DIFFERENTIAL TOPOLOGY

MATH 866-

COURSE'S PRESENTATION

I will discuss: -basic structures of geometry-topology, like differential manifold, Lie groups, vector bundles, smooth dynamics (vector fields), foliations, -the concept of " the general position = transversality", and (very likely) -the calculus with differential forms.

I will show: -how to use "calculus with differential forms" to define and calculate homology (the so called De Rham theory), -how to use homology to count the fixed and periodic points of continuous maps and the rest points of smooth vector fields (the so called Lefschetz fixed point theory and Hopf and Morse type results), -what vector bundles are good for and -how to use "tranvesality" and vector bundle to provide topological invariants.

I will show: -how to use what most of the students learned in several variable calculus (about counting maxima, minima and saddle points) to calculate homology of manifolds and to classify the surfaces (i.e. Morse theory) and other way around. -how to use transversality in order to: 1) derive some basic results of mathematics, like the fundamental theorem of algebra, Brower Fixed point theorem, etc. 2) explain and measure the linking and knotting in three and higher dimensions, 3) differentiate topological spaces and nonhomotopic maps and (may be), -why in dimension two any smooth manifold (surface)has only one smooth structure while in higher dimension, even the spheres can have many.

Background:

I expect the students be familiar with calculus and basic algebra and linear algebra. It will be very useful if they were exposed to the material discussed in elementary topology sequence 655-658.

The book of M. Hirsch (Differential Topology) or Guillemin-Polack (Differential Topology) are quite close from what I have in mind. If you are a beginner and want to prepare your mind (and spirit) for this course take a look at the beautiful little book of J. Milnor, Topology from differential point of view.

References;

1) Differential Topology by Victor Guillemin and Allan Polack- Pretinece Hall 1974

2) Introduction to Differential topology by T. Br/"ocker and K. J/"anich, Springer Verlag

3) Differential Topology by M.Hirsch Springer Verlag

4) J. Milnor Topology from differential point of view,

SYLLABUS:
a) Example of Differential manifolds
1. Calculus in R^n
- maximal rank theorem
-existence and unicity for solutions of ODE
- Frobenius theorem
- Sard Theorem
- Morse Lemma

2) The basic objects of Differential topology
-Differential manifolds, bundles, vector bundles
-diffeomorphisms, embeddings, immersions
-vector fields, foliations
-Lie groups

3) Transversality and Intersection Theory
- Intersection Theory,
- Transfer
- Lefschetz fixed point theory
- (Hopf) infinitesimal fixed point theory

4) Morse Theory (and surgery)
-the cell structure of a smooth manifold
-modifying a Morse function and surgery
-Morse inequalities
-classification of surfaces in dimension 2

5) Calculus with differential forms and integration theory
- De Rham theorem

HOME ASSIGNMENTS

ASSIGNMENT 1

1. Describe the following objects;

n Sphere
n-Torus
n dimensional projective space (over $\mathbb{R}$ and over $\mathbb{C}$)

The grasmannian of $k-$ planes in the n dimensional vector space $G_{n:k}$
The linear group GL_n (over $\mathbb{R}$ and over $\mathbb{C}$)
The orthogonal /unitary group $O(n)$ $U(n)$
Show that they are all topological manifolds, and any two of them homeomorphic.

2. Is "topological manifold with corners" a good concept ? Explain.

ASSIGNMENT 2

Problem 1:
Analyze the critical behavior of the following functions at the origin. Is the critical point nondegenerate ? Is it isolated? Is it a local maximum or local minimum ?
a) $f(x,y) = x^2 + 4y^2$
b) $f(x,y) = x^2 - 2xy + y^2$
c) $f(x,y) = x^2 + y^4$
d) $f(x,y) = x^2 + 11xy + y^2/2 + x^6$
e) $f(x,y) = 10xy + y^2 + 75y^3$

Problem 2:
Prove Morse Lemma for $U \subset \mathbb{R}^1$.

Problem 3:
Prove the Morse Lemma at then critical point $(0,0,0)$ for the function $f(x,y,z) = xy(1+z^2) + z^2(1+xy)$. i.e find the diffeomorphism $\varphi : (U,0) \to (U',0)$, U, U' open neighborhoods of $0 \in \mathbb{R}^3$ so that $f \cdot \varphi^{-1}(y_1, y_2, y_3)$ is sum of squares $\pm x^2 + \pm y^2 + \pm z^2$

Problem 4:
Formulate and prove (using inverse function theorem) the Implicit function Theorem.

Problem 5:
Consider $f : U \subset \mathbb{R}^n \to \mathbb{R}$ with the property that

$$\partial f/\partial x_i(x_1, 0, \cdots, 0) = 0$$

for any $x_1 \in \mathbb{R}$ and the matrix

$$||\partial^2/\partial x_i \partial x_j(x_1, 0, \cdots, 0)||, 2 \leq i, j \leq n$$

is nondegenerate for any $x_1 \in\in BbbR$.
Show that there exists a diffeomorphism $\varphi : (U,0) \to (U',0)$ so that

$$f\cdot\varphi^{-1}(y_1, y_2, \cdots y_n) = f(0) + \sum_{i\geq 2}\epsilon_i y_i^2$$

with $\epsilon_i = \pm 1$.
Characterize the number of negative ϵ_i.

ASSIGNMENT3

Let $v : J\times U \to \mathbb{R}^n$ be a smooth map of class C^r with J an open interval of $\mathbb{R}$ and U a connected open set of $\mathbb{R}^n$. A local flow at $(t_0, x_0)\in J\times U$ is a smooth map $\varphi : J_0\times U_0 \to U$ with J_0 a subinterval of J containing t_0 , U_0 an open subset of U containing x_0 so that

$$\frac{d\varphi}{dt}(t, x) = v(t, \varphi(t, x))$$
$$\varphi(t_0, x) = x$$

The main result of ODE states that given v as above for any (t_0, x_0) there exists local flows and any two such local flows agree on the common domain.

Problem 1: Suppose v is idependent of $t\in J$. Show that for any t_1, t_2

$$\varphi(t_1, \varphi(t_2, x)) = \varphi(t_1 + t_2, x)$$

whenever the formula makes sense.

Problem 2: (Invariance of domain).

1)Let $f : U \to \mathbb{R}^n$, U an open set of $\mathbb{R}^n$, be a map of class C^1 so that $Df(x) : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism for any $x\in U$. Show that $f(U)$ is an open set.

2) Denote by $\mathbb{R}^n_+ := \{(x_1, \cdots x_n)\in\mathbb{R}^n | x_1 \geq 0\}$ and let $U\subset\mathbb{R}^n_+$ be an open set of $\mathbb{R}^n_+$. Suppose that $f : U\to U$ is a diffeomorphism. Snow that $f(U\cap\mathbb{R}^n_+) = U\cap\mathbb{R}^n_+$

Problem 3: (maximal rank theorem). Let $U\subset\mathbb{R}^k\times\mathbb{R}^{(n-k)}$ open neighborhood of $O = (0, 0)\in\mathbb{R}^n$ and $F : (U, O)\to(\mathbb{R}^k\times\mathbb{R}^{(m-k)}, O')$, $O' = (0, 0)\in\mathbb{R}^m$, be a smooth map of class C^r. We write $F = (F^1, F^2)$ with $F^1 : (U, O) \to (\mathbb{R}^k, 0)$ $F^2 : (U, O)\to(\mathbb{R}^{(m-k)}, 0)$. and $(x, y)\in\mathbb{R}^k\times\mathbb{R}^{(n-k)}$for a point in $\mathbb{R}^n$.

1) Suppose that $D_xF^1(0, 0) : \mathbb{R}^k \to \mathbb{R}^k$ is an isomorphism. There exists a neighborhood of O, $U'\subset U$, and a diffeomorphism $\varphi : (U', O)\to \mathbb{R}^n, O)$ so that

$$F\cdot\varphi^{-1}(x, y) = (x, G(x, y)$$

with $G : \varphi(U')\to\mathbb{R}^{(m-k)}$ of class C^k.

2) If in addition $rank DF(x,y) = k$ for any $(x,y) \in U$ one can find as above

i) a neighborhood of O $U' \subset U$ and a diffeomorphism $\varphi : (U', O) \to \mathbb{R}^n, O)$ and in addition

ii) a neighborhood of O' $V' \subset \mathbb{R}^m$ and a diffeomorphism $\psi : (U', O) \to \mathbb{R}^m, O')$ so that

$$\psi \cdot F \cdot \varphi^{-1}(x,y) = (x,0)$$

Problem 4: Let $U \subset \mathbb{R}^n$ and $f : U \to U$ be a smooth mapof class C^r so that $f \cdot f = f$. Show that $f(U)$ is a smooth submanifold of $U \subset \mathbb{R}^n$.

ASSIGNMENT 5

Problem 1.

Let X, Y, S smooth manifolds, X compact, $Z \subset Y$ a smooth submanifold of Y, and $F : X \times S \to Y$ a smooth map.

For any $s \in S$ let $F_s : X \to Y$ denote the restriction of F to $X \times \{s\}$.

Suppose that F_{s_0} satisfies the property $\mathcal{P}$. Then there exists the open set $U \subset S$, $s_0 \in U$ so that for any $s \in U$ the map F_s satisfies $\mathcal{P}$.

$\mathcal{P}$ is one of the following properties:

a) immersion,

b) submersion,

c) embedding,

d) diffeomorphism

e) transversal to Z

Problem 2. *Let $Y \subset \mathbb{R}^N$ be a submanifold. Consider*

$$N(Y) := \{(y,v) \in Y \times \mathbb{R}^N | v \perp T_y(Y)\}$$

and let $h : N(Y) \to \mathbb{R}^N$ be the map defined by

$$h(y,v) := y + v$$

1)Show that $N(Y)$ is a smooth manifold, and h a smooth map.

For a smooth function $\epsilon : Y \to (0, \infty)$ denote by

$$N(Y)^\epsilon := \{(y,v) \in Y \times \mathbb{R}^N | v \perp T_y(Y), ||v|| < \epsilon(y)\}$$

2)Show that there exists a smooth function $\epsilon : Y \to (0,\infty)$ so that $h : N(Y)^\epsilon \to \mathbb{R}^N$ is a diffeomorphism onto the image.

Problem 3. *Suppose X and Y are compact smooth submanifolds of $\mathbb{R}^N$.*

1) If $dimX + dimY = N$ show that for almost any $a \in \mathbb{R}^N$ the submanifolds $X + a$ and Y intersect in finitely many points.

2) If $dimX + dimY < N$ for almost any $a \in \mathbb{R}^N$ then submanifolds $X + a$ and Y are disjoint

Problem 4. *Suppose X is a compact smooth manifolds and $f : M \to M$ a smooth map . Show that one can find $g : X \to X$ arbitrary close to f so that the set of fixed points of f is finite.*

Hint: Interpret the set of fixed points of f as the intersection of the subsets $graphf$ and Δ_M of $M \times M$, where $graphf := \{(x, f(x)) \in M \times M\}$ and $\Delta_M := \{(x, x) \in M \times M\}$

Assignment 6

Problem 1.

1) Let M be a connected manifold, TM its tanget bundle and $s : M \to TM$ be its zero section. The smooth map s is an embedding whose image will be denoted bu $s(M)$. Let $\Delta_M \subset M \times M$ be the diagonal, i.e the subset of points of $M \times M$ of the form (x, x). $\Delta_{\subset} M$. The diagonal is always a closed smooth submanifold of $M \times M$.. Show that there exists open sets $U \subset M \times M$ with $M \subset U$ so that (U, Δ_M) is diffeomorphic to (TM, M).

2) Show that $L(id_M)$, the Lefschetz number of the identity map is the same as $I(s, \Delta_M)$.

3) Show that the Lefschetz number of id_M is zero when M is a compact Lie group, for example $M = T^n$ or $M = S^3$ or $M = SO(n)$.

Problem 2. *Suppose $i : M^n \to \mathbb{R}^{n+1}$, M a compact manifold. Show that $\mathbb{R}^{n+1} \setminus M$ has exactly two connected components.*

Hint: For any point $z \in \mathbb{R}^{n+1} \setminus M$ denote by $W(i, z)$ the winding number of i at z.

1) Show that the function $z \rightsquigarrow W(i, z)$ is locally constant.

2) Show that there exists points $z \in \mathbb{R}^{n+1} \setminus M$ with $W(i, z) = 0$.

3) For $v \in S^n \equiv \{u \in \mathbb{R}^{n+1} | ||u|| = 1\}$ denote by $L(z, v) :\ = \{z + tv | t \in \mathbb{R}\}$ the line passing through z in the direction v. Show that:

a) For z fixed and any v in an open and dense subset of S^n the line $L(z, v)$ is transversal to $i(M)$ (therefore there are finitely many number of t so that $z + tv \in i(M)$.

b) If $L(z,v)$ is transversal to $i(M)$ and $z' \in L(z,v) \setminus i(M)$ then $W(i,z') = W(i,z)+l$ where l is the number of intersection points of $L(z,v)$ and $i(M)$ contained between z and z'.

4)There exists points $z \in \mathbb{R}^{n+1} \setminus M$ with $W(i,z) = 1$. Hence combyining with 1) and 2), $\mathbb{R}^{n+1} \setminus i(M)$ has at least two components

5)For $\epsilon > 0$ small, $M^{\epsilon} \setminus M$ has at most two components.

6) $\mathbb{R}^{n+1} \setminus i(M)$ has at most two components.

Problem3. *Show that there is no compact manifold of dimension ≥ 1 which is contractible.*

Problem4. *Show that at any given time there are two plces at the oposite ends of the earth where both the temperature and the atmospheric presure are the same.*