Solutions to Math 787.03 Mock Exam 1, Summer, 2003

July 22, 2003

Four complete solutions are sufficient to pass. Please use only one side of every sheet, and use distinct sheets for distinct problems.

1. If a, b, c are positive real numbers and $a + b + c = 1$, then show that

$$\left(\frac{1}{a} - 1\right)\left(\frac{1}{b} - 1\right)\left(\frac{1}{c} - 1\right) \geq 8.$$

Solution. Doing a little algebra (and using the fact that a, b, c are positive) shows that the inequality is equivalent to

$$\frac{1-a}{2} \cdot \frac{1-b}{2} \cdot \frac{1-c}{2} \geq abc.$$

Using the fact that $a + b + c = 1$ gives

$$\frac{b+c}{2} \cdot \frac{a+c}{2} \cdot \frac{a+b}{2} \geq abc. \tag{1}$$

This suggests we look at the inequality relating the arithmetic and geometric means:

$$\sqrt{bc} \leq \frac{b+c}{2}, \quad \sqrt{ac} \leq \frac{a+c}{2}, \quad \sqrt{ab} \leq \frac{a+b}{2},$$

Multiplying these gives the inequality (1).

2. Let $f_n \colon \mathbf{R} \to \mathbf{R}$ be differentiable for each $n = 1, 2, \ldots$ with $|f_n'(x)| \leq 1$ for all n and all x. Assume

$$\lim_{n\to\infty} f_n(x) = g(x)$$

for all x. Prove that $g \colon \mathbf{R} \to \mathbf{R}$ is continuous.

Solution. (See Berkeley Problems in Analysis, 1.6.2.) Fix $\epsilon > 0$. Let x, y satisfy $|x - y| < \epsilon/3$. There is an n such that

$$|f_n(x) - g(x)| < \frac{\epsilon}{3} \quad \text{and} \quad |f_n(y) - g(y)| < \frac{\epsilon}{3}.$$

(The n depends on x and y, but we can always find such an n.) Then

$$\begin{aligned}|g(x) - g(y)| &\le |g(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - g(y)| \\ &\le \frac{\epsilon}{3} + |f_n'(\xi)| \cdot |x - y| + \frac{\epsilon}{3} \\ &\le \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}\end{aligned}$$

(using the Mean Value Theorem and the estimate on f_n'). This shows g is (uniformly) continuous on $\mathbf{R}$.

3. Define the function ζ by

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}.$$

Prove that $\zeta(x)$ is defined and has continuous derivatives of all orders on the interval $1 < x < \infty$.

Solution. See Berkeley, problem 1.6.28.

4. Let

$$f(x) = e^{x^2/2} \int_x^{\infty} e^{-t^2/2}\, dt$$

for $x > 0$.

(a) Show that $0 < f(x) < 1/x$.

(b) Show that $f(x)$ is strictly decreasing for $x > 0$.

Solution (See Berkeley Problems in Analysis, 1.5.7). (a) Clearly f is strictly positive because the integral is strictly positive since the integrand is positive for all t. The idea is to change variables to clarify as much as possible the x dependence in the integral. So put $u = t - x$. Then

$$\begin{aligned}0 < f(x) = e^{x^2/2} \int_0^{\infty} e^{-(u+x)^2/2}\, du &= \int_0^{\infty} e^{-u^2/2 - xu}\, du \\ &< \int_0^{\infty} e^{-xu}\, du = \frac{1}{x}.\end{aligned}$$

The last inequality is strict because the inequality relating the integrands is strict for all $u \neq x$.

(b) For $x < y$ and $u > 0$ we have $e^{-yu} < e^{-xu}$. Then

$$f(x) - f(y) = \int_0^{\infty} e^{-u^2/2} \left(e^{-xu} - e^{-yu}\right) du > 0.$$

Another proof could be based on computing $f'(x)$ and using part a) to show $f'(x) < 0$.

5. Prove that any bounded sequence of real numbers has a convergent subsequence.

   Solution. (See Rudin, Principles of Mathematical Analysis, Theorem 3.6). The sequence of numbers $\{a_n\}$ lies in a compact subset of $\mathbf{R}$. If there are only finitely many values of the sequence, then the sequence clearly has a convergent subsequence $\{a_{n_k}\}$ with all a_{n_k} equal to one of the finite values.

   If the range of the sequence is infinite, then use the fact that every infinite subset of a compact set has a limit point (proof: Let A be an infinite subset of a compact set K. If A has no limit point in K, then every point of K has a neighborhood containing only finitely many elements of A. This gives an open cover of K. Then there is a finite subcover which still covers K, and each of these open sets contains only finitely many elements of A. Then there are only finitely many elements of A in K, contradicting the assumption that A is an infinite subset of K). So let p be a limit point of the sequence. Then every neighborhood of p of the form $|p - x| < 1/k$ contains an element of the sequence not equal to p. In this way we can construct a subsequence a_{n_k} with $n_k \uparrow \infty$ converging to p.

6. A sequence of functions defined on a set A is said to be *equicontinuous* on A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all x, $y \in A$ and all positive integers n, if $|x - y| < \delta$ then $|f_n(x) - f_n(y)| < \epsilon$. Prove that if $\{f_n\}$ is a uniformly convergent sequence of continuous functions on a compact set K, then $\{f_n\}$ is equicontinuous on K.

   Solution. (See Rudin, Principles of Mathematical Analysis, Theorem 7.24.) We can use without proof the result that a continuous function on a compact set is uniformly continuous (we should know how to prove that fact also!). Fix $\epsilon > 0$. Using the uniform convergence of the f_n on K, choose N such that for all $n \geq N$, and all $x \in K$,

$$|f_N(x) - f_n(x)| < \frac{\epsilon}{3}.$$

   Choose $\delta > 0$ (depending on N) such that for all x, y in K, if $|x - y| < \delta$, then

$$|f_N(x) - f_N(y)| < \frac{\epsilon}{3}.$$

   (Here we are using the uniform continuity of f_N on K.) Then for all $n \geq N$,

$$\begin{aligned} |f_n(x) - f_n(y)| &\leq |f_n(x) - f_N(x)| + |f_N(x) - f_N(y)| + |f_N(y) - f_n(y)| \\ &< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}. \end{aligned} \tag{2}$$

   Using the uniform continuity of f_n, for all $n < N$ we can find δ_i, $i = 1, \ldots, N - 1$ such that (2) holds for all x, y in K. Then if $\delta_0 =$

$\min(\delta, \delta_1, \ldots, \delta_{N-1})$, the estimate (2) holds for all x, y in K and all positive integers n.