Solution Hints for Spring 99 Exam

1. Find a sequence of function $\{f_n\} \in C([0,1])$ such that $f_n \to f$ as $n \to \infty$ with $f \in C([0,1])$, but $\int_0^1 f_n$ does not converge to $\int_0^1 f$ as $n \to \infty$.

   Solution. The classic example is a sequence of functions $\{f_n\}$ whose graph is an equilateral triangle with base vertices at 0 and $\frac{1}{n}$, and whose altitude is $2n$, so that the functions converge pointwise to zero but the area of the triangles is always 1. Explicitly, let

$$f_n(x) = \begin{cases} 4n^2x & \text{if } 0 \le x \le \frac{1}{2n} \\ 4n(1-nx) & \text{if } \frac{1}{2n} \le x \le \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} \le x \le 1 \end{cases}.$$

2. Let $a_n \in (0,\infty)$. Prove that if $\sum_{j=1}^{\infty} a_j$ converges, then $\sum_{j=1}^{\infty} \frac{\sqrt{a_j}}{j}$ converges as well.

   Solution. (This was solved in class and on the homework. There are a couple of ways to do it, either use the arithmetic-geometric mean inequality or the Schwarz inequality.)

3. Let f be a C^2 function on the real line. For $k = 0,\ 1,\ 2$, define $M_k = \sup_{x\in\mathbf{R}} |f^{(k)}(x)|$, where $f^{(k)}$ denotes the k-th derivative of f. Prove that there exists an absolute constant K [independent of the function f and its derivatives] such that

$$M_1 \le K\sqrt{M_0 \cdot M_2}.$$

   Hint: you may wish to use Taylor's theorem.

   Solution. (This was also solved in class. See also Kaczor and Nowak, V. II, problem 2.3.19.)

4. For $n \in \mathbf{N}$ and $x \in \mathbf{R}^+$, define $f_1(x) = \sqrt{x}$ and $f_{n+1}(x) = \sqrt{x + f_n(x)}$. Prove that $\{f_n\}$ converges uniformly on every interval $[a,b]$ where $0 < a < b < \infty$.

   Solution. As was shown in class, the idea is to use Dini's Theorem. Since $[a,b]$ is compact, it remains to show: (i) for each $x \in [a,b]$, $f_1(x) \le f_2(x) \le \cdots \le f_n(x) \le \cdots$ (or that the opposite inequalities hold for all $x \in [a,b]$) (ii) each f_n is continuous on $[a,b]$, and (iii) the pointwise limit f of the f_n's exists and is continuous on $[a,b]$.

   To show (i): Since $0 < a$, it is clear that for $x \in [a,b]$, we have $f_2(x) = \sqrt{x+\sqrt{x}} \ge \sqrt{x}$. Suppose $f_n(x) \ge f_{n-1}(x)$ for all $x \in [a,b]$. Then for $x \in [a,b]$, we have $f_{n+1}(x) = \sqrt{x + f_n(x)} \ge \sqrt{x + f_{n-1}(x)} = f_n(x)$. To show (ii), we can easily use induction (details omitted). To show (iii), note that since $f_{n+1}(x) \ge f_n(x)$ for all integers n and all $x \in [a,b]$, we have

$$f_n(x) + x \ge f_n^2(x) \text{ so that } x \ge f_n^2(x) - f_n(x).$$

   Completing the square and taking the square root gives

$$-\frac{1}{2} \pm \sqrt{\frac{1}{4} + x} \ge f_n(x).$$

   Since the $f_n(x)$ are increasing and $f_1(x) \ge 0$, we must have $-\frac{1}{2} + \sqrt{\frac{1}{4} + x} \ge f_n(x)$. Thus the sequence $\{f_n(x)\}$ is bounded and increasing so it must converge to some limit $L(x)$. Taking the limit on both sides of the relationship (for fixed x) gives

$$f_{n+1}^2(x) = x + f_n(x)$$

   gives $L(x) = -\frac{1}{2} + \sqrt{\frac{1}{4} + x}$. This is continuous and proves (iii). Now the sequence converges uniformly on $[a,b]$ by Dini's Theorem. (See Rudin, Theorem 7.13, for Dini's Theorem in the case where the f_n are nonincreasing. To pass to the case where the f_n are nondecreasing, consider the sequence $\{-f_n\}$.)

5. Let $f \in C(\mathbf{R})$ and suppose that $\lim_{|x|\to\infty} f(x) = 0$. Show that f is uniformly continuous on the entire real line.

   (This was solved in class).

6. A theorem of Weierstrass says that the uniform limits of polynomials on $[0,1]$ is the set of all continuous function on $[0,1]$. What is the set of uniform limits of polynomials on $[0,1]$ with only even exponents without constant terms, i.e., polynomials of the form $a_2x^2 + a_4x^4 + \cdots + a_{2n}x^{2n}$?

   Solution. First let $\mathcal{U}$ be the set of all polynomials without constant term. We first claim that the closure, $\overline{\mathcal{U}}$, of $\mathcal{U}$ in the topology of uniform convergence on $[0,1]$ is the set, C_0, of all continuous functions on $[0,1]$ which vanish at 0.

   To show $\overline{\mathcal{U}} \subset C_0$: if f is the uniform limit of a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ without constant term, then clearly f is continuous on $[0,1]$. To show that $f(0) = 0$, let $\epsilon > 0$. Choose N such that for all $n > N$ and all $x \in [0,1]$,

$$|f(x) - P_n(x)| < \epsilon.$$

   In particular, $|f(0) - P_n(0)| = |f(0)| < \epsilon$. Since $\epsilon > 0$ was arbitrary, we must have $f(0) = 0$. This shows that $\overline{\mathcal{U}} \subset C_0$.

   To show $C_0 \subset \overline{\mathcal{U}}$, let $f \in C_0$. Fix $\epsilon > 0$ and choose a sequence of polynomials $\{P_n\}_{n=1}^{\infty}$ (possibly with constant term) and an N such that for $n > N$ and all $x \in [0,1]$,

$$|f(x) - P_n(x)| < \epsilon/2.$$

   Then $|f(0) - P_n(0)| = |P_n(0)| < \epsilon/2$. Consider now the sequence of polynomials without constant term given by $Q_n(x) = P_n(x) - P_n(0)$. Then for all $n > N$,

$$|f(x) - Q_n(x)| \leq |f(x) - P_n(x)| + |P_n(0)| \leq \epsilon.$$

   This shows that $C_0 \subset \overline{\mathcal{U}}$ and so $C_0 = \overline{\mathcal{U}}$.

   Now let $\overline{\mathcal{V}}$ be the closure of the set of polynomials without constant term and with only even exponents. We claim that $\overline{\mathcal{V}} = C_0$. To show that $\overline{\mathcal{V}} \subset C_0$: clearly f is continuous, so only need to show that if $f \in \overline{\mathcal{V}}$, then $f(0) = 0$. Fix $\epsilon > 0$ and choose a sequence of polynomials P_n without constant term and with only even exponents such that for all $n > N$ and all $x \in [0,1]$, $|f(x) - P_n(x)| < \epsilon$. Then $|f(0) - P_n(0)| = |f_n(0)| < \epsilon$. So $f \in C_0$. To show that $C_0 \subset \overline{\mathcal{V}}$, let $f \in C_0$ and consider $g(x) = f(\sqrt{x})$. Since $x \in [0,1]$, $g \in C_0$. Fix $\epsilon > 0$. Then by the above there is a sequence of polynomials P_n without constant term and an N such that for all $n > N$ and all $x \in [0,1]$,

$$|g(x) - P_n(x)| < \epsilon.$$

   Since this holds for all $x \in [0,1]$ and since $x^2 \in [0,1]$, we have

$$|f(x) - P_n(x^2)| < \epsilon.$$

   Since $Q_n(x) = P_n(x^2)$ is a polynomial with only even exponents and without constant term, this shows $f \in \overline{\mathcal{V}}$. Thus $C_0 \subset \overline{\mathcal{V}}$ and so $C_0 = \overline{\mathcal{V}}$.