

Math 787.03 Mock Exam 3, Summer, 2003

August 27, 2003

1. Find the radius of convergence of the series

$$(a) \quad \sum_{n=1}^{\infty} \frac{x^{n^2}}{n!} \qquad (b) \quad \sum_{n=1}^{\infty} \frac{x^{n^2}}{(n!)^n}$$

2. Suppose $f : (0, \infty) \rightarrow \mathbf{R}$ is a continuous function such that

$$\lim_{n \rightarrow \infty} f(n^2 x) = a$$

for every $x \in (0, \infty)$. (Of course here n is an integer). Prove that $\lim_{x \rightarrow \infty} f(x) = a$.

3. Suppose $f_n(x)$ is a sequence of non-decreasing functions on $[0, 1]$ (i.e., for all n , if $x_1 < x_2$ then $f_n(x_1) \leq f_n(x_2)$) which converge pointwise to a continuous function $g(x)$. Prove that the convergence is actually uniform on $[0, 1]$.
4. Suppose f and g are positive, Riemann integrable functions on $[0, 1]$ such that $f(x)g(x) \geq 1$ for all $x \in [0, 1]$. Prove that

$$\int_0^1 f(x) dx \cdot \int_0^1 g(x) dx \geq 1.$$

5. Let \mathbf{N} be the set of all positive integers. Suppose $f : \mathbf{N} \rightarrow \mathbf{R}$ is a function such that

$$f(n+m) \leq f(n) + f(m)$$

for all $m, n \in \mathbf{N}$. Prove that $\lim_{n \rightarrow \infty} \frac{f(n)}{n}$ exists and is equal to $\inf_{n \in \mathbf{N}} \frac{f(n)}{n}$.

6. Let f be a differentiable function on $[0, 1]$ such that $\sup_{x \in [0, 1]} |f'(x)| \leq M$. Prove that for all positive integers n ,

$$\left| \sum_{j=0}^{n-1} \frac{f(\frac{j}{n})}{n} - \int_0^1 f(x) dx \right| \leq \frac{M}{2n}.$$