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SOME PROPERTIES OF CONFORMALLY SYMMETRIC MANIFOLDS WHICH ARE NOT RICCI-RECURRENT.

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1. Introduction. An n-dimensional ($n \geq 4$) Riemannian manifold M (whose metric g_{ij} need not be definite) is said to be conformally symmetric [1][1)] if its Weyl's conformal curvature tensor

$$(1)\qquad C_{hijk} = R_{hijk} - (g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik})/(n-2) + R(g_{hk}g_{ij} - g_{ik}g_{hj})/(n-1)(n-2)$$

satisfies the condition

$$(2)\qquad C_{hijk,l} = 0\,,$$

where the comma indicates covariant differentiation with respect to the metric of M. Clearly the class of conformally symmetric manifolds contains all conformally flat as well as all locally symmetric manifolds of dimension $n \geq 4$.

Since a conformally symmetric manifold with a positive definite metric is necessarily conformally flat or locally symmetric ([3], Theorem 2), a natural question arises of the existence of essentially conformally symmetric manifolds, i.e., of conformally symmetric manifolds which lie beyond the two classes mentioned above. The answer to this problem is affirmative and can be stated as follows (see [6], Theorem 3 and [7], Theorem 1):

Theorem 1. *Let M denote the Euclidean n-space ($n \geq 4$) endowed with the metric g_{ij} given by*

$$(3)\qquad g_{ij}dx^i dx^j = \varphi(dx^1)^2 + k_{\lambda\mu}dx^\lambda dx^\mu + 2dx^1 dx^n\,, \qquad \varphi = (Qk_{\lambda\mu} + c_{\lambda\mu})x^\lambda x^\mu\,,$$

where $i, j = 1, 2, \ldots, n$, $\lambda, \mu = 2, 3, \ldots, n-1$, $[k_{\lambda\mu}]$ is a symmetric non-singular matrix and $[c_{\lambda\mu}]$ is a symmetric non-zero matrix satisfying $k^{\lambda\mu}c_{\lambda\mu} = 0$, $[k^{\lambda\mu}]$ being the reciprocal of $[k_{\lambda\mu}]$, and Q is a non-constant function of x^1 only. Then M is essentially conformally symmetric.

The metrics defined by (3) are, moreover, Ricci-recurrent ([6], Theorem 3), i.e., for each point $x \in M$ such that $R_{ij}(x) \neq 0$, there exists a tangent vector ϕ_j at x which satisfies the condition

$$(4)\qquad R_{ij,k}(x) = \phi_k R_{ij}(x)\,.$$

The existence of essentially conformally symmetric non-Ricci-recurrent manifolds has been established in [2] as follows:

Theorem 2. *Let M denote the n-dimensional ($n \geq 4$) Euclidean space endowed with*

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1) Numbers in brackets refer to the references at the end of the paper.

the metric g_{ij} *defined by*

$$g_{ij} = \begin{cases} \exp F_i & \text{if } i+j=n+1 \\ -1 & \text{if } i=j=1 \\ 0 & \text{otherwise}, \end{cases}$$

the functions $F_i = F_{n-i+1}\colon M \to R$ *being given by*

$$F_1(x^1, \ldots, x^n) = F_n(x^1, \ldots, x^n) = 2bx^2 - ax^1 - c(x^1)^2 ,$$
$$F_2(x^1, \ldots, x^n) = F_{n-1}(x^1, \ldots, x^n) = 2c(x^1)^2 + 2ax^1 - bx^2 ,$$
$$F_\lambda(x^1, \ldots, x^n) = 2c(x^1)^2 + 2ax^1 + 2bx^2 , \qquad \lambda = 3, \ldots, n-2 ,$$

where a, b are any real numbers distinct from zero and c is an arbitrary real number. Then M is an essentially conformally symmetric non-Ricci-recurrent Riemannian manifold for which the condition $\operatorname{rank} R_{ij} = 2$ *holds on some open dense subset of M. For $c = 0$, this subset coincides with M.*

In Section 2 of this paper it is shown (Theorem 3) that any essentially conformally symmetric manifold admits a unique function F such that

$$(5) \qquad FC_{hijk} = R_{ij}R_{hk} - R_{ik}R_{hj} .$$

Section 3 contains the main results of this paper. Theorem 4 states that any essentially conformally symmetric non-Ricci-recurrent manifold admits a unique parallel absolute exterior 2-form ω satisfying

$$(6) \qquad C_{hijk} = e\omega_{hi}\omega_{jk}$$

with $|e| = 1$, $\operatorname{rank} \omega = 2$ and $\omega_{ir}\omega^r{}_k = 0$.

Next we prove (Theorem 5) that every essentially conformally symmetric manifold satisfies $\operatorname{rank} R_{ij} \leq 2$. In Theorem 6 we establish the existence of essentially conformally symmetric non-Ricci-recurrent manifolds such that

$$(7) \qquad \operatorname{rank} R_{ij} \leq 1 .$$

At the end of Section 3 we observe (Theorem 8) that the curvature tensor of any essentially conformally symmetric manifold has a simple algebraic structure.

Section 4 deals with certain global properties of analytic essentially conformally symmetric manifolds. Such a manifold always admits a totally isotropic parallel distribution of dimension 1 or 2 (Theorem 9), so that it must admit a 2-dimensional distribution of class C^∞ (Theorem 10).

All manifolds considered below are assumed to be connected, paracompact and of class C^∞ or analytic.

2. Preliminaries. In the sequel we shall need the following lemmas:

Lemma 1. *The Weyl's conformal curvature tensor satisfies the well-known relations*

$$(8) \qquad C_{hijk} = -C_{ihjk} = -C_{hikj} = C_{jkhi} ,$$

$$(9) \qquad C_{hijk} + C_{hjki} + C_{hkij} = 0 , \qquad C^r{}_{ijr} = C^r{}_{irk} = C^r{}_{rjk} = 0 .$$

Lemma 2 (see [6], Lemma 2). *If* a_j and P_{lhmjk} *are tensors satisfying*

$$P_{lhmjk} = -P_{lhmkj} , \qquad 2a_iP_{lhmjk} + a_jP_{lhmik} + a_kP_{lhmji} = 0 ,$$

then $a_j = 0$ *or* $P_{hijkl} = 0$.

Lemma 3. *Any essentially conformally symmetric manifold satisfies the following relations* (see [4], formulae (5), (62) and (63)):

$$\text{(10)}\qquad \text{a)}\ R_{ij,k} = R_{ik,j}\,,\qquad \text{b)}\ R = 0\,,\qquad \text{c)}\ R_{ri}C^r{}_{jkl} = 0\,,$$

$$\text{(11)}\qquad R_{ri}R^r{}_j = 0\,,$$

$$\text{(12)}\qquad \begin{aligned} &R_{hl}C_{mijk} - R_{hm}C_{lijk} + R_{il}C_{hmjk} - R_{im}C_{hljk} \\ &+ R_{jl}C_{himk} - R_{jm}C_{hilk} + R_{kl}C_{hijm} - R_{km}C_{hijl} = 0\,, \end{aligned}$$

$$\text{(13)}\qquad C_{hijk} = R_{hijk} - (g_{ij}R_{hk} - g_{ik}R_{hj} + g_{hk}R_{ij} - g_{hj}R_{ik})/(n-2)\,.$$

The last relation is an immediate consequence of (1) and (10)b).

By an absolute r-form on a manifold we shall mean an r-form, defined at each point up to a sign (see [8], p. 204).

It is clear how to define smoothness and parallelity of absolute forms.

Lemma 4. *Let M be an essentially conformally symmetric manifold. Then the following three conditions are equivalent:* (i) *There exists $x \in M$ and exterior 2-forms A and B at x such that $C_{hijk}(x) = A_{hi}B_{jk}$.* (ii) *$C_{hijk} = e\omega_{hi}\omega_{jk}$, where $|e| = 1$ and ω is a (uniquely determined) parallel absolute exterior 2-form of rank* 2 *on M.* (iii) $C_{hijk}C_{lmpq} = C_{hipq}C_{lmjk}$.

Proof. By (11), (i) implies $A_{hi}B_{jk} = A_{jk}B_{hi}$, so that $B_{jk} = cA_{jk}$ for some $c \neq 0$, since $C_{hijk} \neq 0$. Hence $C_{hijk}(x) = eD_{hi}D_{jk}$, where $|e| = 1$, and $e(D_{jk})^2 = C_{jkjk}(x)$ shows that D_{jk} is unique up to a sign. Since C_{hijk} is parallel, its algebraic shape must be the same at each point of M, which implies (6). Parallelity of ω_{ij} follows from that of C_{hijk} together with the uniqueness of ω_{ij}. Rank $\omega = 2$ since $\omega \wedge \omega = 0$, which is immediate from (9). Thus (ii) follows from (i). The implication (ii)$\rightarrow$(iii) is trivial. Assume now (iii) and choose $x \in M$ and vectors a, b, c, d at x such that $a^h b^i c^j d^k C_{hijk}(x) = 1$. Transvecting (iii) with $a^h b^i c^j d^k$ we obtain (i), as desired.

Theorem 3. *Any essentially conformally symmetric manifold M admits a unique function F such that $R_{ij}R_{hk} - R_{ik}R_{hj} = FC_{hijk}$. Clearly, $F(x) = 0$ if and only if* rank $R_{ij}(x) \leq 1$.

Proof. Our assertion is trivial ($F = 0$) if rank $R_{ij} \leq 1$ everywhere. Suppose now that $x \in M$ and

$$\text{(14)}\qquad \operatorname{rank} R_{ij}(x) > 1\,.$$

We may choose a vector u at x such that $u^r u^s R_{rs} = e$, $|e| = 1$. Setting

$$d_j = u^r R_{rj}(x)\,,\qquad B_{ij} = B_{ji} = u^r u^s C_{rijs}(x)\,,\qquad S_{ijk} = u^r C_{rijk}(x)\,,$$

so that $S_{ijk} = -S_{ikj}$ and

$$\text{(15)}\qquad S_{ijk} + S_{jki} + S_{kij} = 0\,,$$

and transvecting (12) with $u^h u^l$, we obtain

$$\text{(16)}\qquad C_{mijk} = e(d_m S_{ijk} + d_i S_{mkj} + d_j S_{ikm} + d_k S_{imj} + R_{km}B_{ij} - R_{jm}B_{ik})\,,$$

which, in view of $C_{mijk} = C_{jkmi}$ and by a further transvection with u^j, yields

(17) $$S_{kmi} + S_{imk} = e(d_i B_{mk} + d_k B_{mi} - 2d_m B_{ik}) .$$

In virtue of (16) and (17), relation $C_{mijk} - C_{jkmi} = 0$ can be written as

$$B_{ij}(R_{km} - ed_k d_m) = B_{km}(R_{ij} - ed_i d_j) ,$$

which yields, by (14),

(18) $$B_{ij} = G(R_{ij} - ed_i d_j)$$

for some real G. This turns (17) into

$$S_{kmi} + S_{imk} = eG(d_i R_{km} + d_k R_{mi} - 2d_m R_{ki}) ,$$

which states that the tensor

$$T_{kmi} = S_{kmi} - eG(d_i R_{km} - d_m R_{ki})$$

is skew-symmetric in all indices. By (15), $3T_{kmi} = T_{kmi} + T_{mik} + T_{ikm} = 0$, which, together with (16) and (18), implies $C_{mijk} = eG(R_{ij}R_{mk} - R_{ik}R_{mj})$ at x. Since $C_{mijk} \neq 0$, we have $G \neq 0$, which completes the proof.

Lemma 5. *Let M be an essentially conformally symmetric manifold such that*

(19) $$a_l C_{jkhm} + a_m C_{jklh} + a_h C_{jkml} = 0$$

for some field a_i of non-zero vectors. If C_{hijk} is not of the form (6), then

(20) $$a_{i,j} = A_j a_i$$

for a certain vector field A_i on M. Moreover, if

(21) $$a_{i,j} = a_{j,i}$$

then rank $R_{ij} \leq 1$.

Proof. Choose a vector field v^i such that $v^r a_r = 1$. Transvecting (19) which v^l and then with v^j, we find

(22) $$C_{jkhm} = a_m v^r C_{rhkj} - a_h v^r C_{rmkj} ,$$

(23) $$v^r C_{rkhm} = a_m S_{hk} - a_h S_{mk} ,$$

where $S_{ij} = S_{ji} = v^r v^s C_{rijs}$. Substituting (23) into (22), we obtain

(24) $$C_{jkhm} = a_h a_k S_{mj} - a_h a_j S_{mk} + a_m a_j S_{hk} - a_m a_k S_{hj} .$$

Differentiating now (19) covariantly and transvecting the resulting equality with $v^l v^j$, we get

$$A_p v^r C_{rkhm} = a_{m,p} S_{hk} - a_{h,p} S_{mk} ,$$

where $A_p = v^r a_{r,p}$. By (23), this yields

(25) $$(a_{h,p} - A_p a_h) S_{mk} = (a_{m,p} - A_p a_m) S_{hk} .$$

If $a_{h,p} - A_p a_h$ did not vanish identically, then, by (25), we would have rank $S_{ij} \leq 1$ at some point $x \in M$, say $S_{ij}(x) = ec_i c_j$, $|e| = 1$. In view of (24) and Lemma 4, this would imply (6), a contradiction. Thus we obtain (20).

Assume now (21). Contracting (19) with g^{jl} and using (9), we obtain

(26) $$a^r C_{rkhm} = 0\,,$$

which, by a transvection of (19) with a^l, implies $a^r a_r = 0$. Transvecting (19) with $R_p{}^l$ we obtain, by (10)c),

(27) $$a_r R^r{}_k = 0\,.$$

By (20) and (21) we have $a_{i,j} = b a_i a_j$ for some function b. Hence Ricci identity implies $a_r R^r{}_{jkl} = (b_{,k} a_l - b_{,l} a_k) a_j$, which, in view of (13), (26) and (27), can be written as

$$a_l(R_{jk} - (n-2)a_j b_{,k}) = a_k(R_{jl} - (n-2)a_j b_{,l})$$

so that $R_{jk} = (n-2)a_j b_{,k} + c_j a_k$ for some vector field c_j. By Theorem 3,

$$(n-2)(a_i c_h - c_i a_h)(b_{,j} a_k - a_j b_{,k}) = F C_{hijk}\,.$$

If we had rank $R_{ij} > 1$ at some point x, then $F(x) \neq 0$ and, by Lemma 4, we would obtain (6), a contradiction. This completes the proof.

3. Main results.

Lemma 6. *Let M be an essentially conformally symmetric non-Ricci-recurrent manifold whose Ricci tensor satisfies* rank $R_{ij} \leq 1$. *Then its Weyl conformal curvature tensor is of the form* (6).

Proof. Alternating (12) in h, l, m, we obtain

$$\begin{aligned}2(R_{il}C_{hmjk} + R_{im}C_{lhjk} + R_{ih}C_{mljk}) + R_{jm}(C_{lihk} - C_{hilk})\\ + R_{jl}(C_{himk} - C_{mihk}) + R_{hj}(C_{milk} - C_{limk}) + R_{kl}(C_{hijm} - C_{mijh})\\ + R_{km}(C_{lijh} - C_{hijl}) + R_{kh}(C_{mijl} - C_{lijm}) = 0\,,\end{aligned}$$

which, by (9), yields

(28) $$\begin{aligned}2(R_{il}C_{hmjk} + R_{im}C_{lhjk} + R_{ih}C_{mljk}) + R_{jm}C_{lhik}\\ + R_{jl}C_{hmik} + R_{hj}C_{mlik} + R_{kl}C_{hmji} + R_{km}C_{lhji} + R_{kh}C_{mlji} = 0.\end{aligned}$$

In view of our assumption, we may choose $x \in M$ such that $R_{ij}(x) \neq 0$ and (4) is not satisfied by any vector ϕ. Thus, in some neighbourhood of x we have

(29) $$R_{ij} = e a_i a_j\,, \qquad |e| = 1\,,$$

a_i being a C^∞ vector field. Substituting (29) into (28), we obtain

$$\begin{aligned}2a_i(a_l C_{hmjk} + a_m C_{lhjk} + a_h C_{mljk}) + a_j(a_l C_{hmik} + a_m C_{lhik} + a_h C_{mlik})\\ + a_k(a_l C_{hmji} + a_m C_{lhji} + a_h C_{mlji}) = 0\,,\end{aligned}$$

which, in view of Lemma 2, implies

(30) $$a_l C_{hmjk} + a_m C_{lhjk} + a_h C_{mljk} = 0\,.$$

If C_{hijk} were not of the form (6), Lemma 5 would yield (20) and (4) would follow with $\phi_i = 2A_i$, a contradiction. This completes the proof.

Lemma 7. *Let M be an essentially conformally symmetric manifold. If the function F determined in Theorem 3 is not constant, then C_{hijk} is of the form* (6).

Proof. Differentiating (5) covariantly, we obtain

$$F_{,p}C_{hijk} = R_{ij,p}R_{hk} + R_{ij}R_{hk,p} - R_{ik,p}R_{hj} - R_{ik}R_{hj,p} . \tag{31}$$

Alternating this in p, j, k and using (10)a), we obtain (19) with $a_i = F_{,i}$. Choose an open submanifold U of M such that $F \neq 0$ and $F_{,i} \neq 0$ everywhere in U. If C_{hijk} were not of the form (6), then Lemma 5 applied to the manifold U would yield (7) in U, contradicting $F \neq 0$. This completes the proof.

Lemma 8. *Let M be an essentially conformally symmetric manifold. If the function F determined in Theorem 3 satisfies $F = constant \neq 0$, then C_{hijk} is of the form (6).*

Proof. Formula (31) yields

$$R_{ij,p}R_{hk} + R_{ij}R_{hk,p} = R_{ik,p}R_{hj} + R_{ik}R_{hj,p} . \tag{32}$$

Choose an open subset U of M and a vector field u^i on U such that

$$R_{ij,k} \neq 0 \tag{33}$$

everywhere in U and $u^r u^s R_{rs} = e$, $|e| = 1$. Setting

$$d_j = u^r R_{rj} , \qquad D_{kp} = D_{pk} = u^r R_{rk,p} , \qquad T_j = u^r D_{rj} , \qquad T = u^r T_r ,$$

and transvecting (32) with $u^i u^j$, we obtain, by (10)a),

$$R_{hk,p} = e(d_h D_{kp} + d_k D_{hp} - T_p R_{hk}) , \tag{34}$$

which implies, by transvection with u^p,

$$D_{hk} = e(T_k d_h + T_h d_k - T R_{hk}) . \tag{35}$$

Substituting this into (34), we obtain

$$R_{hk,p} = 2T_p d_h d_k + T_k d_h d_p + T_h d_k d_p - eT_p R_{hk} - T d_h R_{kp} - T d_k R_{hp} , \tag{36}$$

which, in view of (10)a), yields

$$d_j d_h (T_k d_p - T_p d_k) = d_j (T d_p - eT_p) R_{hk} - d_j (T d_k - eT_k) R_{hp} .$$

Alternating the last relation in j, h, we obtain

$$(T d_p - eT_p)(d_j R_{hk} - d_h R_{jk}) = (T d_k - eT_k)(d_j R_{hp} - d_h R_{jp}) \tag{37}$$

which, by transvection with u^j, implies

$$(T d_p - eT_p)(R_{hk} - e d_h d_k) = (T d_k - eT_k)(R_{hp} - e d_h d_p) .$$

In the case where $T d_p - eT_p \neq 0$ at some $x \in M$, this yields $R_{ij} = e d_i d_j + d c_i c_j$ for some vector c_i ($|d| = 1$) and (6) follows from (5) combined with Lemma 4. Suppose therefore

$$T d_p - eT_p = 0 . \tag{38}$$

Then (35) takes the form $D_{hk} = T(2 d_h d_k - e R_{hk})$. Substituting this into

$$D_{ij} R_{hk} + R_{ij} D_{hk} = D_{ik} R_{hj} + R_{ik} D_{hj} ,$$

which is an obvious consequence of (32), using (5) and noting that $T \neq 0$ by (36) and (33), we obtain in U

$$F C_{hijk} = e(d_i d_j R_{hk} + d_h d_k R_{ij} - d_i d_k R_{hj} - d_h d_j R_{ik}) . \tag{39}$$

Suppose now that our assertion fails. Multiplying (39) by d_p and alternating the resulting equality in p, h, i, we obtain $d_pC_{hijk} + d_hC_{ipjk} + d_iC_{phjk} = 0$ and, by Lemma 5,

$$d_{i,j} = A_jd_i \tag{40}$$

for some vector field A_j in U.

On the other hand, (36) and (38) imply

$$R_{hk,p} = -T(d_pR_{hk} + d_hR_{kp} + d_kR_{hp} - 4ed_pd_hd_k) . \tag{41}$$

Differentiating (39) covariantly and making use of (40) and (41), we easily obtain

$$(2A_p - Td_p)(d_id_jR_{hk} + d_hd_kR_{ij} - d_jd_hR_{ik} - d_id_kR_{hj}) = 0 .$$

Since $FC_{hijk} \neq 0$, this implies $A_j = \frac{1}{2}Td_j$, so that (40) yields $d_{i,j} = d_{j,i}$. From Lemma 5 it follows now that rank $R_{ij} \leq 1$, i.e., $F = 0$ in U, a contradiction. This completes the proof.

We are now in a position to state the main results of this section.

Theorem 4. *Let M be an essentially conformally symmetric manifold. If M is not Ricci-recurrent, then $C_{hijk} = e\omega_{hi}\omega_{jk}$ where $|e| = 1$ and ω is a (uniquely determined) parallel absolute 2-form satisfying*

$$\text{rank}\, \omega = 2 \quad \text{and} \quad \omega_{ir}\omega^r{}_j = 0 . \tag{42}$$

In fact, all possible cases ($F = 0$, $F = \text{constant} \neq 0$ and F non-constant, F determined by (5)) are covered by Lemmas 6, 8 and 7. Relations (42) are obvious algebraic consequences of (6) (cf. Lemma 4).

Theorem 5. *Every essentially conformally symmetric manifold satisfies the relation* rank $R_{ij} \leq 2$.

Proof. If M is Ricci-recurrent, then at points where $R_{ij,k} \neq 0$ we obtain, from (4) and (10)a), rank $R_{ij} \leq 1$, which extends to the whole of M by an elementary boundary argument. In the non-Ricci-recurrent case, let $x \in M$. If $R_{ij}(x) \neq 0$, we may choose a vector u^i at x such that $R_{ij}u^iu^j = d$, $|d| = 1$. Then we have, by (6), $FC_{hijk}u^hu^k = -eFw_iw_j$, where $w_j = u^r\omega_{rj}$ and $(R_{ij}R_{hk} - R_{hj}R_{ik})u^hu^k = dR_{ij} - d_id_j$, where $d_j = u^rR_{rj}$. Hence, by (5), $R_{ij} = dd_id_j - edFw_iw_j$, which completes the proof.

As shown in the above proof, if a given essentially conformally symmetric manifold is Ricci-recurrent, then the assertion of Theorem 5 can be strengthened to the form rank $R_{ij} \leq 1$. The converse statement, however, fails in general, which can be seen as follows.

Theorem 6. *Let M denote the Euclidean four-space R^4 endowed with the Riemannian metric g_{ij} whose components at any point (x, y, z, u) are given by*

$$g_{11} = g_{12} = g_{22} = g_{23} = 0 , \quad g_{13} = g_{24} = 1 , \quad g_{14} = \tfrac{1}{3}z , \quad g_{33} = 18Ay ,$$
$$g_{34} = x + 6Ayz , \quad g_{44} = \tfrac{2}{3}xz + \tfrac{4}{3}y + 2Ayz^2 - 2\exp(-2u) ,$$

where A is a fixed non-zero real number. Then M is an essentially conformally symmetric manifold which is not Ricci-recurrent but satisfies the condition rank $R_{ij} = 1$.

Proof. The contravariant metric tensor g^{ij} is clearly given by

$$g^{11} = -18Ay\,, \quad g^{12} = -x\,, \quad g^{13} = g^{24} = 1\,, \quad g^{23} = -\tfrac{1}{3}z\,,$$
$$g^{22} = -\tfrac{4}{3}y + 2\exp(-2u)\,, \quad g^{14} = g^{33} = g^{34} = g^{44} = 0\,.$$

It is easy to see that the only non-zero components of the Riemannian connection, curvature tensor, Ricci tensor and Weyl's tensor are those related to

$$\Gamma^1_{14} = \tfrac{1}{3}\,, \quad \Gamma^1_{23} = 9A\,, \quad \Gamma^1_{24} = 3Az\,, \quad \Gamma^1_{33} = 9Ax\,, \quad \Gamma^1_{34} = 6Ay + 3Axz\,,$$
$$\Gamma^1_{44} = 4Ayz + \tfrac{1}{3}x + Axz^2\,, \quad \Gamma^2_{13} = \tfrac{2}{3}\,, \quad \Gamma^2_{14} = \tfrac{2}{9}z\,, \quad \Gamma^2_{24} = \tfrac{2}{3}\,,$$
$$\Gamma^2_{33} = 18Ay - 18A\exp(-2u)\,, \quad \Gamma^2_{34} = \tfrac{2}{3}x + 6Ayz - 6Az\exp(-2u)\,,$$
$$\Gamma^2_{44} = \tfrac{4}{9}xz + \tfrac{8}{9}y + 2Ayz^2 - 2Az^2\exp(-2u) + \tfrac{2}{3}\exp(-2u)\,,$$
$$\Gamma^3_{33} = 3Az\,, \quad \Gamma^3_{34} = -\tfrac{1}{3} + Az^2\,, \quad \Gamma^3_{44} = -\tfrac{1}{9}z + \tfrac{1}{3}Az^3\,,$$
$$\Gamma^4_{33} = -9A\,, \quad \Gamma^4_{34} = -3Az\,, \quad \Gamma^4_{44} = -\tfrac{2}{3} - Az^2\,;$$

$$R_{2334} = 6A\,, \quad R_{2434} = 2Az\,, \quad R_{3434} = -8Ay - 6A\exp(-2u)\,;$$

$$R_{33} = 12A\,, \quad R_{34} = 4Az\,, \quad R_{44} = \tfrac{4}{3}Az^2\,, \tag{43}$$

and, respectively,

$$C_{3434} = -18A\exp(-2u)\,. \tag{44}$$

It is now easy to verify (2). Moreover, M is not conformally flat by (44). Thus, the relations $R_{33,3} = 0$ and $R_{34,3} = 8A$ show that M is essentially conformally symmetric and non-Ricci-recurrent. Now (43) yields $R_{33}R_{44} - R_{34}R_{43} = 0$. This completes the proof.

Theorem 7. *Every essentially conformally symmetric manifold M satisfies the relation*

$$R_{mp}C_{hijk} + R_{mh}C_{ipjk} + R_{mi}C_{phjk} = 0\,. \tag{45}$$

Proof. Let $x \in M$. If $F(x) \neq 0$, then our assertion is an immediate consequence of (5) and of $R_{ij} = da_ia_j + eb_ib_j$, (where $|d| = |e| = 1$), which follows immediately from Theorem 5.

In the case $F(x) = 0$, (45) is an obvious consequence of (29) and (30).

Theorem 8. *Let M be an essentially conformally symmetric non-Ricci-recurrent manifold. Then at each $x \in M$ such that $R_{ij}(x) \neq 0$ we have a relation of the form*

$$R_{hijk} = R_{ij}B_{hk} + R_{hk}B_{ij} - R_{hj}B_{ik} - R_{ik}B_{hj} \tag{46}$$

for some symmetric tensor B_{ij} at x.

Proof. By Theorem 5, we have two cases. If rank $R_{ij}(x) = 1$, say $R_{ij} = da_ia_j$, where $|d| = 1$ and a is a non-zero 1-form, then (45) and (6) yield $a \wedge \omega = 0$, so that $\omega = a \wedge b$ for some 1-form b. From (13) we obtain (46) with $B_{hk} = g_{hk}/(n-2) - edb_hb_k$. Assume now rank $R_{ij}(x) = 2$. In this case relation (46) with $B_{hk} = R_{hk}/2F + g_{hk}/(n-2)$ follows immediately from (13) and (5). This completes the proof.

Remark. An analogous statement holds in the Ricci-recurrent case ([6], Lemma 5).

4. Some global properties. We are now going to derive some consequences of the above results.

Theorem 9. *Let M be an essentially conformally symmetric manifold.* (i) *If M is analytic and Ricci-recurrent, then it admits a parallel field L of tangent isotropic lines such that any vector d_i of the form*

$$d_i = u^r R_{ri} \tag{47}$$

lies in L. (ii) *If M is not Ricci-recurrent, then it admits a parallel field P of totally isotropic tangent 2-planes which contains all vectors of the form*

$$d_i = C_{ijkl} a^j b^k c^l \, . \tag{48}$$

Proof. (i) Let $\bar{M}$ be the Riemannian universal covering of M, so that $M = \bar{M}/\Gamma$, Γ being a group of isometries. Choose $x \in \bar{M}$ with $R_{ij}(x) \neq 0$ and $R_{ij,k}(x) \neq 0$. By Theorem 3 of [6] the metric of $\bar{M}$ is of type (3) in some connected neighbourhood U of x. By an easy computation we verify that the isotropic vector field v with components $(0, \ldots, 0, 1)$ (in the chart determined in (3)) is the unique (up to a factor) parallel vector field in U and that any vector of type (47) is a multiple of v (cf. [6], p. 93). Thus v_x is left invariant by the local holonomy group of $\bar{M}$ at x and therefore it is invariant by the whole holonomy group ([5], Theorem 10.8, p. 101) so that v extends to a parallel isotropic vector field on $\bar{M}$, denoted again by v. For any isometry f of $\bar{M}$ onto itself we have $f_* v = tv$ in U for some real t, since $f_* v$ is parallel. By analyticity, the same remains true on $\bar{M}$, so that the parallel line field determined by v is invariant under the action of Γ and therefore it defines a line field in M.

(ii) Define P to be the set of all vectors of type (48). By (6), and (42), P is a parallel totally isotropic field of 2-planes on M. For any vector d of the form (47), formulae (6) and (45) yield $d \wedge \omega = 0$ which means, geometrically, that d is in the image of ω. This completes the proof.

Theorem 10. *Every analytic essentially conformally symmetric manifold M admits a C^∞ field D of tangent 2-planes.*

Proof. Let M be Ricci-recurrent and denote by L the isotropic line field determined in (i) of Theorem 9. Choose a positive definite C^∞ Riemannian metric h_{ij} on M. Define a line field K by assigning to $x \in M$ the set K_x of all vectors $w^i = g^{ir} h_{rs} d^s$, where d^i runs through L_x. We have $d_i w^i \neq 0$ if $d^i \neq 0$, which proves that $K_x \neq L_x$ for any x. Setting $D = K + L$ we obtain our assertion.

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