ANALYTIC STRUCTURE OF TWO 1D-TRANSPORT EQUATIONS WITH NONLOCAL FLUXES

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Abstract

We replace the flux term in Burger's equation by two simple alternates that contain contributions depending globally on the solution. In one case, the term is in the form of a hyperbolic equation where the characteristic speed is nonlocal, and in the other the term is in conservation form. In both cases, the nonlocality is due to the presence of the Hilbert transform. The equations have a loose analogy to the motion of vortex sheets. In particular, they both form singularities in finite time in the absence of viscous effects. Our motivation then is to study the influence of viscosity. In one case, viscosity does not prevent singularity formation. In the other, we can prove solutions exist for all time, and determine the likely weak solution as viscosity vanishes. An interesting aspect of our work is that singularity formation can be viewed as the motion of singularities in the complex physical plane that reach the real axis in finite time. In one case, the singularity is a pole and causes the solution to blow up when it reaches the real axis. In the other, numerical solutions and an asymptotic analysis suggest that the weak solution contains a square root singularity that reaches the real axis in finite time, and then propagates along it. We hope our results will spur further interest in the role of singularities in the complex spatial plane in solutions to transport equations.

Keywords: nonlocal fluxes, singularity formation, viscous regularization.

1 Introduction

There is much current interest in the possible formation of singularities in finite time for the three dimensional Euler equations of incompressible fluid flow. One of the main reasons for this interest stems from new perspectives gained from studies of singularity formation in several well-studied nonlinear partial differential equations. The simplest example is Burger's equation whose solutions describe the formation of shocks [1].

$$u_t = -uu_x + \nu u_{xx} \tag{1}$$

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or

$$u_t = -(u^2/2)_x + \nu u_{xx} \tag{2}$$

Without the viscous term ($\nu = 0$), solutions steepen and form a vertical slope in finite time. Even the smallest amount of viscosity (ν) will prevent the slope from becoming vertical. Instead a shock will form, and the solution will persist indefinitely. In place of singularities, large scale structures in the solution form as a by-product of viscous regularization. If the viscous term is replaced by the dispersive term, νu_{xxx}, the new equation is essentially the Korteweg-DeVries equation. Shocks are replaced by highly oscillatory wave-trains [2] that are narrowly confined. Thus the form of the regularization places a key role in determining the nature of the large-scale structure that replaces the singularity.

In both the Burger's equation and its dispersive counterpart, weak solutions exist in the limit of vanishing ν, but they are different depending on the form of the regularization. Nevertheless, the existence of weak solutions for vanishing ν raises the prospects of determining appropriate solutions to Euler's equations that relate to the limiting behavior of high Reynold's number flow. More importantly, it might be the formation of singularities in Euler equations that trigger the large-scale structures seen frequently in turbulent fluid flow.

A particular example serves to illustrate the idea. At high Reynold's numbers, a thin shear layer, generated typically by shedding from a solid boundary, takes the asymptotic form of a vortex sheet, that is, a layer of vorticity distributed as a delta function on a surface [3]. We shall restrict our attention to two dimensional flow where the surface is a curve and the vorticity points out of the plane. The equation of motion for the location of the vortex sheet, $\vec{x}(p,t) = (x(p,t), y(p,t))$, is given by [4]

$$\vec{x}_t(p,t) = \frac{1}{2\pi} \mathrm{P}\!\!\oint \gamma(q,t) \frac{(y(q,t) - y(p,t),\ x(p,t) - x(q,t))}{|\ \vec{x}(p,t) - \vec{x}(q,t)\ |^2} dq \tag{3}$$

$$\gamma_t(p,t) = 0. \tag{4}$$

where the integral must be evaluated as a principal-value. The parametrization variable p is a Lagrangian parameter in that the quantity γ remains constant along the trajectory of a marker on the sheet labelled by p. Alternatively, p may be regarded as a characteristic variable, and (3) and (4) are the equations for the characteristics of a partial differential equation that describes the transport of the vorticity along the sheet.

There is now overwhelming evidence that vortex sheets form curvature singularities in finite time. The evidence is both analytic [5–7] and numerical

[8,9]. Typically, studies concentrate on the long time evolution of unstable modes of a slightly perturbed, initially flat vortex sheet, $(p, \varepsilon \sin(kp))$. The linearized motion about a flat sheet with uniform strength $\bar{\gamma}$ indicates that the growth rate for such modes satisfies $\sigma = \bar{\gamma}k/2$, that is, the amplitude grows according to $\varepsilon \exp(\sigma t)$. Modes with largest wave number k grow the fastest. Consequently, the motion is linearly ill-posed [10]. Recent studies, then, have demonstrated that the motion does in fact lead to singularity formation in finite time. Further, a study of three dimensional vortex sheet motion [11] suggest that singularities also form and that the analytic structure of these singularities is essentially that observed in two dimensional flow.

There is evidence that these singularities are the precursors to the formation of large-scale vortex structures. Various regularizations of vortex sheet motion have been used to assess the behavior beyond the time of singularity formation. Layers of finite thickness [12] show similar behavior to those whose motion is viscous [13]: a small vortex core replaces the point of curvature singularity, and a thin layer spirals around this core. Krasny uses a numerical regularization that keeps the vorticity on the curve, but smoothes the velocity of the markers. His results also show the appearance of a spiral, but without a core. When the vortex sheet represents the interface between two immiscible fluids of equal density, surface tension becomes an important physical regularization. Numerical calculations with surface tension effects included also reveal a spiral structure [14], but the arms show oscillations that may lead to a breakup of the spiral into detached drops.

So far, the results suggest that some form of spiral may be the weak solution to vortex sheet motion beyond the time of singularity formation. Rigorous theory [15,16] establishes the global existence in time for vortex sheet motion in a weak class of functions, but it does not clarify the nature of the vortex sheet. The missing part of our understanding of the nature of vortex sheet motion is the precise behavior in the limit of vanishing regularization. For Burger's equation (2), a study of the limiting behavior as ν vanishes leads to jump conditions that control shock motion. The shock speed is given by the jump in flux $(u^2/2)$ over the jump in density (u). We have no such analysis to describe precisely the nature of the vortex sheet after the time of singularity formation. The difficulty is that the motion of the sheet depends globally on its location and the distribution of vorticity along it. This is the same difficulty one finds in attempts to understand the limiting nature of solutions to the Navier-Stokes equation of incompressible flow as viscosity vanishes.

We try here to shed some light on the nature of limiting solutions to transport equations where the velocity is defined globally by designing two model equations with similar features to vortex sheet motion and the transport of vorticity. Dhanak [3] derives an asymptotic equation for a thin shear layer including the effects of weak viscosity:

$$\gamma_t + (V\gamma)_s = \nu\gamma_{ss} \tag{5}$$

where γ is the effective vortex sheet strength assigned to the curve in the center of the layer, s is the arclength along this curve, and V is the tangential component of the velocity generated by the vortex sheet, that is, the tangential component of (3). We replace the tangential component of the Biot-Savart integral (3) by a one dimensional analogue, the Hilbert transform,

$$\frac{dx}{dt} = -H(u) \equiv -\frac{1}{\pi} P\!\!\int_{-\infty}^{\infty} \frac{u(\eta)}{\eta - x} d\eta \tag{6}$$

where we use $u(x)$ in place of $\gamma(p)$. This leads to an equation in conservative form,

$$u_t = -(H(u)u)_x + \nu u_{xx} \tag{7}$$

This equation is also an analogue to the conservative form of Burger's equation (2), and has been considered before in a different form [17].

We obtain our other model equation by interpreting (6) as the "velocity" in the transport equation,

$$u_t = H(u)u_x + \nu u_{xx} \tag{8}$$

The addition of the term νu_{xx} models viscous regularization in analogy to (5). This equation is a transport equation in non-conservative form.

Both equations have a constant $\bar{u}$ as a solution. By considering small perturbations of the form, $u = \bar{u} + \varepsilon \exp(\sigma t) \sin(kx)$, we obtain the growth rates, $\sigma = \bar{u}k - \nu k^2$ for (7) and $\sigma = -\nu k^2$ for (8). In both cases, $\nu \neq 0$ appears to regularize the solution by ensuring high wavenumbers will have decaying amplitudes. When $\nu = 0$, the first model equation has a similar growth rate to that of a vortex sheet, providing additional justification for its consideration. While the analogy of these equations to vortex sheet motion is weak by virtue of their being only one dimensional, both equations do contain nonlocal fluxes simple enough to allow us to examine dissipative effects.

Like many before us, we consider periodic boundary conditions only. The argument is that if an equation develops singularities for periodic boundary conditions, it will do so for any boundary conditions. Boundary conditions will have little influence on the local structure of a singularity (or other features).

We shall show that solutions to both equations can develop singularities in finite time when $\nu = 0$. A previous study of (7) has already shown that it has solutions that blow up in finite time even when $\nu \neq 0$. We restate these results in a form particularly suited to our purposes, that is, for the usual initial conditions used in singularity studies of vortex sheet motion. Thus (7) is unsatisfactory as a model equation for vortex sheet motion, but, as we will show, it does have interesting mathematical properties. We prove that the other equation (8) does have solutions for all time when $\nu \neq 0$, and an asymptotic analysis supports the plausibility of a weak solution containing a moving square root singularity. Numerical calculations support these conclusions.

An interesting interpretation of the origin of singularities comes from previous work on certain nonlinear partial differential equations which examines the behavior of their solutions in the complex plane of the spatial variable x. The body of literature is too large for us to review in depth, but we mention a few specific examples. Pole singularities occur in the solution to the Sivashinsky equation [18], in Burger's equation [19], and in Hele-Shaw flows [20]. Other isolated singularities, usually branch points of half-integer powers, occur in Burger's equation (when $\nu = 0$), in Hele-Shaw flows [21], and in vortex sheet motion [5,7]. In many cases, solutions can be expressed as collections of such singularities, where these singularities move according to mutual interactions.

Of particular interest to us, is the behavior for Burger's equation [22]. When $\nu = 0$, square root singularities are formed in the complex plane of the spatial variable x, and move towards and reach the real axis in finite time. When $\nu > 0$, the singularities are not square roots, but rather poles. As $\nu \to 0$, the poles and zeros of u accumulate on a branch cut and form a square root. Consequently, we have studied the two model equations (7),(8) by looking for solutions with singularities in the complex plane of the spatial variable x.

We construct solutions to (7) when $\nu = 0$, and show that they contain square root singularities in the complex plane which move towards and reach the real axis in finite time. There is also a pole decomposition for the first model equation (7) when $\nu \neq 0$ [17]. Unlike the situation in Burger's equation however, poles can reach the real axis in finite time, causing a blow up in the solution. Details and results are presented in Section 2.

In Section 3, we consider the other model equation (8). We establish the existence of solutions to (8) for all time by following closely the arguments in [23] used to establish existence for Burger's equation. We present an heuristic argument that the solution contains square root singularities in the complex plane when $\nu = 0$, and verify this behavior by direct numerical calculations. There is no pole decomposition for (8) when $\nu \neq 0$. We solve the equation numerically, and from the nature of the numerical solutions, we are led to a guess for the weak limit. The weak solution appears to be also a square root singularity that reaches the real axis, and then propogates along it. In Section 4, we study the weak limit by asymptotic analysis, and confirm our results by numerical calculations.

We point out some related work [24] on a model equation for the effects of vortex stretching in the Euler's three dimensional equations. Singularities also form in finite time in this example, and a viscous term [25] fails to regularize the behavior. The influence of advection with nonlocal fluxes is not considered in this work: It is the advection of vorticity along the vortex sheet that generates its singularities.

2 The First Model Equation

By simple modification of the steps that lead to Theorem 3, we can establish local existence in time for initial data that is 2π-periodic and in C^∞. However, since solutions to (7) do not satisfy a maximum principle, as in Lemma 4, we cannot prove existence for all time. There is good reason for our failure: solutions can blow up in finite time even when $\nu > 0$ [17]. We demonstrate this assertion by an explicit construction of solutions that blow up in finite time, proceeding in a slightly different way to previous work [17].

We need several important properties of the Hilbert Transform:

$$H(fg) = fH(g) + gH(f) + H(H(f)H(g)), \tag{9}$$
$$H(H(f)) = -f, \tag{10}$$
$$H(e^{ikx}) = i\,\mathrm{sign}(k)\,e^{ikx} \tag{11}$$

From (9) and (10), we have

$$H(H(u)u) = \frac{1}{2}((H(u))^2 - u^2) \tag{12}$$

By following previous ideas [24], we use this equation to transform (7) into the Burger's equation,

$$z_t + zz_x = \nu z_{xx}, \tag{13}$$
$$z(x,0) = H(f) - if, \tag{14}$$

for the complex function $z = H(u) - i\,u$. Here $u(x,0) = f(x)$ gives the initial condition for (7). Notice that z^* (star superscript implies complex conjugation) also satisfies Burger's equation, and we may recover u from $u = i(z - z^*)/2$.

First, we consider the case where $\nu = 0$. We choose the initial condition,

$$u(x,0) = 1 + 2\epsilon\cos(x) \tag{15}$$

which is a typical choice in studies of vortex sheet singularities. The corresponding initial condition for (13) is $z(x,0) = -i[1 + 2\epsilon\exp(-ix)]$. We can construct the solution in parametric form by the method of characterisitics,

$$z = -i(1 + 2\epsilon e^{-i\xi}) \tag{16}$$
$$x = \xi - i(1 + 2\epsilon e^{-i\xi})t \tag{17}$$

Singularities occur where the mapping between ξ and x fails, that is, at the zeroes $\xi = 2n\pi - i\ln(2\epsilon t)$ of x_ξ. To understand the nature of these singularities, we pick one ($n = 0$) and approximate (17) by a Taylor series around this point. Since x_ξ vanishes at $-i\ln(2\epsilon t)$ we terminate the series with the quadratic term.

$$x = -i\left[1 + t + \ln(2\epsilon t)\right] + \frac{i}{2}\left[\xi + i\ln(2\epsilon t)\right]^2 \tag{18}$$

Thus ξ, and consequently z, has a square root singularity in terms of x at $x_s = -i[1+t+\ln(2\epsilon t)]$. Initially, this singularity is at $i\infty$, but at some $t = t_c$ it reaches the real axis. Since $u = i(z - z^\star)/2$, u will have square root singularities at x_s and its complex conjugate. Thus u always has square root singularities in the complex plane of x, but it is only when they reach the real axis at t_c that a singularity suddenly becomes apparent. This feature of solutions to certain partial differential equations, including Burger's equation, has been noted before [26].

Since we are interested in 2π-periodic solutions to (7), we may express u in a Fourier series,

$$u(x,t) = \sum_{k=-\infty}^{\infty} A_k(t)\, e^{ikx}.$$

where the requirement $A_{-k} = A_k^*$ (and A_0 is purely real) ensures u is a real function. As a consequence of (11),

$$z(x,t) = -iA_0(t) - 2i\sum_{k=1}^{\infty} A_k^*(t)\, e^{-ikx}. \tag{19}$$

We may construct a "travelling wave" solution [27–29] to (7) by letting $\xi = -ix + \sigma t$, and assuming that $u(x,t) = U(\xi)$: the "speed", σ, is a constant to be determined. Note that this assumption is equivalent to setting $A_k^* = C_k^* \exp(k\sigma t)$, where C_k^* are constants. Thus, (19) becomes

$$z(x,t) = Z(\xi) = -iC_0 - 2i \sum_{k=1}^{\infty} C_k^* \, e^{k\xi}. \tag{20}$$

By substituting (20) into (13) and balancing the coefficients of $\exp(k\xi)$ we obtain a set of equations for C_k^* that can be solved recursively. Specifically, the equation for $k = 1$ is

$$(\sigma + \nu - C_0) \, C_1^* = 0$$

Since we do not want $C_1^* = 0$, we must have

$$\sigma = -\nu + C_0$$

The important aspect of this result is that σ must be real. Rather than solve for the Fourier coefficients recursively, we can actually construct the travelling wave solution in closed form as follows.

In terms of ξ, (13) becomes

$$\sigma \frac{dZ}{d\xi} - iZ \frac{dZ}{d\xi} = -\nu \frac{d^2 Z}{d\xi^2}, \tag{21}$$

This equation may be integrated once, and the constant of integration evaluated by the requirement that $Z \to -iC_0 = -i(\sigma + \nu)$ as $\xi \to -\infty$. We find

$$2\nu \frac{dZ}{d\xi} = i(Z + i\sigma)^2 + i\nu^2$$

The general solution to this equation is

$$Z = -i\sigma - i\nu \frac{1 + B\, e^{\xi}}{1 - B\, e^{\xi}}$$

where B is the constant of integration, which we may write as $B = \rho \exp i\theta$, where ρ and θ are real. Since θ merely sets the origin of x, we may set $\theta = 0$ without loss of generality. Finally, we are led to

$$u(x,t) = \sigma + \nu \frac{1 - \rho^2 \, e^{2\sigma t}}{1 + \rho^2 e^{2\sigma t} - 2\rho e^{\sigma t} \cos(x)}; \tag{22}$$

For $\rho < 1$, these solutions breakdown at $T = -\log(\rho)/\sigma$ and at $x = 2n\pi$. The form of the singularity in the solution is best studied by setting $t = T - \varepsilon$, and taking the limit $\varepsilon \to 0$.

$$\lim_{t\to T} u(x,t) = \sigma + \nu \lim_{\varepsilon\to 0} \frac{1 - e^{-2\sigma\varepsilon}}{1 + e^{-2\sigma\varepsilon} - 2e^{-\sigma\varepsilon}\cos(x)} \tag{23}$$

Clearly, this limit is just σ when $x \neq 2n\pi$. We study the behavior near $x = 2n\pi$ by expanding the cosine function in a Taylor series and retaining only the first two terms. Then

$$\begin{aligned}\lim_{\varepsilon\to 0} \frac{1 - e^{-2\sigma\varepsilon}}{1 + e^{-2\sigma\varepsilon} - 2e^{-\sigma\varepsilon}\cos(x)} &= \lim_{\varepsilon\to 0} \frac{(1 - e^{-\sigma\varepsilon})(1 + e^{-\sigma\varepsilon})}{(1 - e^{-\sigma\varepsilon})^2 + e^{-\sigma\varepsilon}(x - 2n\pi)^2} \\ &= \lim_{\alpha\to 0} \frac{2\alpha}{\alpha^2 + (x - 2n\pi)^2} \\ &= 2\pi\delta(x - 2n\pi)\end{aligned} \tag{24}$$

The other way to interpret these results is to recognize that they are a special case of a pole decomposition [17]. By a simple modification of the results in [17], or by using a pole decomposition [19] for (13), we can express solutions to (7) in the form,

$$u(x,t) = \sigma - i\nu \sum_{j=1}^{n} \frac{1}{x - z_j(t)} + i\nu \sum_{j=1}^{n} \frac{1}{x - z_j^*(t)} \tag{25}$$

where the poles located at z_j are assumed to lie in the upper half plane. The trajectories of the poles are given by

$$\frac{dz_j}{dt} = -i\sigma - 2\nu \sum_{\substack{k=1 \\ k\neq j}}^{n} \frac{1}{z_j - z_k} \tag{26}$$

These results can be extended to the case of 2π-periodic solutions by constraining the poles to be on 2π-periodic arrays and by replacing the z^{-1} interaction between poles by $\cot(z/2)$ interactions between the arrays.

Pole trajectory equations of this type have been studied by [18] in regard to the Sivashinsky equation. Their results indicate the poles attract each other parallel to the real axis, and repel each other parallel to the imaginary axis. The constant term $-i\sigma$ adds a drift towards the real axis. However, there is an important difference between (26) and the equation for the trajectory of the poles in the Sivashinsky equation. In the latter case, the summation must include pairs of poles at complex conjugate locations. Thus, a pole and its conjugate will not meet on the real axis because of the tendency for poles to repel parallel to the imaginary axis. In contrast, the summation in (26) is only for poles in the upper half plane. The conjugates appear only in (25). The tendency for poles to repel parallel to the imaginary axis, coupled with the drift towards the real axis, means that poles will always reach the real axis in finite time. The result (22) can be obtained by picking a single 2π-periodic array of poles:

$$\begin{aligned} u(x,t) &= \sigma + \nu i \cot\left(\frac{x-Z}{2}\right) - \nu i \cot\left(\frac{x-Z^*}{2}\right) \\ &= \sigma + \nu \frac{1-e^{-2Y}}{1+e^{-2Y}-e^{-Y}\cos(x-X)} \end{aligned}$$

where $Z = X + iY$ gives the location of the pole inside the strip $0 < x < 2\pi$. The trajectory of this pole is

$$\frac{dZ}{dt} = -i\sigma \tag{27}$$

We find (22) directly with the choice of initial condition, $X = 0$, $Y = -\ln(\rho)$, with $\rho < 1$.

By constructing solutions from two different approaches, the "travelling wave" approach [27],[28] and the pole decomposition approach [19],[17], we have shown an interesting connection between them. The "travelling wave" solution contains singularities that move constantly towards the real axis. In general, these singularities need not be poles [28], but for this model equation (7) they are, and the solution corresponds to the simplest choice for a pole decomposition, namely a single pole and its complex conjugate.

We conclude this section by noting that the general behavior of a pole decomposition is likely to be a generic feature [22], and that means solutions will form singularities in general in finite time. Viscous regularization has failed to prevent singularity formation in solutions of (7).

3 The Second Model Equation

In Appendix A, we provide the details of a proof for the global existence in time of solutions to (8) when the initial data is C^∞ and 2π-periodic, and $\nu > 0$. Morevover, the solution is infintely many times differentiable. The key difference in being able to establish existence for all time is that solutions to (8) satisfy a maximum principle, whereas those of (7) do not. However, we are unable to construct exact analytic solutions to (8), and we must use asymptotic and numerical methods to uncover features of the solutions.

We start first by considering $\nu = 0$. We present an asymptotic argument that indicates solutions to (8) contain square root singularities in the complex plane of x. First, we rewrite the Hilbert transform as

$$\frac{1}{\pi} P\!\!\!\int_{-\infty}^{\infty} \frac{u(\eta) - \overline{u}}{\eta - x} d\eta = \frac{1}{\pi} \int_{C} \frac{u(z) - \overline{u}}{z - x} dz + i[u(x) - \overline{u}]$$

where $\overline{u}$ is the mean value of u, and where the contour C lies above the point $z = x$. Actually, we move x far below the real axis, keep the contour C on the real axis, and note that the contour integral becomes asymptotically small. Thus, far below the real axis, (8) takes the simpler form,

$$u_t - i[u - \overline{u}]\, u_x = 0$$

For the initial condition (15) we have $\overline{u} = 1$ and u has the form $u = 1 + \epsilon \exp(ix)$ for $\mathrm{Im}(x) \ll 0$. The method of characteristics provides the solution in parametric form,

$$u = 1 + \epsilon e^{i\xi} \tag{28}$$

$$x = \xi - i\epsilon e^{i\xi} t \tag{29}$$

From an analysis similar to that applied to (16) and (17), we deduce the presence of a square root singularity following the trajectory,

$$x_s = \pi + i[1 + \ln(\epsilon t)] \tag{30}$$

This trajectory is valid for small times when the singularity is still far from the real axis. To confirm this asymptotic prediction, we turn to direct numerical solution of (8) with $\nu = 0$ and use the method designed by [26] to detect the presence of singularities in the complex plane.

Before proceeding with our discussion of the behavior of solutions to (8) with $\nu = 0$, we describe briefly the numerical method that we used on both equations, whether $\nu = 0$ or not. Since we are interested in periodic solutions to (8) and (7), we use spectral methods to construct the solutions numerically. The coefficients of a Fourier series satisfy the system,

$$\frac{dA_k}{dt} = -\nu k^2 A_k + \hat{G}_k. \tag{31}$$

where $\hat{G}_k$ are the coefficients for $H(u)\,u_x$ or $-(H(u)\,u)_x$. For large k and $\nu \neq 0$, this system is stiff, so we use an alternative form instead,

$$\frac{d}{dt}(e^{\nu k^2 t} A_k) = e^{\nu k^2 t} \hat{G}_k. \tag{32}$$

We apply the Adams-Moulton fourth order predictor-corrector method to either form, and to do so, we evaluate $\hat{G}_k$ by pseudo-spectral techniques. Given the Fourier coefficients at some time level, we use the fast inverse Fourier transform to obtain u at evenly spaced points; to obtain u_x, and $H(u)$ at evenly spaced points, we must first multiply the Fourier coefficients by ik and $i\text{sign}(k)$ respectively before using the inverse transform. We then form the products, $H(u)\,u_x$ or $H(u)\,u$, at evenly spaced points. By using the fast Fourier transform, we obtain the Fourier coefficients for these products. The Fourier coefficients of the derivative of $H(u)\,u$ are now simply obtained by multiplying the Fourier coefficients of $H(u)\,u$ with ik. Obviously, we must use a truncated Fourier series to perform these operations, and we update only those coefficients in the truncated series through either (31) or (32). For all calculations reported in this paper, we took $N = 2^p$ Fourier coefficients, and we increased p in a series of calculations until the pointwise difference between successive results were less than 10^{-7}. Similarly, we decreased the time-step until the pointwise differences were also less than 10^{-7}. We find that a timestep of 10^{-3} is usually adequate for this level of accuracy, but we must use larger and larger values of p for smaller values of ν.

As a check of our method, we use the travelling wave solution (22) for (7) with $\rho = 1/e$, $\sigma = 1.0$ and $\nu = 10^{-3}$. For these values of the parameters, the solution becomes singular at $t_c = 1.0$. We find complete agreement with the analytic solution to the degree of accuracy in our numerical calculations (10^{-7}). Also, we find that our code breaks down at the singularity time t_c as expected, since the numerical solution cannot resolve a delta function.

We turn now to a brief description of the procedure designed by [26] to detect singularities in the complex plane by analysis of the Fourier coefficients of the solution known on the real axis. If the closest singularity to the real axis is an isolated branch point of the form,

$$u(x) \sim (x - z_s)^{\mu}\,, \tag{33}$$

then the Fourier coefficients have the form,

$$A_k \sim |k|^{-(\mu+1)} e^{-k\delta} e^{i\beta k}, \qquad k \to +\infty \tag{34}$$

where $z_s = \beta - i\delta$. There is a similar form for $k \to -\infty$. This fact has been used to good effect in singularity studies of vortex sheet and interfacial motion by several researchers [28,8,9,12,30,29]. By fitting the numerically calculated Fourier coefficients to the form (34), they are able to track the motion of the nearest singularity as it approaches the real axis. The procedure that appears most reliable is based on finding values for C, α, δ so that

$$|A_m| = \frac{C}{m^\alpha} e^{-\delta m} \tag{35}$$

holds for just three adjacent values, $m = k-1, k, k+1$. Note that we have used the information from (18) that $\beta = 0$, and we expect $\alpha = 1+\mu$ to be real. We label these local solutions by k, and then examine their behavior for large k. In particular, they should asymptote to constant values. We tested this procedure for the singularity trajectory predicted by (18) and found perfect agreement. Thus we have some confidence in applying this technique to the other model equation (8).

To obtain the numerical solution to (8) with $\nu = 0$ and $\epsilon = 0.05$ in (15), we start with 512 Fourier coefficients and advance them using (31) with a time step of 0.001. At first, the Fourier coefficients decay rapidly and reach round-off levels for low values of k. As time advances, the rate of decay slows and more of the coefficients have values above round-off. We stop our calculation at $t = 10.156$ when the tail of the spectrum first reaches round-off at $k = 509$, which is just 3 short of the largest k (512) present in our numerical calculations. Recall that for a real function, the Fourier coefficients with negative values of k are the conjugates of the Fourier coefficients with positive values for k. Thus our tests on the decay of the spectrum need only be conducted on the range with positive k. We double the number of active Fourier coefficients by adding more with zero amplitude. We now continue the calculation, but double the number of Fourier coefficients whenever the tail of the spectrum reaches round-off at a value of k just 3 before the maximum allowable k in our calculations. Doublings occurred at $t = 10.156, 10.886, 11.416, 11.800, 12.079, 12.150$, but were stopped when we reached 32,768 Fourier coefficients. As Shelley [9] points out, this process ensures that the spatial errors are at round-off levels. Errors resulting from time stepping are controlled by decreasing the time step. For a time step of 0.001, the errors are less than 10^{-10}.

At each time level, we apply our procedure to calculate α and δ in (35). We show their variations with k in Fig. 1 at $t = 11.0$, but the behavior is typical. First observe that α is approaching the value 3/2, agreeing with the asymptotic predictions of square root singularities. Secondly, note that there is a range in k where δ is essentially constant. At higher values, the form-fit deteriorates because of round-off errors reducing the number of accurate digits. In Table 1, we show the average value of δ, the range in k for which the average value of δ is calculated, and the standard deviation of δ from the average, at representative times. We find no change in the values for δ, when

Table 1
The range in k used to determine the average value of δ and its standard deviation.

time	k range	average δ	standard deviation
1.00	5 - 10	2.00099	9.7×10^{-4}
2.00	5 - 10	1.31865	8.6×10^{-4}
3.00	10 - 15	0.929871	1.4×10^{-4}
4.00	10 - 20	0.666978	1.7×10^{-4}
5.00	15 - 25	0.475559	5.2×10^{-5}
6.00	25 - 35	0.331906	8.9×10^{-6}
7.00	30 - 50	0.223189	8.7×10^{-6}
8.00	40 - 60	0.141603	5.9×10^{-6}
9.00	50 - 100	8.20244×10^{-2}	1.4×10^{-5}
10.00	65 - 110	4.07822×10^{-2}	9.7×10^{-6}
11.00	70 - 120	1.50173×10^{-2}	1.8×10^{-5}
11.20	80 - 150	1.15549×10^{-2}	1.6×10^{-5}
11.40	80 - 150	8.55600×10^{-3}	1.6×10^{-5}
11.60	100 - 200	5.89328×10^{-3}	7.4×10^{-6}
11.80	120 - 230	3.85706×10^{-3}	3.6×10^{-6}
12.00	200 - 350	2.27317×10^{-3}	4.0×10^{-6}
12.19	250 - 400	1.31190×10^{-3}	3.6×10^{-6}

we double the time step.

In Fig. 2 we compare the temporal variation in the numerically determined δ with the asymptotic prediction (30). When δ is large, the agreement is very good, but for smaller δ there is a noticeable difference. As the singularity approaches the real axis, it slows down considerably. Nevertheless, it reaches the real axis in finite time, as evident by the plot of $-1/\ln(\delta)$ versus time. The slope of this curve is steepening, indicating that it will cross the axis at a finite value of time. Consequently, $\delta = 0$ at this time, and the singularity in $u(x,t)$ becomes apparent. Similar behavior has been observed for square root singularities in Burger's equation [26], where the singularity approaches the real axis as $\delta \sim (t_c - t)^{3/2}$. Thus singularities can approach the real axis with vanishing speed, but still reach it in finite time.

We acknowledge that the evidence is not conclusive, but we proceed with the belief that a singularity is generated in finite time for (8) with initial condition (15) when $\nu = 0$. Since (8) has solutions for all time when $\nu > 0$, the question arises whether there is a weak solution for (8) beyond the time of singularity formation when $\nu = 0$. We use our code to obtain numerical solutions for a range of decreasing values of ν to help us assess what is the limiting behavior of the solution as $\nu \to 0$.

The numerical solutions to (8) with $\nu > 0$ and with initial condition (15) exhibit no singularities, and we are able to run our calculations for long times. In Fig. 3, we show two sets of profiles, one for $\nu = 0.1$ and one for $\nu = 0.0001$. In both cases, $\epsilon = 0.05$, and the time step is 0.001 using (32). For $\nu = 0.1$, we use 256 Fourier modes, and for $\nu = 0.0001$, we use 512 Fourier modes. For the larger value of ν, the solution simply decays away. For the smaller value of ν, we see the development in time of a small-scale structure that is almost a cusp. As part of our tests, we check that the curvature is also accurate to a few digits, and we find that it has finite, but large values at $x = \pi$. By running our code for various ν, it is clear that the curvature at the tip of the spike increases with smaller ν, raising the possibility that the spike is a cusp in the limit of vanishing ν.

4 Weak Solutions to the Second Model Equation

First we study whether (8) allows singularity propagation when $\nu = 0$. Extrapolating from the numerical calculations for small ν, we see a tendency for the solution to form a cusp. This suggests to us that a singularity of the form $|x - x_s(t)|^q$ might occur, where q lies between 0 and 1. So we assume that the solution has the form,

$$u(x,t) = a_0(t) + a_1(t)|\eta|^q + a_2(t)\,\eta + o(\eta) \tag{36}$$

near the singularity, where $\eta = x - x_s(t)$ measures distance from the singularity. Before we can substitute (36) into (8), we need to assess the influence of the Hilbert transform on this form. We split the Hilbert transform into two parts; one accounts for local contributions around $\eta = 0$ and the other for the rest.

$$H(u(x,t)) = I_1 + I_2, \tag{37}$$

$$I_1 = \frac{1}{\pi} \mathcal{P}\!\!\int_{-\delta}^{+\delta} \frac{u(\xi,t)}{\xi - \eta} d\xi \tag{38}$$

$$I_2 = \frac{1}{\pi} \int_{-\infty}^{-\delta} \frac{u(\xi,t)}{\xi - \eta} d\xi + \frac{1}{\pi} \int_{+\delta}^{+\infty} \frac{u(\xi,t)}{\xi - \eta} d\xi \tag{39}$$

where δ is a fixed positive number such that the local expansion (36) is valid in $(-\delta, \delta)$, and we restrict η by $|\eta| \ll \delta \ll 1$. Without loss of generality, we consider the case $\eta > 0$. Then we may change the integration variable in the range $[-\delta, 0]$ to obtain

$$\begin{aligned} I_1 &= \frac{2}{\pi} P\!\!\!\!\int_0^\delta \frac{\eta(a_0 + a_1 \xi^q) + O(\eta\xi) + O(\xi^2)}{\xi^2 - \eta^2} d\xi \\ &= \frac{a_0(t)}{\pi} \ln \left| \frac{\delta - \eta}{\delta + \eta} \right| + \frac{2a_1(t)\eta^q}{\pi} P\!\!\!\!\int_0^{\delta/\eta} \frac{y^q}{y^2 - 1} dy + O(\delta) + O(\eta) \end{aligned} \tag{40}$$

We have used the substitution $\xi = \eta y$ to obtain the second integral. We have checked that if the higher order terms in (36) are of the form η^p with $p > 1$, then the corrections to (40) are truly of higher order. Finally, we note that for $\eta \ll \delta$, the integral in (40) asymptotes to a constant, C say. Thus we have

$$I_1 = \frac{2}{\pi} C a_1 \eta^q + O(\delta) + O(\eta) \tag{41}$$

The other integral I_2 is an analytic function of η in a small neighborhood of 0. So it can be expanded in Taylor's series,

$$I_2 = b_0(t) + b_1(t)\eta + o(\eta) \tag{42}$$

Note in particular that

$$b_0(t) = \frac{1}{\pi} \int_{|\xi|>\delta} \frac{u(\xi, t)}{\xi} d\xi \tag{43}$$

By combining the results, (41) and (42), we obtain

$$H(u) = b_0 + \frac{2}{\pi} C a_1 |\eta|^q + O(\eta) + O(\delta) \tag{44}$$

where we have generalized the result to include the case $\eta < 0$.

Together with (36), we substitute (44) into the inviscid form of (8) to obtain

$$\begin{aligned} u_t - H(u)\, u_x = &\frac{da_0}{dt} - q a_1 \frac{dx_s}{dt} \frac{|\eta|^q}{\eta} - a_2 \frac{dx_s}{dt} \\ &- \left(b_0 + \frac{2}{\pi} C a_1 |\eta|^q + O(\delta) \right) \left(q a_1 \frac{|\eta|^q}{\eta} + a_2 \right) + O(\eta^q) \end{aligned}$$

The terms of lowest order are those with $|\eta|^q/\eta$. Thus we must set

$$\frac{dx_s}{dt} = -b_0 + O(\delta) \to \frac{1}{\pi} P\!\!\int_{-\infty}^{\infty} \frac{u(\xi,t)}{\xi} d\xi \qquad \text{as} \qquad \delta \to 0 \tag{45}$$

Thus the speed of the singularity is determined by (45) which is the Hilbert transform evaluated at the singularity. This result is consistent with our numerical results in Fig. 3 since $b_0 = 0$ due to the symmetry in the solution. To obtain a balance of the next order terms, we set $2q - 1 = 0$, or $q = 1/2$. This result is not unexpected, and suggests that a square root singularity in the complex plane of x moves towards the real axis, reaches it in finite time, and and propagates along it. The balance of the next order terms gives

$$\frac{da_0}{dt} = \frac{C}{\pi} a_1^2 \tag{46}$$

where

$$C = P\!\!\int_0^{\infty} \frac{\sqrt{y}}{y^2 - 1} dy = \frac{\pi}{2}.$$

Thus the motion of the singularity is determined without recourse to information from the viscous solutions. In other words, no "jump conditions" are needed.

In Fig. 4, we show the consequences of choosing initial conditions without symmetry. We set $u(x,0) = 1 + 0.1\cos(x) + 0.1\sin(x)$, and take $\nu = 0.0001$. We use 512 Fourier coefficients to advance the solution with time step 0.001. Our resolution studies indicate an accuracy of 10^{-7} at least. The profiles give the solution at times $t = 3, 6, 9, \ldots, 50$. The results show clearly the formation of two singularities that move together and merge. Curiously enough, the profile appears to develop symmetry, and the new singularity that emerges after the merger appears stationary. We offer no explanation for this behavior, but leave it as a matter for further study. The profiles do suggest, however, that square root singularities underlie the behavior.

Next we show that the numerical results are consistent with a square root singularity as $\nu \to 0$. Fortunately, the initial condition (15) leads to singularity formation at $x = \pi$, and the singularity remains there subsequently, simplifying the comparison between our asymptotic analysis and the numerical results. We choose ($\epsilon = 0.05$) and select data at $t = 13.5$ which is after the time of singularity formation. As we study a sequence of solutions for decreasing values of ν, we expect convergence to the square root singularity, but the convergence will not be uniform. There is a region around $x = \pi$ in which viscosity will act to smooth the profile, and this region will shrink for choices of smaller ν. We ignore this region for the moment, and assume that the solution will behave like

$$u(x,t) \sim u(\pi,t) + a_1(t)|x-\pi|^q$$

in a region near $x = \pi$. Then we use the numerical data to determine q as follows.

$$q = \frac{\ln(u(x_{i+1},t) - u(\pi,t)) - \ln(u(x_i,t) - u(\pi,t))}{\ln(x_{i+1}-\pi) - \ln(x_i - \pi)} \tag{47}$$

where x_i and $u(x_i,t)$ are the grid points and corresponding numerical values of the solution. Fig. 5 shows the plot of the estimation for q for different values of ν. The dotted line gives $q = 0.5$. There are two important features to the curves. First, note that q takes the value 2 when $x = \pi$. The reason is that the solution has a minimum at $x = \pi$, so it is locally a quadratic. Secondly, the curves are clearly approaching the line $q = 0.5$, but of course non-uniformily. Thus we believe that the square root singularity is the weak solution selected in the limit of vanishing ν.

It is only near $x = x_s$ that we expect small values of ν to have an important effect on the solution. We examine the nature of the solution in the immediate vicinity of $x = x_s$ by introducing an inner variable $X = [x - x_s(t)]/\nu^n$ and assuming the solution in the form

$$u(x,t) = a_0(t) + \nu^m U\Big(\frac{x - x_s(t)}{\nu^n}, t\Big). \tag{48}$$

Here $a_0(t)$ is the leading order term of the outer solution from (36). Note that we do not scale the time variable: the numerical results show no rapid changes in the solutions with time as $\nu \to 0$. Again we choose δ such that $\nu^n \ll \delta \ll 1$. Just as we did in (37), we split $H(u)$ into two parts I_1 and I_2. By a change of the integration variable, we may express I_1 in terms of the new variables as

$$I_1 = \frac{\nu^m}{\pi} P\int_{-\delta/\nu^n}^{\delta/\nu^n} \frac{U(\eta,t)}{\eta - X} d\eta.$$

Its leading order as $\nu \to 0$ is

$$I_1 = \nu^m H(U) = \frac{\nu^m}{\pi} P\int_{-\infty}^{\infty} \frac{U(\eta,t)}{\eta - X} d\eta \tag{49}$$

As before, the other integral may be expressed as (42); in terms of the inner variable,

$$I_2 = b_0 + b_1 \nu^n X + \dots$$

Thus, after combining these two results, we have

$$H(u) = b_0 + b_1 \nu^n X + \nu^m H(U) + \dots$$

Upon substitution of this result and the expression (48) into (8), we find

$$\begin{aligned} \frac{da_0}{dt} + \nu^m U_t - \nu^{m-n} \frac{dx_s}{dt} U_X = (b_0 + b_1 \nu^n X + \nu^m H(U)) \nu^{m-n} U_X \\ + \nu^{m-2n+1} U_{XX} + \dots \end{aligned} \tag{50}$$

From (45), the terms with ν^{m-n} cancel. We choose the following balance of the remaining exponents,

$$m - 2n + 1 = 2m - n = 0$$

which gives $m = 1/3$ and $n = 2/3$. The inner equation then becomes

$$\frac{da_0}{dt} = H(U)\, U_X + U_{XX} \tag{51}$$

In addition, the inner solution at large values of $|X|$ must match with the outer solution (36) at small values of η. If we express (36) in terms of the inner variable, we find $a_0 + \nu^{1/3} a_1 |X|^{1/2} + \nu^{2/3} a_2 X \dots$. Thus $U(X) \to a_1 \sqrt{|X|}$ as $|X| \to \infty$.

By introducing a further change of variables,

$$\xi = \left(\frac{da_0}{dt} \right)^{1/3} X \qquad \text{and} \qquad V = \left(\frac{da_0}{dt} \right)^{-1/3} U$$

we can rewrite (51),

$$H(V)\, V_\xi + V_{\xi\xi} = 1 \tag{52}$$

$$V \to \sqrt{2|\xi|} \quad \text{as} \quad |\xi| \to +\infty \tag{53}$$

We proceed to solve this nonlinear problem by numerical techniques.

If there is a unique solution to (52) and (53), it must satisfy the symmetry $V(-\xi) = V(\xi)$. Thus, we reduce the range to $0 \le \xi \le \infty$, and introduce the boundary condition $V_\xi(0) = 0$. To solve (52), we divide $[0, \infty)$ into two parts, $[0, 1]$ and $[1, \infty)$. On the latter range, we set

$$V(\xi) = \sqrt{2}(\sqrt{\xi} - 1) + W(\xi).$$

Our choice is motivated by making $W(\xi)$ connect continuously with $V(\xi)$ at $\xi = 1$. Note that the matching condition (53) implies $W(\xi) \to \sqrt{2}$ for large ξ.

The Hilbert transform splits into several contributions.

$$H(V) = \frac{2\xi}{\pi} \fint_0^1 \frac{V(\eta)}{\eta^2 - \xi^2} d\eta + \frac{2\xi}{\pi} \fint_1^\infty \frac{\sqrt{2}(\sqrt{\eta} - 1)}{\eta^2 - \xi^2} d\eta + \frac{2\xi}{\pi} \fint_1^\infty \frac{W(\eta)}{\eta^2 - \xi^2} d\eta.$$

The second integral can be performed exactly, but we split the last integral again over the ranges $[1, L]$ and $[L, \infty)$. In the latter range, we set $W = \sqrt{2}$ and integrate exactly. We replace all remaining principal-value integrals by

$$\fint \frac{f(\eta)}{\eta^2 - \xi^2} d\eta = \int \frac{f(\eta) - f(\xi)}{\eta^2 - \xi^2} d\eta + \frac{f(\xi)}{2\xi} \ln \left| \frac{\eta - \xi}{\eta + \xi} \right|$$

The integrand is now well-defined for all ξ, and we apply the trapezoidal rule. Of course the correct limiting value for the integrand must be used when $\eta = \xi$. The derivatives in (52) are replaced by central differences, and we use the method of fictitious points to approximate $V_\xi(0) = 0$. We set $W(L) = \sqrt{2}$ as the other boundary condition. Consequently, we obtain a large system of nonlinear, algebraic equations which we solve by Newton's method.

To obtain the solution with an accuracy better than 0.5×10^{-4}, we first fix L, decrease the spatial mesh size h systematically, and use extrapolation based on the truncation error being of second order to improve the numerical results. Then we increase L, determine the solution accurately as before and use extrapolation again based on the assumption that the solution will converge as L^{-1}. We find that $L = 10$ is sufficient to obtain our desired level of accuracy. In Fig. 6, we show the inner solution as a solid curve.

We compare our numerical solution for the asymptotic equation with the numerical solutions of the full equations in the following way. We pick a time $t = 13.5$, and use the values of $u(\pi, 13.5)$ for various ν to determine a_0 from the form, $u(\pi, 13.5) = a_0 + \nu^{1/3} C$. We also use $u_{xx}(\pi, 13.5)$, calculated spectrally, to determine the time derivative of a_0, $\dot{a}_0$ say, from the form $\nu u_{xx}(\pi, 13.5) = \dot{a}_0$. Then, we are able to plot the numerical solution for the full equations in terms of $(u(\xi, 13.5) - a_0)/(\nu \dot{a}_0)^{1/3}$ as a function of ξ. We show the result as discrete points in Fig. 6. The comparison is very good; we believe the small dicrepancies are due to higher order effects.

5 Discussions and Conclusions

Our studies of the first model equation (7) have shown an interesting connection between a pole decomposition [17] and a "travelling wave" solution [28]. Either a single pole or a single periodic array of poles in the upper half plane move with constant speed (the "wave speed") towards the real axis and reach it in finite time. Poles will reach the real axis in finite time for all pole decompositions, and so singularity formation is probably generic for (7).

We have proved global existence of solutions in time to (8) provided $\nu > 0$. We have provided asymptotic and numerical evidence that solutions to (8) with $\nu = 0$ can contain square root singularities in the complex plane which move towards and reach the real axis in finite time. Further asymptotic and numerical results suggest that the weak limit for such solutions is the continued propagation of the square root singularities along the real axis. By choosing asymmetrical initial conditions, we are able to find a solution for small, but non-zero ν that forms two sharp structures that subsequently merge.

By differentiating (8) we can obtain an equation for $v = u_x$ which is another transport equation with a nonlocal flux.

$$v_t + (I(v)\, v)_x = \nu v_{xx} \tag{54}$$

where $I(v)$ is the global operator,

$$I(v) = \frac{1}{\pi} \int_{-\infty}^{\infty} v(\xi) \ln |\xi - x| d\xi$$

From the properties of the solutions to (8) that we have established in Appendix A, we know that solutions to (54) exist for all time provided $\nu > 0$. In this case, we expect them to develop inverse square root singularities in finite time.

We note some important differences between the solutions to the transport equations (8),(54) with nonlocal fluxes and Burger's equation. The weak solution to Burger's equation after a shock has formed must satisfy certain jump conditions at the shock. The form and motion of the square root singularities in (8) are determined by the inviscid equation itself without additional conditions.

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A Existence and Uniqueness for the Second Model Equation

Our proof of the existence and uniqueness of 2π-solutions globally in time for (8) follows very closely the proof of existence of solutions for (1) as presented in [23]. We present only the changes needed in their lemmas and theorems. First, we recall some results related to the Hilbert transform [31,32].

Let f be a C^∞ function that is also 2π-periodic. Then $H(f)$ is a C^∞, 2π-periodic function. Also,

$$(H(f))_x = H(f_x)\,. \tag{A.1}$$

The L_2-norm of $H(f)$ satisfies the bound,

$$||H(f)|| \le ||f||\,. \tag{A.2}$$

We will also use the following two different bounds on the inner product $(f, H(g)h)$, where g, h are also C^∞, 2π-periodic functions.

$$\begin{aligned} |(f, H(g)h)| &\le |H(g)|_\infty ||f||\;||h|| && \text{(A.3)}\\ &\le |f|_\infty ||g||\;||h|| && \text{(A.4)} \end{aligned}$$

Estimates on the norms of derivatives of f satisfy very simple inequalities as a consequence of Parseval's relation. We introduce the notation that an integer subscript refers to the number of spatial derivatives taken. Then

$$||f_j|| \le ||f_k|| \quad \text{for } 0 < j \le k \tag{A.5}$$

Note that if u has no mean value, then Parseval's relation gives $||u|| \le ||u_x||$ as well. As a consequence of this result, we can use a Sobolev inequality (B.3) to establish that

$$|f_x|_\infty \le \sqrt{2}\,||f||_{H^2} \tag{A.6}$$

Here H^2 is the Sobolev space of functions which are in L_2 and whose first two spatial derivatives are also in L_2.

In imitation of the line of argument used in [23], we introduce the iteration,

$$u_t^{n+1} = H(u^n)u_x^{n+1} + \nu u_{xx}^{n+1}, \tag{A.7}$$
$$u^{n+1}(x,0) = f(x), \tag{A.8}$$

with $n = 0, 1, \ldots$, and $u^0(x,t) = f(x)$. Note that each iterate u^n satisfies a linear equation, so if $f(x) \in C^\infty$, a unique solution exists in C^∞ for each iterate. It will be shown that the sequence $(u^n)_{n\ge 0}$ converges to a function u in C^∞ that solves (8) subject to the initial condition $u(x,0) = f(x)$.

The two lemmas below establish bounds on u^n and its derivatives independently of n and ν, irrespect of whether ν vanishes or not.

Lemma 1 *For a suitable time* $T_1 = T_1(||f||_{H^2})$,

$$||u^n(\cdot,t)||_{H^2} \le 2\,||f||_{H^2}, \tag{A.9}$$

in $0 \le t \le T_1$.

Proof: This Lemma replaces Lemma 4.1.5 in [23]. In their notation, $v = u^{n+1}$ and $w = u^n$. We need estimates for the inner products, $(v, H(w)v_x)$, $(v_x, (H(w)v_x)_x)$, and $(v_{xx}, (H(w)v_x)_{xx})$. First, from integration by parts, we have $(v, H(w)v_x) = -1/2\,(v, H(w_x)v)$, from which it follows by (A.3), (A.6), and (A.2) that

$$|(v, H(w)v_x)| \le \frac{\sqrt{2}}{2}\,||w||_{H^2}\,||v||^2 \tag{A.10}$$

Again, from integration by parts, we have $(v_x, (H(w)v_x)_x) = -(v_{xx}, H(w)v_x)$ $= 1/2\,(v_x, H(w_x)v_x)$. We may follow the same steps that led to (A.10) to obtain

$$|(v_x, (H(w)v_x)_x)| \le \frac{\sqrt{2}}{2}\,||w||_{H^2}\,||v_x||^2 \tag{A.11}$$

By using integration by parts, the last inner product may be expressed as

$$(v_2, (H(w)v_x)_2) = 3/2\,(v_2, H(w_x)v_2) + (v_x, H(w_2)v_2).$$

The first inner product on the right hand side is treated as in (A.11). We proceed differently for the second inner product. We use (A.4) and (B.3) to get

$$\begin{aligned}|(v_x, H(w_2)v_2)| &\le \sqrt{2}\,||v_x||^{1/2}||w_2||\,||v_2||^{3/2} \\ &\le \sqrt{2}\,||w||_{H^2}\,||v_{xx}||^2\end{aligned}$$

Consequently, we find

$$|(v_2, (H(w)v_x)_2)| \le \frac{5\sqrt{2}}{2}\,||w||_{H^2}\,||v_{xx}||^2 \tag{A.12}$$

As described in [23], the above estimates on the inner products lead directly to

$$\frac{d}{dt}||v||^2_{H^2} \le 10\sqrt{2}\,||f||_{H^2}||v||^2_{H^2} \tag{A.13}$$

The Lemma is proved provided $T_1 > 0$ and $\exp\left(10\sqrt{2}\,||f||_{H^2}T_1\right) \le 4$. ■

Now we show that all spatial derivatives of u^n can be estimated on the interval $[0, T_1]$.

Lemma 2 *For each $j = 2,\ 3\ ,\ \ldots,$ there exists K_j such that*

$$||u^n(\cdot, t)||_{H^j} \le K_j, \tag{A.14}$$

for $t \in [0, T_1]$. The constant K_j depends on $||f||_{H^j}$, but is independent of n and ν.

Proof: This Lemma replaces Lemma 4.1.6 of [23]. We need to obtain estimates for the inner product, $(v_j, H(w_k)v_{j+1-k})$. For $1 \le k \le j-2$, we use (A.3), (B.3), and (A.5) to get

$$\begin{aligned}|(v_j, H(w_k)v_{j+1-k})| &\le \sqrt{2}\,||v_j||\,||w_k||^{1/2}||w_{k+1}||^{1/2}||v_{j+1-k}||, \\ &\le \sqrt{2}\,K_{j-1}||v_j||^2.\end{aligned} \tag{A.15}$$

For $k = j-1, j$, we use (A.4), (B.3), and (A.5):

$$
\begin{aligned}
|(v_j, H(w_{j-1})v_2)| = |(v_2, H(w_{j-1})v_j)| &\le \sqrt{2}\, K_{j-1}||v_j||^2\,, \\
|(v_j, H(w_j)v_1)| = |(v_1, H(w_j)v_j)| &\le \frac{\sqrt{2}}{2} K_{j-1}\Big(||v_j||^2 + ||w_j||^2\Big).
\end{aligned}
$$

Finally, for $k = 0$, we integrate by parts to obtain $(v_j, H(w)v_{j+1}) = -1/2$ $(v_j, H(w_1)v_j)$ which is a special case of (A.15).

The rest of the details of the Lemma follow those in Lemma 4.1.6 of [23]. ■

As pointed out in [23], these two Lemmas imply that

$$
\left|\frac{\partial^{j+m}}{\partial x^j \partial t^m} u^n(x,t)\right| \le K(j,m), \tag{A.16}
$$

on the interval $[0, T_1]$. Here the constant $K(j,m)$ is independent of n.

Now we can show that the sequence $(u^n)_{n\ge 0}$ has a limit in $[0, T_1]$.

Theorem 3 *Let $f(x)$ be a 2π-periodic function in C^∞ and let $T_1 = T_1$ $(||f||_{H^2})$ be determined as in Lemma 2.1. For any $\nu \ge 0$, equation (8) has a 2π-periodic, C^∞ solution defined on $[0, T_1]$.*

Proof: This theorem replaces Theorem 4.1.7 in [23]. Let the sequence $(u^n)_{n\ge 0}$ be the solution of the iteration (A.7) and let $v = u^{n+1} - u^n$, $w = u^n - u^{n-1}$. Then the function v satisfies

$$
\begin{aligned}
v_t &= H(u^n)v_x + H(w)u^n_x + \nu v_{xx}, \\
v(x,0) &= 0.
\end{aligned}
$$

To continue the proof as in [23], we need estimates for the inner products, $(v, H(u^n)v_x)$ and $(v, H(w)u^n_x)$. The first inner product can be rewritten as $-1/2\,(v, H(u^n_x)v)$ as a consequence of integration by parts. Thus, from (A.3), (B.3), (A.5), and (A.9), we have

$$
\begin{aligned}
|(v, H(u^n)v_x)| &\le \sqrt{2}\, ||u^n_{xx}||\; ||v||^2 \\
&\le 2\sqrt{2}\, ||f||_{H^2}\, ||v||^2
\end{aligned}
$$

For the other inner product, we note that it is equivalent to $(u^n_x, H(w)v)$. Then we use (A.4), (B.3), (A.5), and (A.9) to obtain

$$
\begin{aligned}
|(v, H(w)u^n_x)| &\le \sqrt{2}\, ||u^n_{xx}||\; ||v||\; ||w|| \\
&\le 2\sqrt{2}\, ||f||_{H^2}\, ||v||\; ||w|| \\
&\le \sqrt{2}\, ||f||_{H^2}\Big(||v||^2 + ||w||^2\Big)
\end{aligned}
$$

By using these estimates, we are led directly to

$$\frac{1}{2}\frac{d}{dt}||v||^2 \leq 3\sqrt{2}\,||f||_{H^2}\left(||v||^2 + ||w||^2\right)$$

and the proof proceeds as in [23]. ■

So far, we have allowed $\nu \geq 0$. In order to prove the existence of a solution for all time, we require bounds on the solution to hold independent of the length of the time interval. We can get such bounds using the following lemma which establishes the maximum principle for (8), but with the requirement that $\nu > 0$.

Lemma 4 *Consider $\nu > 0$ and the initial data $f(x)$ to be C^∞ and 2π-periodic. Let u be a C^∞ solution to (8) that exists for some $[0,T]$. Then*

$$|u|_\infty \leq |f|_\infty, \tag{A.17}$$
$$||u|| \leq \sqrt{2\pi}\,|f|_\infty. \tag{A.18}$$

Proof: The first inequality follows directly from Lemma 4.2.3 in [23]. The bound on the L_2 norm of u is immediately deduced from the definition of the L_2 norm and the maximum norm estimate. ■

The following theorem replaces Theorem 4.2.1 in [23], but the proof does not follow their work. Actually, the approach in our theorem can be modified to apply to (1). The important common ingredient in the proof that applies to (1) and (8) is the maximum principle.

Theorem 5 *Let f be a 2π-periodic C^∞ initial condition and let u be a C^∞ solution of (8) defined on $[0,T]$. Then there is a constant K, dependent on the H^2 norm of the initial condition and on the viscosity ν, but independent of T, such that*

$$||u(\cdot,t)||_{H^2} \leq K, \tag{A.19}$$

with $t \in [0,T]$.

Proof: Let $v = u_x$. The equation for v is

$$v_t = H(v)\,v + H(u)\,v_x + \nu v_{xx}$$

By multiplying with v and integrating over 2π, we are led to

$$\frac{1}{2}\frac{d}{dt}||v||^2 = (v, H(v)\,v) + (v, H(u)\,v_x) - \nu||v_x||^2 \tag{A.20}$$

We need estimates for both inner products. The first one may be integrated by parts to give,

$$(v, H(v)\,v) = -(u, H(v_x)\,v) - (u, H(v)\,v_x)$$

By using (A.4) and (A.2), we find

$$|(v, H(v)\,v)| \leq 2\,|u|_\infty\,||v||\,||v_x|| \tag{A.21}$$

By using (B.1), we find

$$2\,|u|_\infty\,||v||\,||v_x|| \leq 2\sqrt{2}\,|u|_\infty\,||u||^{1/2}\,||v_x||^{3/2} \tag{A.22}$$

Recall Young's inequality that states for $a, b > 0$,

$$a\,b \leq \frac{1}{p}a^p + \frac{1}{q}b^q \quad \text{with } \frac{1}{p} + \frac{1}{q} = 1 \tag{A.23}$$

Take $a = 2^6\nu^{-3/4}|u|_\infty\,||u||^{1/2}$ and $b = 2^{-3/2}\nu^{3/4}\,||v_x||^{3/2}$. Then, with $p = 4$ and $q = 4/3$, (A.21) becomes

$$|(v, H(v)\,v)| \leq \frac{2^{22}}{\nu^3}\,|u|_\infty^4\,||u||^2 + \frac{3\nu}{16}\,||v_x||^2 \tag{A.24}$$

The other inner product may be estimated by using (A.4), (B.3), and (B.1):

$$\begin{aligned} |(v, H(u)\,v_x)| &\leq |v|_\infty\,||u||\,||v_x|| \\ &\leq 2^{3/4}||u||^{5/4}\,||v_x||^{7/4} \end{aligned} \tag{A.25}$$

Take $a = 2^{5/2}\nu^{-7/8}||u||^{5/4}$, $b = 2^{-7/4}\nu^{7/8}||v_x||^{7/4}$, $p = 8$, and $q = 8/7$. Then, (A.23) applied to (A.25) gives

$$|(v, H(u)\,v_x)| \leq \frac{2^{17}}{\nu^7}\,||u||^{10} + \frac{7\nu}{32}\,||v_x||^2 \tag{A.26}$$

By combining (A.24) and (A.26) and substituting into (A.20), we are led to the differential inequality

$$\frac{d}{dt}||v||^2 \leq \frac{2^{23}}{\nu^3}\,|u|_\infty^4\,||u||^2 + \frac{2^{18}}{\nu^7}\,||u||^{10} - \nu||v_x||^2$$

From (A.5), we may write this differential inequality as

$$\frac{d}{dt}||v||^2 + \nu||v||^2 \leq \frac{2^{23}}{\nu^3}\,|u|_\infty^4\,||u||^2 + \frac{2^{18}}{\nu^7}\,||u||^{10}$$

We may integrate this differential inequality in a standard way to obtain

$$||v||^2 \leq \frac{2^{23}}{\nu^4}\,|u|_\infty^4\,||u||^2 + \frac{2^{18}}{\nu^8}\,||u||^{10} + ||f_x||^2 \qquad \text{(A.27)}$$

Now, we need an estimate for the L_2-norm of $w = u_{xx}$, which satisfies the equation,

$$w_t = H(w)v + 2H(v)w + H(u)w_x + \nu w_{xx} \qquad \text{(A.28)}$$

To proceed, we need estimates for several inner products. First, we write one of the inner products, after integrating by parts, as

$$\begin{aligned}|(w, H(w)v)| &\leq |(u, H(w_x)w)| + |(u, H(w)w_x)| \\ &\leq 2\,|u|_\infty\,||w||\,||w_x|| \\ &\leq 4\,|u|_\infty\,||u||^{1/3}\,||w_x||^{5/3}\end{aligned} \qquad \text{(A.29)}$$

where we use (A.4) in the second step, and (B.2) in the last step. The other two inner products may be combined into one through integration by parts;

$$2\,(w, H(v)w) + (w, H(u)w_x) = -3\,(w, H(u)w_x)$$

By using (A.4), (B.3), (B.1), and (B.2), we get

$$|(w, H(u)w_x)| \leq 2\,||u||^{7/6}||w_x||^{11/6} \qquad \text{(A.30)}$$

By multiplying (A.28) with w and integrating, we derive the differential inequality,

$$\frac{d}{dt}||w||^2 \leq 8\,|u|_\infty\,||u||^{1/3}||w_x||^{5/3} + 12\,||u||^{7/6}||w_x||^{11/6} - 2\nu||w_x||^2$$

By applying Young's inequality (A.23) to $8\,|u|_\infty\,||u||^{1/3}||w_x||^{5/3}$ with $a = 2^{23/6}$ $\nu^{-5/6}|u|_\infty||u||^{1/3}$, $b = 2^{-5/6}\nu^{5/6}||w_x||^{5/3}$, $p = 6$, and $q = 6/5$, and to $12\,||u||^{7/6}$ $||w_x||^{11/6}$ with $a = 3 \times 2^{35/12}\nu^{-11/12}||u||^{7/6}$, $b = 2^{-11/12}\nu^{11/12}||w_x||^{11/6}$, $p = 12$, and $q = 12/11$, we may rewrite this differential inequality as

$$\frac{d}{dt}||w||^2 \leq \frac{2^{22}}{3\nu^5}\,|u|_\infty^6\,||u||^2 + \frac{3^{11}2^{33}}{\nu^{11}}\,||u||^{14} - \nu||w_x||^2$$

$$\frac{d}{dt}||w||^2 + \nu||w||^2 \leq \frac{2^{22}}{3\nu^5}\,|u|_\infty^6\,||u||^2 + \frac{3^{11}2^{33}}{\nu^{11}}\,||u||^{14}$$

where the last step uses (A.5). By integrating this differential inequality, we are led to

$$||w||^2 \leq \frac{2^{22}}{3\nu^6}\,|u|_\infty^6\,||u||^2 + \frac{3^{11}2^{33}}{\nu^{12}}\,||u||^{14} + \,||f_{xx}||^2 \tag{A.31}$$

By combining (A.18), (A.27), and (A.31), and by using (A.6), (A.17), and (A.18), we find

$$||u||_{H^2} \leq K$$

where K depends on $1/\nu$ and $||f||_{H^2}$, but not on T. ∎

We are now in a position to prove the major result of this section:

Theorem 6 *Let the initial condition be C^∞ and 2π-periodic, and let $\nu > 0$. The equation (8) has a unique, 2π-periodic solution u on $[0, \infty)$, which is infinitely many times differentiable.*

Proof: Proof of existence follows directly from the arguments in Theorem 4.2.2 in [23]. We have only to show uniqueness. Let u and v be solutions of (8) that satisfy the same initial condition. Their difference $w = u - v$ satisfies

$$\begin{aligned} w_t &= H(w)u_x + H(v)w_x + \nu w_{xx}, \\ w(x,0) &= 0. \end{aligned}$$

The inner product $(w, H(w)u_x) = (u_x, H(w)w)$ may be bounded by (A.4). Since $(w, H(v)w_x) = -1/2\,(w, H(v_x)w)$, we may use (A.3), to obtain the bound $|H(v_x)|_\infty||w||^2$. These two estimates may be used to obtain the differential inequality,

$$\frac{d}{dt}||w||^2 \leq \Big(2|u_x|_\infty + |H(v_x)|_\infty\Big)||w||^2.$$

Gronwall-Bellman's inequality then implies that $w = 0$. ∎

Finally, we can show that all spatial and temporal derivatives of the solution to (8) remain bounded for all time provided $\nu > 0$. The proof follows the ideas already expressed; the details can be found in [33]. This result indicates that there is a limit to how distorted the solution can become.

B Some Sobolev Inequalities

We establish here some of the inequalities used in this paper.

Lemma 7 *Let u be a function that is 2π-periodic and C^∞. Then,*

$$||u_x||^2 \leq 2\,||u||\,||u_{xx}|| \tag{B.1}$$

Proof: From Lemma A.3.1 in [23],

$$||u_j||^2 \leq \epsilon\,||u_{j+k}||^2 + \epsilon^{-j/k}||u||^2$$

for $j, k \geq 1$ and $\epsilon > 0$. The result (B.1) follows directly by taking $j = k = 1$ and $\epsilon = ||u||/||u_2||$. ∎

Lemma 8 *Let u be a function that is 2π-periodic and C^∞. Then,*

$$||u_2|| \leq 2\,||u||^{1/3}\,||u_3||^{2/3} \tag{B.2}$$

Proof: Substitute u by u_x in (B.1). Then

$$\begin{aligned} ||u_2|| &\leq \sqrt{2}\,||u_x||^{1/2}\,||u_3||^{1/2} \\ &\leq 2^{3/4}\,||u||^{1/4}\,||u_2||^{1/4}\,||u_3||^{1/2} \end{aligned}$$

where we use (B.1) again in the last step. The inequality (B.2) then follows directly. ∎

Lemma 9 *Let u be a function that is 2π-periodic and C^∞. Then,*

$$|u|_\infty^2 \leq 2\,||u||\,||u_x|| \quad \textit{for } ||u|| \leq ||u_x|| \tag{B.3}$$

or

$$|u|_\infty^2 \leq 2\,||u||^2 \quad \textit{for } ||u|| > ||u_x|| \tag{B.4}$$

Proof: From Lemma A.3.4 in [23], we find, with $k = 1$ and $L = 2\pi$,

$$|u|_\infty^2 \leq \epsilon ||u_x||^2 + \frac{1}{\epsilon}||u||^2$$

where $0 < \epsilon \leq 1$. (It is easy to show that $C_1 < 1$). If $||u|| \leq ||u_x||$, set $\epsilon = ||u||/||u_x||$, and (B.3) follows. If $||u|| > ||u_x||$, set $\epsilon = 1$, and (B.4) follows.
■

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List of Figure Captions

1 A typical result for the fit of the parameters α and δ to the form (35). These results are at $t = 11.0$.

2 The temporal behavior of δ.

3 Solutions to (8) for two values of ν, shown at various times; $t = 2, 4, 6, \ldots, 30$ for $\nu = 0.1$ and $t = 3, 6, 9, \ldots, 60$ for $\nu = 0.0001$.

4 Solutions to (8) with $\nu = 0.0001$ and with asymmetric initial conditions, shown at times $t = 3, 6, 9, \ldots, 50$.

5 The behavior of q as determined by (47) as a function of x for $\nu = 1.6 \times 10^{-4}, 8.0 \times 10^{-5}, 4.0 \times 10^{-5}, 2.0 \times 10^{-5}, 1.0 \times 10^{-5}, 5.0 \times 10^{-6}$. Decreasing ν corresponds to decreasing profiles.

6 The asymptotic solution in the region close to the singularity.