Solution Hints for Autumn 99 Exam

1. Prove that if $\{s_n\}_{n\in\mathbf{N}}$ is a sequence of real numbers, and if the finite limit $\lim_{n\to\infty} s_n$ exists, then

$$\sum_{n=0}^{\infty} s_n x^n$$

converges for all $x \in (-1,1)$, and, furthermore,

$$\lim_{x\to 1^-} (1-x)\sum_{n=0}^{\infty} s_n x^n = \lim_{n\to\infty} s_n.$$

Solution. Let $\lim_{n\to\infty} s_n = L$. Since $\sum_{n\to\infty} |x^n|$ converges for all $x \in (-1,1)$ and $\lim_{n\to\infty} \frac{|s_n x^n|}{|x^n|} = L$ we conclude from the limit comparison test that $\sum_{n=0}^{\infty} s_n x^n$ converges (absolutely) for $x \in (-1,1)$.

To show $\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty} s_k x^k = L$, note that the above shows that the series is absolutely convergent for all $x \in (-1,1)$. Then, since $(1-x)\sum_{k=0}^{\infty} x^k = 1$ for $x \in (-1,1)$, we can subtract absolutely convergent series to get

$$(1-x)\sum_{k=0}^{\infty} s_k x^k - L = (1-x)\sum_{k=0}^{\infty}(s_k - L)x^k.$$

Now given $\epsilon > 0$ choose N such that for all $k > N$, we have $|s_k - L| < \epsilon$. Then

$$\begin{aligned}
\left|(1-x)\sum_{k=0}^{\infty} s_k x^k - L\right| &= \left|(1-x)\sum_{k=0}^{\infty}(s_k - L)x^k\right| \\
&\leq (1-x)\sum_{k=0}^{N} |s_k - L|\cdot|x^k| + (1-x)\sum_{k=N+1}^{\infty} |s_k - L|\cdot|x^k| \\
&\leq (1-x)\sum_{k=0}^{N} |s_k - L|\cdot|x^k| + \epsilon.
\end{aligned}$$

This holds for all $x \in (-1,1)$ (with N fixed!). Taking the limit as $x \to 1^-$ gives

$$\lim_{x\to 1^-}\left|(1-x)\sum_{k=0}^{\infty} s_k x^k - L\right| \leq \epsilon.$$

This shows that $\lim_{x\to 1^-}(1-x)\sum_{k=0}^{\infty} s_k x^k = L$.

2. Give an example of two sequences of real numbers $\{a_n\}_{n\in\mathbf{N}}$ and $\{b_n\}_{n\in\mathbf{N}}$ such that $\sum_{n=0}^{\infty} a_n$ converges, $\lim_{n\to\infty} b_n = 0$, each b_n is positive, and $\sum_{n=0}^{\infty} a_n b_n$ diverges.

Solution. (Many examples are possible). The idea is to start with a conditionally convergent series such as $\sum_{k=1}^{\infty}(-1)^k/k$. This converges, loosely speaking, because the negative terms balance out the positive terms. If we disrupt this balance by, for example, multiplying the terms corresponding to odd values of k by $1/2$ then the resulting series ought to diverge.

So let $a_0 = b_0 = 1$, and for $k > 0$ let $a_k = (-1)^k/\sqrt{k}$ and

$$b_k = \begin{cases} \frac{1}{\sqrt{k}} & \text{if } k \text{ is even} \\ \frac{1}{2\sqrt{k}} & \text{if } k \text{ is odd.} \end{cases}$$

Then $\sum_{k=0}^{\infty} a_k$ converges, $b_k > 0$, but we claim the series $\sum_{k=0}^{\infty} a_k b_k$ diverges. Suppose by way of contradiction it does converges. Then $\lim_{n\to\infty} s_{2n}$ exists as a finite limit where s_n is the n^{th} partial sum. But

$$s_{2n} = 1 + \sum_{k=1}^{n} \left(\frac{-1}{2(2k-1)} + \frac{1}{2k} \right) = 1 + \frac{1}{2} \sum_{k=1}^{n} \frac{1}{2k-1} - \frac{1}{2} \sum_{k=1}^{n} \frac{1}{k(2k-1)}.$$

If $\lim_{n\to\infty} s_{2n}$ exists, then the above shows that $\sum_{k=1}^{\infty} \frac{1}{2k-1}$ converges, which is a contradiction.

3. Let f be a real valued function such that $f \in C([0,\infty))$, and the improper Riemann integral $\int_1^\infty f(x)/x\,dx$ converges. Prove that for all $a > 0$ and $b > 0$ we have

$$\lim_{\epsilon\to 0^+} \int_\epsilon^\infty \frac{f(ax) - f(bx)}{x}\,dx = f(0) \log \frac{b}{a}.$$

Solution. Note by the change of variable $u = ax/\epsilon$ we have, for $a > 0$ and all $N > \epsilon > 0$,

$$\int_\epsilon^N \frac{f(ax)}{x}\,dx = \int_a^{N\cdot\left(\frac{a}{\epsilon}\right)} \frac{f(\epsilon u)}{u}\,du.$$

So the improper integrals $\int_\epsilon^\infty f(ax)/x\,dx$ and $\int_\epsilon^\infty f(bx)/x\,dx$ converge and, if $0 < a < b$, then

$$\int_\epsilon^\infty \frac{f(ax) - f(bx)}{x}\,dx = \int_a^b \frac{f(\epsilon u)}{u}\,du.$$

Since f is uniformly continuous on $[0,b]$ we can take the limit inside the integral* to get

$$\lim_{\epsilon\to 0^+} \int_\epsilon^\infty \frac{f(ax) - f(bx)}{x}\,dx = \lim_{\epsilon\to 0^+} \int_a^b \frac{f(\epsilon u)}{u}\,du = f(0) \int_a^b \frac{1}{u}\,du = f(0) \ln\left(\frac{b}{a}\right).$$

The same proof works if $0 < b < a$.

4. Prove that $\left(\frac{n}{n+1}\right)^n$, $n \in \mathbf{N}$, is a decreasing sequence.

Solution. It suffices to show that the sequence $\left(\frac{n+1}{n}\right)^n$ is increasing. For each $n = 1, 2, \ldots$ we have the Arithmetic-Geometric Mean Inequality

$$\left(1 \cdot \left(\frac{n+1}{n}\right)^n\right)^{\frac{1}{n+1}} \le \frac{1}{n+1}\left(1 + n \cdot \frac{n+1}{n}\right).$$

It follows immediately that $\left(\frac{n+1}{n}\right)^n \le \left(\frac{n+2}{n+1}\right)^{n+1}$. This shows the sequence is increasing. In fact we can use calculus to show the original sequence is strictly decreasing. Let $f(x) = \left(\frac{x}{x+1}\right)^x$ for $x \ge 1$. Then

$$\frac{f'(x)}{f(x)} = \frac{1}{x+1} + \ln x - \ln(x+1)$$

By the Mean Value Theorem there is a ξ in the open interval $(x, x+1)$ such that $\ln(x+1) - \ln x = \frac{1}{\xi} > \frac{1}{x+1}$. This shows that $f'(x)$ is negative, so f is strictly decreasing.

---

* This step should be justified more carefully. One can apply Lebesgue's Dominated Convergence Theorem since on $[a,b]$ the integrand is dominated by the integrable function $\frac{M}{u}$ where $M = \sup_{[a,b]} f(x)$. Without using the Dominated Convergence Theorem, we could argue as follows. The limit exists if and only if for every sequence $\epsilon_n \to 0$, $\lim_{n\to\infty} \int_a^b \frac{f(\epsilon_n u)}{u}\,du$ exists and is independent of the chosen sequence. Since f is uniformly continuous on $[0,b]$, the sequence converges uniformly to $f(0)$. Then we can take the limit inside the integral and proceed as before.

5. Let $f : \mathbf{R} \to \mathbf{R}$ be differentiable in $\mathbf{R}$. Suppose $f'(-1) = -1$ and $f'(1) = 1$. Prove that f must vanish at some point in $(-1, 1)$ (do not use Daroboux's Theorem).

   Solution. See, e.g., Rudin, Principles of Mathematical Analysis, Theorem 5.12.

6. Let $\{f_n\}_{n \in \mathbf{N}}$ be defined by

$$f_n(x) = \frac{n^2x^2}{1+n^2x^2}, \quad -1 \le x \le 1.$$

   (i) Find $f \stackrel{\text{def}}{=} \lim_{n\to\infty} f_n$ in $[-1, 1]$.

   Solution. As was shown in class, write $\frac{n^2x^2}{1+n^2x^2} = 1 - \frac{1}{n^2x^2+1}$. Then it is clear that the pointwise limit is

$$f(x) = \begin{cases} 1 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

   (ii) Prove or disprove that

$$\lim_{n\to\infty} \int_{-1}^{1} f_n(t)\, dt = \int_{-1}^{1} f(t)\, dt.$$

   Solution. We have

$$\int_{-1}^{1} \frac{n^2x^2}{n^2x^2+1}\, dx = \int_{-1}^{1} \left(1 - \frac{1}{n^2x^2}\right) dx = 2 - 2\frac{\arctan n}{n} \to 2 \text{ as } n \to \infty.$$

   which is equal to $\int_{-1}^{1} f(x)\, dx$.

   (iii) Prove or disprove that $\lim_{n\to\infty} f_n = f$ is uniform in $[-1, 1]$.

   Solution. If the convergence was uniform, then the pointwise limit function f would be continuous, but it's not. However, as was pointed out in class, one should be wary of interpreting these problems in a way that makes them trivial. So it's a good idea to prove directly that the convergence is not uniform. If the convergence was uniform, then given $\epsilon = 1/4$ we would be able to find an N such that for all $n > N$ and all $x \in [-1, 1]$, we have $|f(x) - f_n(x)| < 1/4$. Consider the sequence $x_n = \frac{1}{n}$. Then $f_n(x_n) = \frac{1}{2}$ and $|f(x_n) - f_n(x_n)| = \frac{1}{2}$ for all $n \in \mathbf{N}$. So the convergence can not be uniform.