Solution Hints for Autumn 96 Exam

1. Since $\lim_{x\to\infty} f(x+1)/f(x)$ exists and is less than 1, we can chose an $\epsilon > 0$ and an $M > 0$ such that if $x > M$ then $f(x+1)/f(x) < 1-\epsilon$. It suffices to show the integral $\int_M^\infty f(x)\,dx$ converges. To do this choose $x_k \in [M+k, M+k+1]$ such that $f(x_x) = \sup_{[M+k,M+k+1]} f(x)$ (can do this since f is continuous). Note that since $x_{k+1}-1 \in [M+k, M+k+1]$, we have $f(x_{k+1}-1) < f(x_k)$ and so $f(x_{k+1}) < (1-\epsilon)f(x_{k+1}-1) < (1-\epsilon)f(x_k)$. Thus by the ratio test the series $\sum_{k=0}^\infty f(x_k)$ converges. Let $G(x)$ be the function whose value on $[M+k, M+k+1)$ is $f(x_k)$. Since $f(x_{k+1}-1) < f(x_k)$, $G(x)$ is monotonically decreasing (and of course nonnegative). The integral test implies $\int_M^\infty G(x)\,dx$ converges. Since $f(x) \le G(x)$, $\int_M^\infty f(x)\,dx$ converges also.

   Another solution (courtesy of our colleagues). Choose an integer M and an $\epsilon \in (0,1)$ such that for all $x \ge M$, $f(x+1) \le (1-\epsilon)f(x)$. Then for all $x \ge M$, we have $x+2 \ge M$ also so $f(x+2) \le (1-\epsilon)f(x+1) \le (1-\epsilon)^2 f(x)$. Inductively we have for all $x \ge M$ and all natural numbers k, $f(x+k) \le (1-\epsilon)^k f(x)$. Now

$$\int_0^\infty f(x)\,dx = \lim_{N\to\infty}\int_0^{N+M} f(x)\,dx = \lim_{N\to\infty}\left(\int_0^M f(x)\,dx + \sum_{k=0}^{N-1}\int_{M+k}^{M+k+1} f(x)\,dx\right).$$

   We have $\int_{M+k}^{M+k+1} f(x)\,dx = \int_M^{M+1} f(u+k)\,du$. For all $u \ge M$ we have $f(u+k) \le (1-\epsilon)^k f(u)$ so

$$\int_0^{N+M} f(x)\,dx \le \int_0^M f(x)\,dx + (1-\epsilon)^k \int_M^{M+1} f(u)\,du$$

   so

$$\int_0^{N+M} f(x)\,dx \le \int_0^M f(x)\,dx + \left(\int_M^{M+1} f(u)\,du\right)\sum_{k=0}^{N-1}(1-\epsilon)^k.$$

   Since f is continous on $[0,\infty)$, the integrals on the right are bounded independent of N. The sum is a geometric series and is bounded above by $1/\epsilon$. So the integral on the left is bounded independent of N. Since it is a bounded increasing function of N (f is positive), the limit exists and the integral $\int_0^\infty f(x)\,dx$ converges.

2. Choose $M_1, M_2 \ldots$ such that $\sum_{M_j}^\infty a_n \le 3^{-j}$. Let $\lambda_n = 1$ if $n < M_1$ and $\lambda_n = 2^j$ if $n \in [M_j, M_{j+1})$. To show $\sum_{n=1}^\infty \lambda_n a_n$ converges, it suffices to show the sum is bounded (since the terms are positive, the bounded increasing sequence of partial sums will then have a limit). So

$$\begin{aligned}\sum_{n=1}^\infty \lambda_n a_n &= \sum_{n=1}^{M_1-1} a_n + \sum_{n=M_1}^{M_2-1} 2a_n + \cdots + \sum_{n=M_j}^{M_{j+1}-1} 2^j a_n + \cdots\\ &\le \sum_{n=1}^{M_1-1} a_n + 2\cdot 3^{-1} + \cdots + 2^j\cdot 3^{-j} + \cdots\\ &\le \sum_{n=1}^{M_1-1} a_n + 2.\end{aligned}$$

   (For another solution, see Kaczor and Nowak, "Problems in Mathematical Analysis I," problem 3.2.46.)

3. For all $s < t$ we have $2\int_s^t f(x)\,dx = (t-s)(f(s)+f(t))$. Differentiating with respect to t gives

$$2f(t) = f(s) + f(t) + (t-s)f'(t), \quad \text{or} \quad f(t) - f(s) = (t-s)f'(t).$$

   Differentiating this with respect to s gives, for all $s < t$,

$$f'(s) = f'(t).$$

Thus f' is constant and $f = \alpha x + \beta$ for some constants α, β (it seems to me you don't need the C^2 hypothesis, just C^1).

4. The limit $\lim_{x\to\infty} e^{-x^2}/x^k$ is zero for any $k > 0$, so the integral $\int_0^\infty e^{-u^2}\,du$ converges and both $\int_x^\infty e^{-u^2}\,dx$ and e^{-x^2}/x go to 0 as $x \to \infty$. Then use L'Hôpital's rule to evaluate
$$\lim_{x\to\infty} \frac{\int_x^\infty e^{-u^2}\,dx}{\frac{e^{-x^2}}{x}} = \lim_{x\to\infty} \frac{1}{2+\frac{1}{x^2}} = \frac{1}{2}.$$

5. a. We have
$$\log n = \int_1^2 \frac{1}{t}\,dt + \cdots \int_{n-1}^n \frac{1}{t}\,dt < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1} + \frac{1}{n}$$
so $s_n > 0$. Also
$$s_{n+1} - s_n = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{t}\,dt < \frac{1}{n+1} - \frac{1}{n+1} = 0$$
so the sequence $\{s_n\}$ is decreasing and bounded below. This shows it converges.

b. We will show $\lim_{n\to\infty} \sum_{k=n}^{2n} = \log 2$. Fix $\epsilon > 0$. Note $s_{2n} - s_n = \sum_{k=n}^{2n} \frac{1}{k} - \log 2$. Since $\{s_k\}$ is convergent, it is Cauchy and we can find an N such that for all n, $m > N$, $|s_n - s_m| < \epsilon$. In particular, if $n > N$ then $|s_{2n} - s_n| = |\sum_{k=n}^{2n} \frac{1}{k} - \log 2| < \epsilon$.

Another solution would be to interpret $\sum_{k=n}^{2n} \frac{1}{k}$ as a Riemann sum for $\int_0^1 \frac{1}{t+1}\,dt$ (see the Berkeley book, problem 1.3.12 for this and other solutions). For yet another solution see Kaczor and Nowack, problem 2.5.8.

c. From the Taylor's series we see $\sin(x) = x + x^2 F(x)$ where $F(x) = \sum_{k=0}^\infty \frac{(-1)^{k+1} x^{2k+1}}{(2k+3)!}$. By the ratio test the series for F is uniformly convergent on any bounded interval in $\mathbf{R}$ and so F is continuous (everywhere) hence bounded on $[0, \pi]$. Thus the series $\sum_{k=1}^\infty (\pi/k)^2 F(\pi/k)$ converges and so
$$\lim_{n\to\infty} \sum_{k=n}^{2n} \left(\frac{\pi}{k}\right)^2 F(\frac{\pi}{k}) = 0.$$
Thus
$$\lim_{n\to\infty} \sum_{k=n}^{2n} \sin(\frac{\pi}{k}) = \lim_{n\to\infty} \sum_{k=n}^{2n} \frac{\pi}{k} + \lim_{n\to\infty} \sum_{k=n}^{2n} \left(\frac{\pi}{k}\right)^2 F(\frac{\pi}{k}) = \pi \log 2.$$
For another (possibly more elegant) solution see Kaczor and Nowack, problem 2.5.4

6. (a) Consider the sequence of functions $f_n(x) = \begin{cases} 0 & \text{if } \frac{1}{n} \le x \le 1 \\ 1 - nx & \text{if } 0 \le x \le \frac{1}{n} \end{cases}$. Each f_n is continuous on $[0,1]$. Each f_n converges pointwise on $[0,1]$: for x fixed, $\lim_{n\to\infty} f_n(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } 0 < x \le 1 \end{cases}$. But $\lim_{n\to\infty} \lim_{x\to 0} f_n(x) = \lim_{n\to\infty} 1 = 1$ and $\lim_{x\to 0} \lim_{n\to\infty} f_n(x) = 0$.

(b) Let $f(x) = \lim_{n\to\infty} f_n(x)$ and let $L = \lim_{n\to\infty} f_n(0)$. We know, by the continuity of the f_n, that $\lim_{x\to 0} f_n(x) = f_n(0)$. We need to show that $\lim_{x\to 0} f(x) = L$.

Fix $\epsilon > 0$. Choose N such that for all $n \ge N$, and all $x \in [0,1]$, $|f_n(x) - f(x)| < \epsilon/3$ (using the uniform convergence). Since $f_n(0) \to L$, we can choose an n larger than the N above so that $|f_n(0) - L| < \epsilon/3$. Finally we can, for this fixed n, choose a $\delta > 0$ such that $|x| < \delta$ implies that $|f_n(x) - f_n(0)| < \epsilon/3$ (using the continuity of f_n at 0). Then for $|x| < \delta$,
$$|f(x) - L| \le |f(x) - f_n(x)| + |f_n(x) - f_n(0)| + |f_n(0) - L| \le \epsilon.$$