MATH 6702, SPRING 2024

Tractor Connections

[**DG**] stands for *Differential Geometry* at
`https://people.math.osu.edu/derdzinski.1/courses/851-852-notes.pdf`
[**AC**] for *Algebraic Curvature Tensors* at
`https://people.math.osu.edu/derdzinski.1/courses/7711/ac.pdf`
[**SB**] for *Consequences of the Second Bianchi Identity* at
`https://people.math.osu.edu/derdzinski.1/courses/7711/sb.pdf`
[**CF**] for *Conformal Flatness* at
`https://people.math.osu.edu/derdzinski.1/courses/7711/cf.pdf`

Given a torsion-free connection ∇ and a smooth vector field v on a manifold M, by contracting the Ricci identity $v^k{}_{,ij} - v^k{}_{,ji} = R_{ijl}{}^k v^l$ in $j = k$ we see that

$$v^k{}_{,ik} - v^k{}_{,ki} = R_{ik} v^k \tag{1}$$

or, in coordinate-free notation, $\operatorname{div}\nabla v - d(\operatorname{div} v) = r(\,\cdot\,, v)$.

Lemma 1. *Let smooth functions α and ψ on a pseudo-Riemannian manifold (M,g) of any dimension m with the Ricci tensor r satisfy the "Ricci-Hessian equation"*

$$\nabla d\alpha + q\alpha r = \psi g \tag{2}$$

where q is a constant and ∇ denotes the Levi-Civita connection. Then

$$(m-1)d\psi = -(q+1)r(\nabla\alpha,\,\cdot\,) + qs\,d\alpha + q\alpha\,ds/2,$$

s being the scalar curvature, or, in coordinates,

$$(m-1)\psi_{,i} = -(q+1)R_{ik}\alpha^{,k} + qs\alpha_{,i} + q\alpha s_{,i}/2. \tag{3}$$

Proof. The g-trace of (2) yields $m\psi = \alpha^{,k}{}_{,k} + qs\alpha$. Differentiating this, we obtain

$$m\psi_{,i} = \alpha^{,k}{}_{ki} + qs\alpha_{,i} + q\alpha s_{,i}. \tag{4}$$

Applying div to the coordinate form $\psi g_{ij} = \alpha_{,ij} + q\alpha R_{ij}$ of (2) we get $\psi^{,k} g_{ik} = \alpha_{,ik}{}^k + qR_{ik}\alpha^{,k} + q\alpha R_{ik}{}^{,k}$. As symmetry of the Hessian $\nabla d\alpha$ and (1) give $\alpha_{,ik}{}^k = \alpha_{,ki}{}^k = \alpha^{,k}{}_{,ik} = \alpha^{,k}{}_{ki} + R_{ik}\alpha^{,k}$, while $2R_{ik}{}^{,k} = s_{,i}$ from the Bianchi identity for the Ricci tensor [**DG**, formula (38.13)], the last equality amounts to $\psi_{,i} = \alpha^{,k}{}_{ki} + (q+1)R_{ik}\alpha^{,k} + q\alpha s_{,i}/2$, as $\psi^{,k} g_{ik} = \psi_{,i}$. Subtracted from (4), this yields (3).

Corollary 2. *Under the assumptions of Lemma 1, $\overline{\nabla}_w(v,\alpha,\psi) = 0$ for $v = \nabla\alpha$ and any vector field w, where $\overline{\nabla}$ is the connection given by*

$$\begin{aligned}&\overline{\nabla}_w(v,\alpha,\psi)\\ &= \left(\nabla_w v - \psi w + q\alpha r w,\; d_w\alpha - g(w,v),\; d_w\psi + \frac{2(q+1)r(w,v) - 2qs\,g(w,v) - q\alpha\, d_w s}{2(m-1)}\right)\end{aligned}$$

in the vector bundle $E = TM \oplus [M \times \mathrm{I\!R}^2]$ *over* M *obtained as the direct sum of* TM *and the product plane bundle* $M \times \mathrm{I\!R}^2$.

More precisely, α *and* ψ *satisfy* (2) *if and only if* $(v, \alpha, \psi) = 0$*, with* $v = \nabla\alpha$*, is a* $\overline{\nabla}$*-parallel section of* E.

When $m \geq 3$ and $q = 1/(m-2)$, we call $E = TM \oplus [M \times \mathrm{I\!R}^2]$ the *tractor bundle* of the m-dimensional pseudo-Riemannian manifold (M, g), and refer to $\overline{\nabla}$ as the *tractor connection* in E. Explicitly, the tractor connection of (M, g) is the linear connection $\overline{\nabla}$ in E given by

$$\begin{aligned} &\overline{\nabla}_u(v, \alpha, \psi) \,=\, (\hat{v}, \hat{\alpha}, \hat{\psi}) \quad \text{for any vector field } u, \text{ where} \\ &\hat{v} \,=\, \nabla_u v \,-\, \psi u \,+\, \frac{\alpha r u}{m-2}, \qquad \hat{\alpha} \,=\, d_u\alpha \,-\, g(u, v), \\ &\hat{\psi} \,=\, d_u\psi \,+\, \frac{r(u,v)}{m-2} \,-\, \frac{2sg(u,v) \,+\, \alpha\, d_u s}{2(m-1)(m-2)}. \end{aligned} \tag{5}$$

Lemma 3. *For* M, g, m *as above and a smooth function* α *on* M*, one has*

$$\nabla d\alpha + \frac{\alpha r}{m-2} = \psi g \qquad \text{with some smooth function } \psi \tag{6}$$

if and only if the triple $(\nabla\alpha, \alpha, \psi)$ *is a* $\overline{\nabla}$*-parallel section of the tractor bundle* E.

Proof. Apply Lemma 1 and Corollary 2 to $q = 1/(m-2)$.

Lemma 4. *Under the assumptions of Lemma* 3*, one has* (6) *if and only if* $\tilde{g} = g/\alpha^2$*, defined on the open subset on which* $\alpha \neq 0$*, is an Einstein metric.*

Proof. Use formula (8) in [**CF**] and Schur's theorem [**DG**, Section 41].

As in [**SB**], given a torsion-free connection ∇ and a (not necessarily symmetric) twice-covariant smooth tensor field b on a manifold M, we define the *exterior derivative* of b to be the $(0,3)$ tensor field db with $[db]_{ijk} = b_{jk,i} - b_{ik,j}$. When ∇ is the Levi-Civita connection of a pseudo-Riemannian metric g on M, one also has the raised-index version of db, here denoted by Db, for which

$$[Db]_{ij}^k \,=\, g^{kl}(b_{jl,i} \,-\, b_{il,j}). \tag{7}$$

Lemma 5. *The curvature tensor* $\bar{R}$ *of the tractor connection* $\overline{\nabla}$ *is given by*

$$\bar{R}(u, u')(v, \alpha, \psi) \,=\, (\tilde{v}, \tilde{\alpha}, \tilde{\psi}),$$

for any vector fields u, u' *tangent to* M*, where*

$$\tilde{v} = W(u, u')v - \frac{\alpha}{m-2}[Dh](u, u'), \qquad \tilde{\alpha} = 0, \qquad \tilde{\psi} = -\frac{g(v, [Dh](u, u'))}{m-2}$$

with h *denoting the Schouten tensor, and* Dh *as in* (7).

Proof. We may assume that at the point x in question $d\alpha$, $d\psi$ and the covariant derivatives of u, u', v all vanish (and hence so does $[u, u']$). Thus,

$$\bar{R}(u,u')(v,\alpha,\psi) \;=\; \overline{\nabla}_{u'}(\hat{v},\hat{\alpha},\hat{\psi}) \;-\; \ldots,$$

with $(\hat{v},\hat{\alpha},\hat{\psi})$ defined by (5) and $\ldots$ standing for the result of switching u with u' in the expression for $\overline{\nabla}_{u'}(\hat{v},\hat{\alpha},\hat{\psi})$ at x obtained from (5). Consequently,

$$\begin{aligned}\tilde{v} \;=\; & R(u,u')v + \alpha\frac{[\nabla_{u'}r]u - [\nabla_u r]u'}{m-2} + \frac{r(u',v)u - r(u,v)u'}{m-2}\\ &+ \frac{s[g(u,v)u' - g(u',v)u]}{(m-1)(m-2)} + \frac{\alpha[(d_u s)u' - (d_{u'}s)u]}{2(m-1)(m-2)} + \frac{g(u',v)ru - g(u,v)ru'}{m-2},\end{aligned}$$

while $\tilde{\alpha} = 0$ and

$$\tilde{\psi} \;=\; \frac{[\nabla_{u'}r](u,v) - [\nabla_u r](u',v)}{m-2} \;+\; \frac{(d_u s)g(u',v) - (d_{u'}s)g(u,v)}{2(m-1)(m-2)}.$$

Our assertion is now immediate from the expressions for h and W in [**AC**, the formula preceding (5)], combined with (7).

Lemma 6. *For a pseudo-Riemannian manifold (M,g) of any dimension $m \geq 3$, the following four conditions are equivalent.*

(a) *The tractor connection $\overline{\nabla}$ is flat.*
(b) *The Weyl tensor W and dh, for the Schouten tensor h of g, vanish identically.*
(c) *The metric g is conformally flat.*
(d) *Either $m \geq 4$ and $W = 0$, or $m = 3$ and $dh = 0$, everywhere in M.*

Proof. From (a) we get $\tilde{\psi} = 0$ and $\tilde{v} = 0$ in Lemma 5, for any vector fields v, u, u' tangent to M, so that $Dh = 0$ and, consequently, $W = 0$, which implies (b). Lemma 5 clearly yields the converse implication. Assuming (c) we obtain (b): namely, $W = 0$ due to conformal invariance of the type $(1,3)$ Weyl tensor [**CF**, formula 6]; the conformally-Einstein property of the metric allows us – via Lemmas 3 and 4 – to choose, locally, $\overline{\nabla}$-parallel sections $(\nabla\alpha, \alpha, \psi)$ of the tractor bundle E having $\alpha \neq 0$, while, by $\overline{\nabla}$-parallelity, $\bar{R}(\,\cdot\,,\,\cdot\,)(v,\alpha,\psi) = 0$, and so the formula for $\tilde{v}$ (see Lemma 5) with $W = 0$ and $\alpha \neq 0$ shows that $dh = 0$. On the other hand, if (a) holds, (c) follows

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Finally, condition (b) trivially leads to (d), while (d) gives (b) as a consequence of [**AC**, Remark 2] and the identity $(m-2)\,\mathrm{div}W = -(m-3)\,dh$ in [**SB**].