\sup S$ is the least of the upper bounds of S.

First of all, it needs to be emphasized that in order for a nonempty set S in $\mathbb{R}$ to have a supremum, it must have an upper bound. Thus, not every subset of $\mathbb{R}$ has a supremum; similarly, not every subset of $\mathbb{R}$ has an infimum. Indeed, there are four possibilities for a nonempty subset S of $\mathbb{R}$: it can

(i) have both a supremum and an infimum,

(ii) have a supremum but no infimum,

(iii) have an infimum but no supremum,

(iv) have neither a supremum nor an infimum.

We also wish to stress that in order to show that $u = \sup S$ for some nonempty subset S of $\mathbb{R}$, we need to show that *both* (1) and (2) of Definition 2.3.2(a) hold. It will be instructive to reformulate these statements.

The definition of $u = \sup S$ asserts that u is an upper bound of S such that $u \leq v$ for any upper bound v of S. It is useful to have alternative ways of expressing the idea that u is the "least" of the upper bounds of S. One way is to observe that any number *smaller* than u is *not* an upper bound of S. That is, if $z < u$, then z is *not* an upper bound of S. But to say that z is not an upper bound of S means there exists an element s_z in S such that $z < s_z$. Similarly, if $\varepsilon > 0$, then $u - \varepsilon$ is smaller than u and thus fails to be an upper bound of S.

The following statements about an upper bound u of a set S are equivalent:

(1) if v is any upper bound of S, then $u \leq v$,

(2) if $z < u$, then z is not an upper bound of S,

(3) if $z < u$, then there exists $s_z \in S$ such that $z < s_z$,

(4) if $\varepsilon > 0$, then there exists $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.

Therefore, we can state two alternate formulations for the supremum.

2.3.3 Lemma *A number u is the supremum of a nonempty subset S of $\mathbb{R}$ if and only if u satisfies the conditions:*

(1) *$s \leq u$ for all $s \in S$,*

(2) *if $v < u$, then there exists $s' \in S$ such that $v < s'$.*

For future work with limits, it is useful to have this condition expressed in terms of $\varepsilon > 0$. This is done in the next lemma.

2.3.4 Lemma *An upper bound u of a nonempty set S in $\mathbb{R}$ is the supremum of S if and only if for every $\varepsilon > 0$ there exists an $s_\varepsilon \in S$ such that $u - \varepsilon < s_\varepsilon$.*

Proof. If u is an upper bound of S that satisfies the stated condition and if $v < u$, then we put $\varepsilon := u - v$. Then $\varepsilon > 0$, so there exists $s_\varepsilon \in S$ such that $v = u - \varepsilon < s_\varepsilon$. Therefore, v is not an upper bound of S, and we conclude that $u = \sup S$.

Conversely, suppose that $u = \sup S$ and let $\varepsilon > 0$. Since $u - \varepsilon < u$, then $u - \varepsilon$ is not an upper bound of S. Therefore, some element s_ε of S must be greater than $u - \varepsilon$; that is, $u - \varepsilon < s_\varepsilon$. (See Figure 2.3.2.) Q.E.D.

It is important to realize that the supremum of a set may or may not be an element of the set. Sometimes it is and sometimes it is not, depending on the particular set. We consider a few examples.

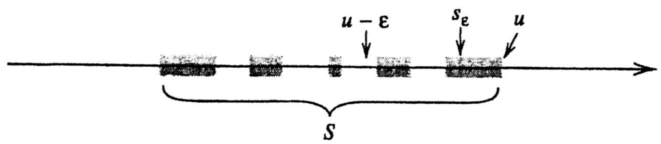

Figure 2.3.2 $u = \sup S$

2.3.5 Examples **(a)** If a nonempty set S_1 has a finite number of elements, then it can be shown that S_1 has a largest element u and a least element w. Then $u = \sup S_1$ and $w = \inf S_1$, and they are both members of S_1. (This is clear if S_1 has only one element, and it can be proved by induction on the number of elements in S_1; see Exercises 12 and 13.)

(b) The set $S_2 := \{x : 0 \le x \le 1\}$ clearly has 1 for an upper bound. We prove that 1 is its supremum as follows. If $v < 1$, there exists an element $s' \in S_2$ such that $v < s'$. (Name one such element s'.) Therefore v is not an upper bound of S_2 and, since v is an arbitrary number $v < 1$, we conclude that $\sup S_2 = 1$. It is similarly shown that $\inf S_2 = 0$. Note that both the supremum and the infimum of S_2 are contained in S_2.

(c) The set $S_3 := \{x : 0 < x < 1\}$ clearly has 1 for an upper bound. Using the same argument as given in (b), we see that $\sup S_3 = 1$. In this case, the set S_3 does *not* contain its supremum. Similarly, $\inf S_3 = 0$ is not contained in S_3. □

The Completeness Property of ℝ

It is not possible to prove on the basis of the field and order properties of ℝ that were discussed in Section 2.1 that every nonempty subset of ℝ that is bounded above has a supremum in ℝ. However, it is a deep and fundamental property of the real number system that this is indeed the case. We will make frequent and essential use of this property, especially in our discussion of limiting processes. The following statement concerning the existence of suprema is our final assumption about ℝ. Thus, we say that ℝ is a *complete ordered field.*

2.3.6 The Completeness Property of ℝ *Every nonempty set of real numbers that has an upper bound also has a supremum in* ℝ.

This property is also called the **Supremum Property** of ℝ. The analogous property for infima can be deduced from the Completeness Property as follows. Suppose that S is a nonempty subset of ℝ that is bounded below. Then the nonempty set $\bar{S} := \{-s : s \in S\}$ is bounded above, and the Supremum Property implies that $u := \sup \bar{S}$ exists in ℝ. The reader should verify in detail that $-u$ is the infimum of S.

Exercises for Section 2.3

1. Let $S_1 := \{x \in \mathbb{R} : x \ge 0\}$. Show in detail that the set S_1 has lower bounds, but no upper bounds. Show that $\inf S_1 = 0$.
2. Let $S_2 := \{x \in \mathbb{R} : x > 0\}$. Does S_2 have lower bounds? Does S_2 have upper bounds? Does $\inf S_2$ exist? Does $\sup S_2$ exist? Prove your statements.
3. Let $S_3 = \{1/n : n \in \mathbb{N}\}$. Show that $\sup S_3 = 1$ and $\inf S_3 \ge 0$. (It will follow from the Archimedean Property in Section 2.4 that $\inf S_3 = 0$.)
4. Let $S_4 := \{1 - (-1)^n/n : n \in \mathbb{N}\}$. Find $\inf S_4$ and $\sup S_4$.