

787.03, Su 2003 Problem Set 5

(The problem set was to do the Spring, 2002, qualifying exam.)

1. Prove this form of Dini's Theorem: Let  $\{f_n\}$  be a sequence of continuous real-valued functions on the closed bounded interval  $[a, b]$ . Suppose that for each  $t \in [a, b]$ , we have  $f_n(t) \geq f_{n+1}(t)$  for all  $n$  and  $\lim_{n \rightarrow \infty} f_n(t) = 0$ . Prove that  $\{f_n\}$  converges uniformly to 0 on  $[a, b]$ .

**Solution** See, e.g., Rudin, Principles of Mathematical Analysis, Theorem 7.12.

2. Let  $f$  be a differentiable function from  $\mathbf{R}$  to  $\mathbf{R}$ . Suppose that for each  $x \in \mathbf{R}$  we have

$$0 \leq f(x) \leq \frac{1}{1+x^2}.$$

Show that there exists a  $c \in \mathbf{R}$  such that

$$f'(c) = \frac{-2c}{(1+c^2)^2}.$$

**Solution.** Let  $G(x) = \frac{1}{1+x^2} - f(x)$ . The hypotheses imply that  $G$  is differentiable on  $\mathbf{R}$ ,  $G(x) \geq 0$  on  $\mathbf{R}$  and  $\lim_{x \rightarrow \pm\infty} f(x) = 0$ , hence  $\lim_{x \rightarrow \pm\infty} G(x) = 0$ . We need to show there exists a  $c \in \mathbf{R}$  such that  $G'(c) = 0$ . Suppose  $G'(x) \neq 0$  for all  $x \in \mathbf{R}$ . Since derivatives have the intermediate value property, we may assume that either  $G'(x) > 0$  for all  $x \in \mathbf{R}$ , or  $G'(x) < 0$  for all  $x \in \mathbf{R}$ . Suppose first that  $G'(x) > 0$  for all  $x \in \mathbf{R}$ . Then  $G$  is non-negative and (strictly) increasing on  $\mathbf{R}$ , and  $\lim_{x \rightarrow \infty} G(x) = 0$ . We claim that this implies  $G(x) = 0$  for all  $x \in \mathbf{R}$ . To see this fix  $\epsilon > 0$  and choose  $N$  such that for all  $x \geq N$  we have  $|G(x)| \leq \epsilon$ . Then for all  $y \leq x$  we have  $0 \leq G(y) \leq G(x) \leq \epsilon$  so that  $|G(y)| \leq \epsilon$  also. Since  $\epsilon$  was arbitrary conclude that  $G(x) = 0$  for all  $x \in \mathbf{R}$ . Similarly it is easy to show that if  $G' < 0$  then  $G$  is identically zero (using the fact that  $\lim_{x \rightarrow -\infty} G(x) = 0$ ). In either case we contradict the assumption that  $G'(x) \neq 0$  for all  $x \in \mathbf{R}$ .

3. Define a function  $f$  on the interval  $(0, 1)$  by

$$f(x) = \sum_{k=1}^{\infty} \frac{\sin(kx)}{k^2}.$$

Prove that  $f$  is differentiable on  $(0, 1)$ .

**Solution.** Let  $x_0 \in (0, 1)$ . Choose a compact interval  $[a, b] \subset (0, 1)$  such that  $x_0 \in (a, b)$ . Recall that if i) a series converges pointwise on a set  $E$ , ii) the series obtained by term-by-term differentiation converges uniformly on  $E$ , then the series is differentiable on  $E$  and its derivative can be computed by term-by-term differentiation. It's clear that the original

series converges pointwise (in fact uniformly by the Weierstrass  $M$ -test, since  $|\sin(kx)/k^2| \leq 1/k^2$ ). The derived series is

$$\sum_{k=1}^{\infty} \frac{\cos(kx)}{k}.$$

We will use the Dirichlet test for uniform convergence (see Kaczor and Nowak, V.2, page 91, second part of problem 3.2.13): let  $E \subseteq \mathbf{R}$  and assume that  $f_k, g_k: E \rightarrow \mathbf{R}$  satisfy:

- (a) for each fixed  $x \in E$ , the sequence  $\{f_n(x)\}$  is monotonic
- (b)  $\{f_k(x)\}$  converges uniformly to 0 on  $E$ .
- (c) the sequence of partial sums of  $\sum_{k=1}^{\infty} g_k(x)$  is uniformly bounded on  $E$ .

Then the series  $\sum_{k=1}^{\infty} f_k(x)g_k(x)$  converges uniformly on  $E$ .

To apply this to the present problem we let  $E = [a, b]$ ,  $f_k(x) = 1/k$ , and  $g_k(x) = \cos(kx)$ . Clearly  $f_n$  satisfies the first two conditions above. It remains to show that  $g_n$  satisfies the third condition. We use the identity

$$\sum_{k=1}^n \cos(kx) = \frac{\sin\left(\frac{2n+1}{2}x\right) - \sin\frac{x}{2}}{2\sin\frac{x}{2}}.$$

Since  $[a, b]$  is a compact subset of  $(0, 1)$ , there is a number  $A > 0$  such that  $A \leq |\sin\frac{x}{2}|$  for all  $x \in [a, b]$ . Thus

$$\left| \sum_{k=1}^n \cos(kx) \right| \leq \frac{1}{A} < \infty.$$

Thus  $g_n$  satisfies the third condition above. Thus the series obtained by term-by-term differentiation converges uniformly on  $[a, b]$ , so it is differentiable on  $[a, b]$ . Since  $[a, b]$  was an arbitrary compact subinterval of  $(0, 1)$ , it is differentiable on  $(0, 1)$  (although it may not converge uniformly on  $(0, 1)$ ).

4. Let  $f$  be a continuously differentiable function from the interval  $[0, 1]$  to  $\mathbf{R}$  (use one-sided derivatives at the endpoints of the interval). Suppose that  $f(1/2) = 0$ . Show that

$$\int_0^1 |f(x)|^2 dx \leq \int_0^1 |f'(x)|^2 dx.$$

**Solution.** Integrating by parts (which is legitimate since  $f'$  is continuous) gives

$$\int_0^{1/2} f(x)^2 dx = - \int_0^{1/2} 2xf'(x)f(x) dx$$

and

$$\int_{1/2}^1 f(x)^2 dx = - \int_{1/2}^1 2(x-1)f'(x)f(x) dx.$$

So

$$\left| \int_0^1 f(x)^2 dx \right| \leq 2 \int_0^{1/2} |xf'(x)f(x)| dx + 2 \int_{1/2}^1 |(x-1)f'(x)f(x)| dx.$$

Since  $|x| \leq 1/2$  on  $[0, 1/2]$  and  $|x-1| \leq 1/2$  on  $[1/2, 1]$ ,

$$\begin{aligned} \left| \int_0^1 f(x)^2 dx \right| &\leq \int_0^{1/2} |f'(x)f(x)| dx + \int_{1/2}^1 |f'(x)f(x)| dx \\ &= \int_0^1 |f'(x)f(x)| dx. \end{aligned}$$

Applying the Cauchy-Schwarz inequality gives

$$\left| \int_0^1 f(x)^2 dx \right| \leq \sqrt{\int_0^1 f'(x)^2 dx} \sqrt{\int_0^1 f(x)^2 dx}.$$

If  $\int_0^1 f(x)^2 dx = 0$  then there is nothing to prove. Otherwise, dividing the last inequality by  $\sqrt{\int_0^1 f(x)^2 dx}$  and squaring both sides gives the conclusion.

5. Let  $f$  be a continuous function from the interval  $[0, 1]$  to  $\mathbf{R}$ . Compute

$$\lim_{n \rightarrow \infty} n \int_0^1 x^n f(x) dx.$$

**Solution.** See Berkeley problem 1.2.12.

6. Let  $\{x_n\}$  be a sequence of real numbers. For each  $n$ , let

$$A_n = \frac{x_1 + \cdots + x_n}{n}.$$

Suppose  $x_n \rightarrow \infty$  as  $n \rightarrow \infty$ . Show that  $A_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

**Solution.** Fix  $N > 0$ . Choose  $L$  (depending on  $N$ ) such that for all  $k > L$ , we have  $x_k > N$ . Then for  $n > L$ ,

$$\begin{aligned} A_n &= \frac{x_1 + \cdots + x_L}{n} + \frac{x_{L+1} + \cdots + x_n}{n} \\ &> \frac{x_1 + \cdots + x_L}{n} + \frac{N(n-L)}{n}. \end{aligned}$$

Letting  $n \rightarrow \infty$  gives

$$\lim_{n \rightarrow \infty} A_n \geq N.$$

Since  $N$  was arbitrary we conclude that  $\lim_{n \rightarrow \infty} A_n = \infty$ .