Solution Hints for Autumn 97 Exam

Revised 7/03

1. Prove that $n! \leq \left(\frac{n+1}{2}\right)^n$, for all integers $n \geq 1$.

Solution. By the inequality between the arithmetic and geometric mean and the well-known formula for the sum of the first n integers,

$$(1 \cdot 2 \cdots n)^{\frac{1}{n}} \leq \frac{1+2+\cdots n}{n} = \frac{n(n+1)}{2n} = \frac{n+1}{2}.$$

The result follows immediately.

2. Prove that

$$\sup_{-\infty<a,b<\infty} \left| \int_a^b \frac{\sin x}{x}\, dx \right| < \infty$$

Solution. Since $\frac{\sin x}{x}$ is an even function, it suffices to assume that $0 \leq a < b < \infty$. Since $\frac{\sin x}{x}$ is continuous on $\mathbf{R}$, the integral of $\frac{\sin x}{x}$ over any subinterval of $[0,1]$ (including $[0,1]$ itself) is a finite number. So it suffices to assume that $1 \leq a < b < \infty$. For $1 \leq a < b < \infty$, integration by parts gives

$$\left| \int_a^b \frac{\sin x}{x}\, dx \right| = \left| -\frac{\cos b}{b} + \frac{\cos a}{a} - \int_a^b \frac{\cos x}{x^2}\, dx \right| \leq 2 + \int_a^b \frac{1}{x^2}\, dx \leq 3.$$

Therefore $\sup_{-\infty<a<b<\infty} \left| \int_a^b \frac{\sin x}{x}\, dx \right|$ is finite.

3. Let $f : \mathbf{R} \to \mathbf{R}$ be a differentiable function such that $\lim_{x\to\infty} f'(x) = 0$. Show that $\lim_{x\to\infty} \frac{f(x)}{x} = 0$. (You may not quote L'Hôpital's rule without proving it).

Solution. Fix $\epsilon > 0$ and choose $M > 0$ such that $x \geq M$ implies $|f'(x)| \leq \epsilon$. For $x > M$ we have, for some $\xi \in (M, x)$,

$$\left| \frac{f(x)}{x} \right| \leq \left| \frac{f(M)}{x} + \frac{x-M}{x} \cdot f'(\xi) \right| \leq \left| \frac{f(M)}{x} \right| + \frac{x-M}{x} \cdot \epsilon.$$

The $\limsup_{x\to\infty} \left| \frac{f(x)}{x} \right| \leq \epsilon$. Since $\epsilon > 0$ was arbitrary, $\lim_{x\to\infty} \frac{f(x)}{x} = 0$.

4. Let $g : \mathbf{R} \to \mathbf{R}$ be a bounded, continuous function. For each $x \in \mathbf{R}$, compute the limit

$$\lim_{\lambda\to\infty} \lambda \int_{-\infty}^{\infty} g(x+y) e^{-\lambda|y|}\, dy$$

Solution. There's an easy solution using the Lebesgue's dominated convergence Theorem: Change variables to $u = \lambda y$. Then we must compute

$$\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} g(x + \frac{u}{\lambda}) e^{-|u|}\, du.$$

Let $\sup_{x\in\mathbf{R}} |g(x)| = M < \infty$. Then for all $\lambda > 0$, x, u in $\mathbf{R}$ we have the bound $|g(x + \frac{u}{\lambda})e^{-|u|}| \leq Me^{-|u|}$. Since $Me^{-|u|}$ is integrable, we can use the dominated convergence theorem to justify taking the limit inside the integral to get

$$\lim_{\lambda\to\infty} \int_{-\infty}^{\infty} g(x + \frac{u}{\lambda}) e^{-|u|}\, du = g(x) \int_{-\infty}^{\infty} e^{-|u|}\, du = 2g(x).$$

Here's a direct proof (without using the dominated convergence theorem): Fix $x \in \mathbf{R}$ and $\epsilon > 0$. As above let $M = \sup_{x\in\mathbf{R}} |g(x)|$. Since $e^{-|x|}$ is integrable, we can choose an $N > 0$ such that $\int_{|u|>N} 2Me^{-|u|}\,du < \epsilon/2$. Since g is continuous at x, there exists a $\delta > 0$ such that if $|x-y| < \delta$ then $|g(x)-g(y)| < \epsilon/2N$. Now choose $\Lambda > 0$ such that for all $\lambda \geq \Lambda$ and all $u \in [-N, N]$, we have $|\frac{u}{\lambda}| \leq \delta$. Then for all $\lambda \geq \Lambda$, using the change of variables above,

$$\begin{aligned}\left|\int_{-\infty}^{\infty} g(x+\frac{u}{\lambda})e^{-|u|}\,du - 2g(x)\right| &= \left|\int_{-\infty}^{\infty}\left(g(x+\frac{u}{\lambda}) - g(x)\right)e^{-|u|}\,du\right| \\ &\leq \int_{|u|\geq N} 2Me^{-|u|}\,du + \int_{|u|<N} |g(x+\frac{u}{\lambda}) - g(x)|e^{-|u|}\,du \\ &\leq \frac{\epsilon}{2} + 2N\cdot\frac{\epsilon}{2N} = \epsilon.\end{aligned}$$

Since $\epsilon > 0$ was arbitrary, $\lim_{\lambda\to\infty}\int_{-\infty}^{\infty} g(x+\frac{u}{\lambda})e^{-|u|}\,du = 2g(x)$.

5. Find the minimal area among triangles having vertices $A = (0,-2)$, $B = (4,0)$, and third vertex C on the parabola $y = x^2$.

Solution. The triangle is spanned by the vectors $\mathbf{v} = (4-0, 4-(-2)) = (4,2)$ and $\mathbf{w} = (x, x^2) - (0,-2) = (x, x^2+2)$. The area of the triangle spanned by these vectors is $\frac{1}{2}\|\mathbf{v}\times\mathbf{w}\| = |2x^2 - x + 4|$. By elementary calculus the minimum occurs when $x = \frac{1}{4}$. The minimum area is $3\frac{7}{8}$.

6. The Taylor series expansion of $f(x) = \log(1+x)$ about the point $x = 0$ is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}$. Show that the series diverges at every point of $(-\infty,-1]\cup(1,\infty)$ and converges uniformly on every interval of the form $[-1+\epsilon, 1]$, $0 < \epsilon < 1$.

Solution. The ratio test shows easily that the series diverges for $|x| > 1$ and converges for $|x| < 1$. When $x = -1$ the series is the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ which diverges. When $x = 1$ the series is the alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$ which converges.

It remains to show the series converges uniformly on every interval of the form $[-1+\epsilon, 1]$, $0 < \epsilon < 1$. Uniform convergence on $[0,1]$ is an immediate consequence of Abel's Theorem:

Theorem (Abel). *If $\sum_{k=0}^{\infty} a_k$ converges, then $\sum_{k=0}^{\infty} a_k x^k$ converges uniformly for $0 \leq x \leq 1$.*

Here is a direct proof of the uniform convergence without Abel's Theorem. On the interval $(0,1]$ the series is alternating, and the absolute value of the remainder is less than or equal to the absolute value of the first term in the remainder: if $L(x)$ is the sum of the series, then

$$\left|\sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k} - L(x)\right| \leq \frac{|x|^{n+1}}{n+1} \leq \frac{1}{n+1}.$$

This estimate clearly holds also when $x = 0$. So the series converges uniformly on $[0,1]$.

It remains to show the series converges uniformly on any interval of the form $[-1+\epsilon, 0]$. On the interval $[-1+\epsilon, 0]$ we have $|x| \leq 1-\epsilon$ so

$$\left|\sum_{k=1}^{n}(-1)^{k-1}\frac{x^k}{k} - L(x)\right| = \sum_{k=n+1}^{\infty}\frac{|x|^k}{k} \leq \sum_{k=n+1}^{\infty}(1-\epsilon)^k.$$

Since the geometric series on the right hand side converges,we get uniform convergence on $[-1+\epsilon, 0]$. This proves the series converges uniformly on $[-1+\epsilon, 1]$.