



Quantum lines over non-cocommutative cosemisimple Hopf algebras

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0. Introduction

The recognition of the importance of braided tensor categories came by the remarkable recent developments of quantum group theory. An especially important example of such a category is the braided tensor category ${}^H_H\mathcal{YD}$ of Yetter–Drinfeld modules over a Hopf algebra H with bijective antipode. It is interesting and delightful to see that old results on ordinary Hopf algebras, such as those by Nichols [18] (1978) and by Radford [19] (1985), are reproduced in terms of the new notion of braided Hopf algebra in ${}^H_H\mathcal{YD}$. Given a braided Hopf algebra R in ${}^H_H\mathcal{YD}$, one can construct an ordinary Hopf algebra $R \# H$ of biproduct; this naturally forms a triple $(R \# H, \iota, \pi)$ called a *Hopf algebra triple over H* , such that $\iota : H \rightarrow R \# H$ and $\pi : R \# H \rightarrow H$ are Hopf algebra maps with $\pi \circ \iota = \text{id}$, if we define $\iota(h) = 1 \# h$, $\pi(x \# h) = \varepsilon(x)h$. It was essentially proved by Radford [19] and then reproduced by Majid [13] that $R \mapsto R \# H$ gives a category equivalence, called ‘bosonisation,’ from the category of braided Hopf algebras in ${}^H_H\mathcal{YD}$ to the category of Hopf algebra triples over H ; see Proposition 1.1.

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Motivated by the work of Nichols, Andruskiewitsch and Schneider started their vigorous joint work on the classification of pointed Hopf algebras; see [2–4,6] (1998–2002). Roughly speaking, their method is first to classify the strictly graded, braided Hopf algebras R in ${}^kG\mathcal{YD}$, where kG is a group Hopf algebra over the ground field k , and then to *lift* each $R \# kG$ to possibly all ordinary Hopf algebras A such that $\text{gr } A \simeq R \# kG$. This is based on the following observation. If A is a pointed Hopf algebra, then the graded coalgebra $\text{gr } A = \bigoplus_{n \geq 0} A_n/A_{n-1}$ arising from the coradical filtration $A_0 \subseteq A_1 \subseteq A_2 \subseteq \dots$ of A forms a graded Hopf algebra, or in fact a *graded* Hopf algebra triple over $A_0 = kG(A)$. It follows by the graded version of the Radford–Majid bosonisation (see Proposition 1.2) that there exists uniquely a strictly graded, braided Hopf algebra R in ${}^kG\mathcal{YD}$, where $G = G(A)$, such that $\text{gr } A \simeq R \# kG$ as graded Hopf algebras over kG .

As is noted in [3], their method is valid to some extent, more generally, for Hopf algebras with the Chevalley property [1], that is, for those Hopf algebras A whose coradical $H = A_0$ is a (not necessarily cocommutative) Hopf subalgebra, since $\text{gr } A$ then forms a graded Hopf algebra which is of the form $R \# H$, where R is a strictly graded, braided Hopf algebra in ${}^H_H\mathcal{YD}$. This will be realized by this paper in the case where R is the simplest, special form R_q , which is defined as follows.

Let $N > 1$ be an integer, and let q be a primitive N th root of 1 in k . Let R_q denote the graded algebra $k[y]/(y^N)$ endowed with the (strictly graded) coalgebra structure such that $1, y, y^2/(2)_q!, \dots, y^{N-1}/(N-1)_q!$ form a divided power sequence; see Section 2. This is called a *quantum line* in [3]. Fix a Hopf algebra H with bijective antipode. It is proved in Proposition 3.4, slightly generalizing [3, Theorem 3.2], that if the characteristic of k does not divide N and if R is a braided Hopf algebra of dimension N in ${}^H_H\mathcal{YD}$ such that the coradical R_0 and the space $P(R)$ of primitives in R are both 1-dimensional, then R is of the form R_q as an algebra and a coalgebra. The following three problems rise from here.

- Classification problem: *Classify all possible biproducts $R_q \# H$ up to an isomorphism of graded or ungraded Hopf algebras.*
- Selfduality problem (for finite dimensional H): *Decide whether each $R_q \# H$ is selfdual or not.*
- Lifting problem (for cosemisimple H): *Classify all Hopf algebras A with coradical H such that $\text{gr } A$ is isomorphic to a given $R_q \# H$ as a graded Hopf algebra over H .*

Given a root $q (\neq 1)$ of 1 in k , we call (H, g, χ) a *Yetter–Drinfel’d (or YD) datum* for R_q , or simply for q , if H is a Hopf algebra with bijective antipode, g is a grouplike in H , and $\chi : H \rightarrow k$ is an algebra map such that $\chi(g) = q$ and $(h \rightharpoonup \chi)g = g(\chi \leftarrow h)$ for all $h \in H$; see Definition 2.1. It is essentially proved in [2] that any structure that makes R_q into a braided Hopf algebra in ${}^H_H\mathcal{YD}$ arises uniquely from a YD datum for R_q . We show that if H is cosemisimple (and semisimple), the first two problems above are equivalent to the corresponding problems for YD data; see Propositions 2.3 and 2.5. For the last problem, a result (Theorem 3.1) from [16] plays the important role which the Taft–Wilson theorem plays in the Andruskiewitsch–Schneider theory when H is a group Hopf algebra; see the proof of Proposition 3.1.

Let us suppose that $\text{ch } k \neq 2$ and $\sqrt{-1} \in k$. As our main results, we explicitly solve the three problems when H is each of the following non-cocommutative cosemisimple Hopf algebras

$$\widehat{\mathcal{D}}_{2n} \quad (n \geq 3), \quad \widehat{\mathcal{T}}_{4m} \quad (m \geq 2), \quad \mathcal{A}_{4m} \quad (m \geq 3), \quad \mathcal{B}_{4m} \quad (m \geq 2) \quad (1)$$

of finite dimension, $2n$ or $4m$, defined in [15]; see Theorems 4.1–4.4. If k contains a primitive $2n$ th or $4m$ th root of 1, then $\widehat{\mathcal{D}}_{2n} \simeq (kD_{2n})^*$ and $\widehat{\mathcal{T}}_{4m} \simeq (kT_{4m})^*$, where D_{2n} is the dihedral group, and T_{4m} is the dicyclic group. If k is algebraically closed \mathcal{B}_8 is the unique non-trivial (co)semisimple Hopf algebra of dimension 8, due to Kac and Paljutkin [11]. To solve the selfduality problem for $\mathcal{A}_{4m}, \mathcal{B}_{4m}$ we prove that some braidings on them that were discovered by Suzuki [22] are non-degenerate as bilinear forms; see Proposition 4.5.

Suppose k is an algebraically closed field of characteristic $\neq 2$. The results thus obtained classify especially those Hopf algebras of dimension 16 whose coradical is a non-cocommutative Hopf subalgebra of dimension 8; see Theorem 5.1. Combined with known results from [10], classifying pointed Hopf algebras of dimension 16, and from [12], classifying semisimple Hopf algebras of dimension 16, this completes the classification of the 16-dimensional Hopf algebras with the Chevalley property over an algebraically closed field of characteristic zero. Note that 16 is the smallest dimension of Hopf algebras that was not completely classified; see [8].

1. Preliminaries—the Radford–Majid bosonisation

Throughout we work over a fixed ground field k . Let H be a Hopf algebra. We assume that the antipode S of H is bijective. This necessarily holds true, either if H is finite-dimensional or if it is cosemisimple, or more generally if it has the Chevalley property. We say that H has *the Chevalley property* [1] if its coradical is a Hopf subalgebra. This is equivalent to the fact that the tensor product of any two semisimple H -comodules is semisimple as an H -comodule.

Let ${}^H_H\mathcal{YD}$ denote the category of Yetter–Drinfeld modules over H . An object of this category is a left H -module V with a left H -comodule structure $\delta: V \rightarrow H \otimes V$, $\delta(v) = \sum v_{-1} \otimes v_0$ such that $\delta(h \cdot v) = \sum h_1 v_{-1} S(h_3) \otimes h_2 \cdot v_0$, where $h \in H$, $v \in V$; see [3, Section 2] or [17, Section 10.6]. For two such objects V and W , the tensor product $V \otimes W$ is an object in the category, endowed with the diagonal action and the codiagonal coaction by H . ${}^H_H\mathcal{YD}$ forms a braided tensor category, endowed with the tensor product just given and the braiding

$$c: V \otimes W \xrightarrow{\sim} W \otimes V, \quad c(v \otimes w) = \sum v_{-1} \cdot w \otimes v_0.$$

Following M. Takeuchi, we call a triple (A, ι, π) a *Hopf algebra triple over H* , if A is a Hopf algebra, and $\iota: H \rightarrow A$ and $\pi: A \rightarrow H$ are Hopf algebra maps such that $\pi \circ \iota = \text{id}$, the identity map of H . Those triples form a category, in which a morphism $(A, \iota, \pi) \rightarrow (A', \iota', \pi')$ is a Hopf algebra map $f: A \rightarrow A'$ such that $f \circ \iota = \iota', \pi = \pi' \circ f$.

Let R be a braided Hopf algebra in the braided tensor category ${}^H_H\mathcal{YD}$. Thus it is in particular a left H -module algebra and left H -comodule coalgebra such that

$$\Delta(xy) = \sum x_1((x_2)_{-1} \cdot y_1) \otimes (x_2)_0 y_2,$$

where $x, y \in R$. Hence, $R \otimes H$ is made into an algebra by the smash product and a coalgebra by the smash coproduct. By [19, Theorem 1], this is in fact an ordinary Hopf algebra, denoted by $R \# H$, which further forms a Hopf algebra triple over H in the obvious way that $\iota(h) = 1 \# h$, $\pi(x \# h) = \varepsilon(x)h$.

Proposition 1.1 [19, Theorem 3]. $R \mapsto R \# H$ gives an equivalence, called ‘bosonisation,’ from the category of braided Hopf algebras in ${}^H_H\mathcal{YD}$ to the category of Hopf algebra triples over H .

A graded Yetter–Drinfeld module over H is an object $V = (V, \cdot, \delta)$ in ${}^H_H\mathcal{YD}$ which is at the same time a graded vector space $V = \bigoplus_{n \geq 0} V(n)$ such that $H \cdot V(n) \subset V(n)$, $\delta(V(n)) \subset H \otimes V(n)$ for each $n \geq 0$. We denote by $\text{Gr}_H^H\mathcal{YD}$ the category of graded Yetter–Drinfeld modules over H . This also forms a braided tensor category in the same way as above. In [3] and also in the Introduction above, braided Hopf algebras in $\text{Gr}_H^H\mathcal{YD}$ are called graded braided Hopf algebras in ${}^H_H\mathcal{YD}$.

A graded Hopf algebra over H is a pair (A, ι) of a graded Hopf algebra $A = \bigoplus_{n \geq 0} A(n)$ and an isomorphism $\iota: H \xrightarrow{\sim} A(0)$ of Hopf algebras. Notice then that the composite $\pi: A \rightarrow A(0) \xrightarrow{\sim} H$ of the natural projection with ι^{-1} is the unique graded Hopf algebra map, where H is supposed to be trivially graded so that $H(0) = H$, which makes (A, ι, π) into a (graded) Hopf algebra triple over H . It will often be the case that ι is the identity map. A map $(A, \iota) \rightarrow (A', \iota')$ of graded Hopf algebras over H is a graded Hopf algebra map $f: A \rightarrow A'$ such that $f \circ \iota = \iota'$.

If a braided Hopf algebra $R = \bigoplus_{n \geq 0} R(n)$ in $\text{Gr}_H^H\mathcal{YD}$ is irreducible as a coalgebra so that $R(0) = k1$, then $R \# H$ is a graded Hopf algebra over H with homogeneous component $(R \# H)(n) = R(n) \otimes H$.

Proposition 1.2. $R \mapsto R \# H$ gives an equivalence from the category of irreducible braided Hopf algebras in $\text{Gr}_H^H\mathcal{YD}$ to the category of graded Hopf algebras over H .

This can be proved in the same way as Proposition 1.1.

For a coalgebra C , we let $C_0 \subset C_1 \subset C_2 \subset \dots$ denote the coradical filtration of C , and let $\text{gr} C = \bigoplus_{n \geq 0} C_n/C_{n-1}$ ($C_{-1} = 0$) denote the associated graded coalgebra. In particular, C_0 denotes the coradical of C . If C is pointed irreducible (or $\dim C_0 = 1$), we let $P(C)$ denote the space of primitives in C .

Let A be a Hopf algebra with the Chevalley property, and write $H = A_0$. Then by [17, Lemma 5.2.8], $\text{gr} A$ forms a graded Hopf algebra over H . This is *coradically graded* (see [3, p. 669]) in the sense that $(\text{gr} A)_n = \bigoplus_{i=0}^n (\text{gr} A)(i)$ for $n = 0, 1$ and then necessarily for all $n \geq 0$. By Proposition 1.2, there exists a unique (up to isomorphism) irreducible braided Hopf algebra R in $\text{Gr}_H^H\mathcal{YD}$ such that $\text{gr} A \simeq R \# H$ as graded Hopf algebras over H ; this

is necessarily *strictly graded* [23, p. 232] in the sense that $R(0) = k1$, $R(1) = P(R)$, and then necessarily $R_n = \bigoplus_{i=0}^n R(i)$ for all $n \geq 0$.

Definition 1.3 [3, p. 659]. R is called the *diagram* of A .

2. The Yetter–Drinfel’d datum

Let $N > 1$ be an integer. Suppose k contains a primitive N th root q of 1. Let R_q denote the graded algebra $k[y]/(y^N)$ endowed with a coalgebra structure such that

$$d_0 = 1, \quad d_1 = y, \quad d_2 = \frac{y^2}{(2)_q!}, \quad \dots, \quad d_{N-1} = \frac{y^{N-1}}{(N-1)_q!}$$

form a divided power sequence in the sense that $\Delta(d_n) = \sum_{i=0}^n d_i \otimes d_{n-i}$ for $0 \leq n < N$. Here,

$$\binom{n}{q} = \frac{q^n - 1}{q - 1}, \quad (n)_q! = (n)_q(n-1)_q \cdots (1)_q.$$

The homogeneous component $R_q(n)$ is $ky^n (= kd_n)$ if $0 \leq n < N$, and is 0 if $n \geq N$. One sees that R_q is a strictly graded coalgebra; see Section 1.

Definition 2.1. A *Yetter–Drinfeld* (or *YD*) *datum* for R_q , or simply for q , is a triple (H, g, χ) which consists of a Hopf algebra H with bijective antipode, a grouplike g in H and an algebra map $\chi : H \rightarrow k$ such that $\chi(g) = q$ and

$$(\chi \rightharpoonup h)g = g(h \leftarrow \chi) \quad (h \in H). \tag{2}$$

Here \rightharpoonup (respectively \leftarrow) denotes the natural left (respectively right) H^* -module structure on H , so that $\chi \rightharpoonup h = \sum h_1 \chi(h_2)$. When q and H are fixed, we just say that (g, χ) is a *YD datum*.

Let (H, g, χ) be such a datum. Then the action (respectively the coaction) induced by

$$h \cdot y = \chi(h)y \quad (h \in H) \quad (\text{respectively } \delta(y) = g \otimes y)$$

well defines on R_q a structure of left H -module algebra and coalgebra (respectively of left H -comodule algebra and coalgebra). Thereby R_q is further a braided Hopf algebra in $\text{Gr}_H^H \mathcal{YD}$; this will be denoted by $R_q(H, g, \chi)$. It is essentially proved in [2, Remark following Lemma 8.1] that any structure that makes R_q into a braided Hopf algebra in $\text{Gr}_H^H \mathcal{YD}$ or even in ${}^H \mathcal{YD}$ arises uniquely from a YD datum for q . In [3], $R_q(H, g, \chi)$ is denoted by $\mathcal{R}(g; \chi)$, and is called a *quantum line*.

Suppose $R_q = R_q(H, g, \chi)$ as above. Then by Proposition 1.2 we have the graded Hopf algebra $R_q \# H$ over H , in which $y (= y \# 1)$ is $(1, g)$ -*primitive* in the sense that

$$\Delta(y) = g \otimes y + y \otimes 1. \quad (3)$$

As an algebra over H (or an H -ring), it is generated by y , and is defined by the relations

$$y^N = 0, \quad hy = \sum \chi(h_1)yh_2 \quad (h \in H).$$

One sees that any automorphism of R_q in $\text{Gr}_H^H \mathcal{YD}$ is of the form

$$\alpha \cdot : R_q \rightarrow R_q, \quad \alpha \cdot (y^n) = \alpha^n y^n \quad (0 \leq n < N), \quad (4)$$

where α is a non-zero scalar.

The YD data for q form a category, in which a morphism $(H, g, \chi) \rightarrow (H', g', \chi')$ is a Hopf algebra map $f: H \rightarrow H'$ such that $f(g) = g'$, $\chi = \chi' \circ f$. The assignment $(H, g, \chi) \mapsto R_q \# H$, where $R_q = R_q(H, g, \chi)$, gives a functor from the category just given to the category of graded Hopf algebras. Assigned to a map f as above is the map

$$F := \text{id} \otimes f : R_q \# H \rightarrow R_q \# H'$$

of graded Hopf algebras. This is *non-degenerate* in the sense that the image $\text{Im } F$ is not included in the 0-component H' of $R_q \# H'$.

Lemma 2.2. *Suppose*

$$R_q = R_q(H, g, \chi), \quad R_{q'} = R_{q'}(H', g', \chi'),$$

where (H, g, χ) , (H', g', χ') are YD data for some roots q, q' of 1, respectively. If $F: R_q \# H \rightarrow R_{q'} \# H'$ is a non-degenerate map of graded Hopf algebras, then $q = q'$ and $F = \alpha \cdot \otimes f$, where α is a non-zero scalar, and $f: (H, g, \chi) \rightarrow (H', g', \chi')$ is a map of YD data for q .

Proof. Since F preserves the 0-component, it restricts to a Hopf algebra map $f: H \rightarrow H'$. We see that the diagram

$$\begin{array}{ccc} R_q \# H & \longrightarrow & H \\ F \downarrow & & \downarrow f \\ R_{q'} \# H' & \longrightarrow & H' \end{array}$$

commutes, where the horizontal arrows are the natural projections.

Regard $R_q \# H$ as a right H -comodule algebra along the natural projection $R_q \# H \rightarrow H$. Then the subalgebra $(R_q \# H)^{coH}$ of H -coinvariants equals R_q . It follows that F is restricted to a graded algebra map $R_q \rightarrow R_{q'}$ between the subalgebras of coinvariants. Hence, $F(y) = \alpha y$ for some scalar, which is non-zero since F is non-degenerate. Since F is a Hopf algebra map, we see that $f(g) = g'$, $\chi = \chi' \circ f$, and necessarily $q = q'$, $F = \alpha \cdot \otimes f$. \square

Proposition 2.3. Let $R_q = R_q(H, g, \chi)$, $R_{q'} = R_{q'}(H', g', \chi')$ be as in Lemma 2.2. Then the conditions (i) and (ii) below are equivalent to each other.

- (i) $q = q'$ and $(H, g, \chi) \simeq (H', g', \chi')$;
- (ii) $R_q \# H \simeq R_{q'} \# H'$ as graded Hopf algebras.

These equivalent conditions imply

- (iii) $R_q \# H \simeq R_{q'} \# H'$ as ungraded Hopf algebras.

Conversely, (iii) implies (i) and (ii), if H and H' are both cosemisimple.

Proof. The equivalence (i) \Leftrightarrow (ii) follows immediately from Lemma 2.2. Obviously, (ii) implies (iii). Notice that if H is cosemisimple, $R_q \# H$ is coradically graded (see Section 1) so that $\text{gr}(R_q \# H) = R_q \# H$. By applying gr to such an isomorphism as in (iii), we obtain an isomorphism as in (ii). \square

Let (H, g, χ) be a YD datum for q . Suppose H is finite-dimensional. Then χ is a grouplike in the dual Hopf algebra H^* of H , and g is regarded as an algebra map $H^* \rightarrow k$.

Lemma 2.4. (1) (H^*, χ, g) is a YD datum for q .

(2) Write

$$R_q = R_q(H, g, \chi), \quad R_q^* = R_q(H^*, \chi, g).$$

Then, $(R_q \# H)^* \simeq R_q^* \# H^*$ as graded Hopf algebras.

Proof. (1) This follows since one sees that Condition (2) in Definition 2.1 is equivalent to that $(g \rightarrow \varphi)\chi = \chi(\varphi \leftarrow g)$ for any $\varphi \in H^*$, where \rightarrow (respectively \leftarrow) denotes the natural left (respectively right) H -module structure on H^* .

(2) It is easy to see that the pairing $\langle \cdot, \cdot \rangle : R_q \times R_q \rightarrow k$ given by

$$\langle y^n, y^l \rangle = \delta_{n,l}(n)_q! \quad (0 \leq n, l < N)$$

induces an isomorphism of graded algebras and coalgebras from R_q onto its dual. We see that through this duality, the action and the coaction

$$\begin{aligned} H \otimes R_q &\rightarrow R_q, & h \otimes y &\mapsto \chi(h)y, \\ R_q &\rightarrow H \otimes R_q, & y &\mapsto g \otimes y \end{aligned}$$

are dualized so that

$$\begin{aligned} H^* \otimes R_q^* &\leftarrow R_q^*, & \chi \otimes y &\leftarrow y, \\ R_q^* &\leftarrow H^* \otimes R_q^*, & \varphi(g)y &\leftarrow \varphi \otimes y. \end{aligned}$$

This proves Part (2). \square

Proposition 2.5. *Let (H, g, χ) and R_q be as above. Then the conditions (i) and (ii) below are equivalent to each other.*

- (i) (H, g, χ) is selfdual in the sense that $(H, g, \chi) \simeq (H^*, \chi, g)$;
- (ii) $R_q \# H$ is selfdual as a graded Hopf algebra.

These equivalent conditions imply

- (iii) $R_q \# H$ is selfdual as an ungraded Hopf algebra.

Conversely, (iii) implies (i) and (ii), if H is semisimple and cosemisimple.

Proof. This follows from Proposition 2.3 and Lemma 2.4. \square

3. Hopf algebras with diagram $R_q(H, g, \chi)$

Let A be a Hopf algebra with the Chevalley property, and write $H = A_0$.

Proposition 3.1. *Suppose the diagram of A is $R_q(H, g, \chi)$, where (H, g, χ) is a YD datum for some root q of 1.*

- (1) *We have an isomorphism $\text{gr } A \simeq R_q \# H$ of graded Hopf algebras over H , which is unique up to automorphism, $\alpha \cdot \text{id}$ (see (4)), of $R_q \# H$. We shall identify $\text{gr } A$ with $R_q \# H$ through such an isomorphism.*
- (2) *There exists a $(1, g)$ -primitive z in A (see (3)) such that*
 - (i) $gzg^{-1} = qz$,
 - (ii) A is generated by z and H , and
 - (iii) z is mapped to the canonical generator y in R_q through the natural projection $A_1 \rightarrow \text{gr } A(1)$.

Part (1) follows by Proposition 1.2 and Definition 1.3. To prove Part (2), we need a result from [16]; it will play the role which the Taft–Wilson theorem [17, Theorem 5.4.1] plays in [3] when H is a group algebra.

Let A, H be as in the beginning of this section. Set $Q = A/AH^+$; this is a quotient left A -module coalgebra of A . It is proved in [16, Theorem 3.1] that there is a right H -linear coalgebra map $\gamma : A \rightarrow H$ whose restriction $\gamma|_H$ to H is the identity map. Hence we have the isomorphism

$$A \simeq Q \triangleright \triangleleft H, \quad a \mapsto \sum \bar{a}_1 \otimes \gamma(a_2) \quad (5)$$

of right H -module coalgebras, where \bar{a} denotes the natural image of $a \in A$ in Q . $Q \triangleright \blacktriangleleft H$ denotes the smash coproduct arising from the left H -comodule coalgebra structure on Q

$$\lambda : Q \rightarrow H \otimes Q, \quad \lambda(\bar{a}) = \sum \gamma(a_1)S(\gamma(a_3)) \otimes \bar{a}_2.$$

One sees that the coradical filtration of $Q \triangleright \blacktriangleleft H$ is given by $(H =) Q_0 \otimes H \subset Q_1 \otimes H \subset Q_2 \otimes H \subset \dots$, and λ preserves the filtration. It follows that the diagram R of A is, as a graded left H -comodule coalgebra, identified with $\text{gr } Q$ whose comodule structure is given by $\text{gr } \lambda : \text{gr } Q \rightarrow H \otimes \text{gr } Q$. Moreover, (5) induces an isomorphism $\text{gr } A \simeq R \# H$ of graded Hopf algebras over H .

Proof of Proposition 3.1(2). We identify A with $Q \triangleright \blacktriangleleft H$ through (5). Suppose $R = R_q(H, g, \chi)$. Since $P(Q) \simeq P(\text{gr } Q) = P(R)$, $P(Q)$ is spanned by a primitive z which is mapped to y . Since $P(Q)$ is H -costable, $\lambda(z) = g' \otimes z$ for some grouplike g' in H . Since then $(\text{gr } \lambda)(y) = g' \otimes y$, we must have $g' = g$, and so $z (= z \otimes 1)$ is a $(1, g)$ -primitive in A .

We have $A_1 = H \oplus zH$. It is then straightforward to see that the space $P_{1,g}(A)$ of $(1, g)$ -primitives in A (see (3)) is given by

$$P_{1,g}(A) = kz \oplus k(1 - g). \tag{6}$$

This is stable under the g -conjugation $u \mapsto gug^{-1}$. Since $gyg^{-1} = qy$, we have $gzg^{-1} = qz + \alpha(1 - g)$ for some scalar α . Replacing z with $z + \alpha(q - 1)^{-1}(1 - g)$, we have $gzg^{-1} = qz$. We have obtained a $(1, g)$ -primitive z satisfying (i), (iii). Condition (ii) necessarily follows from (iii); see [3, Lemma 2.2]. \square

Remark 3.2. Let A be a Hopf algebra, and let $\{C_\tau\}_\tau$ denote the set of all simple sub-coalgebras in A , so that $A_0 = \bigoplus_\tau C_\tau$. Andruskiewitsch and Schneider [5, Conjecture 4.1] conjecture that

$$A_1 = A_0 + \sum_\tau (C_\tau \wedge k1)A_0 = A_0 + \sum_{\tau, \mu} C_\tau C_\mu \wedge C_\mu.$$

Suppose A has the Chevalley property. Then the conjecture holds true, since by the isomorphism (5), the proof of [5, Lemma 4.2] is valid; the lemma proves that the conjecture holds true for $\text{gr } A$.

Next we will slightly generalize [3, Theorem 3.2]; it characterizes $R_q(H, g, \chi)$, assuming that H is a finite-dimensional semisimple Hopf algebra in characteristic zero.

Let H be a Hopf algebra with bijective antipode. The structure of any object in ${}^H_H\mathcal{YD}$ or in $\text{Gr } {}^H_H\mathcal{YD}$ will be denoted by \cdot, δ .

Lemma 3.3 (cf. [3, Lemma 3.1 (i)]). *Let R be a finite-dimensional braided Hopf algebra in ${}^H_H\mathcal{YD}$ such that the dimension $\dim R$ is not divided by the characteristic $\text{ch } k$ of k : $\text{ch } k \nmid \dim R$. If z is a primitive in R such that $g \cdot z = z$, $\delta(z) = g \otimes z$ for some grouplike g in H , then $z = 0$.*

Proof. Suppose z satisfies the assumptions above. It follows by [21, Theorem 2.2] that R is free over the braided Hopf subalgebra $k[z]$ generated by z , and so that $\text{ch } k \nmid \dim k[z]$. The ordinary Hopf subalgebra $k[z, g]$ in $R \# H$ which is generated by z and g is commutative since $g \cdot z = z$. Hence, $k[z] = k[z, g]/(g - 1)$; this is a finite-dimensional, ordinary Hopf algebra generated by a primitive z . If $\text{ch } k = 0$, we must have $k[z] = k$, and so $z = 0$. If $\text{ch } k = p > 0$, then $\dim k[z] = p^r$ for some $r \geq 0$. Since $\text{ch } k \nmid \dim k[z]$, we have $r = 0$, and so $z = 0$. \square

Proposition 3.4 (cf. [3, Theorem 3.2]). *Let H be a Hopf algebra with bijective antipode. Let R be a finite-dimensional braided Hopf algebra in ${}^H_H\mathcal{YD}$ of dimension $N = \dim R$ such that $\text{ch } k \nmid N$. Suppose $\dim R_0 = \dim P(R) = 1$. Then R is of the form $R_q(H, g, \chi)$, where (H, g, χ) is a YD datum for a primitive N th root of 1.*

Proof. Suppose $P(R) = kz$. Since $P(R)$ is H -stable and H -costable, there exist a grouplike g in H and an algebra map $\chi : H \rightarrow k$ such that

$$h \cdot z = \chi(h)z \quad (h \in H), \quad \delta(z) = g \otimes z.$$

See [3, p. 672, lines 1–4]. Write $q = \chi(g)$. Then, $g \cdot z = qz$.

Write $S = \bigoplus_{n \geq 0} S(n)$ for $\text{gr } R$, so that $S(0) = k1$, $S(1) = P(R) = kz$. Since S is strictly graded, we have a divided power sequence of non-zero elements in S ,

$$d_0 = 1, \quad d_1 = z, \quad d_2, \dots, \quad d_{N-1},$$

such that $S(n) = kd_n$ if $n < N$, and $S(n) = 0$ if $n \geq N$. Since one sees that $H \cdot R_n \subset R_n$, $\delta(R_n) \subset H \otimes R_n$ for each n , S is an object in $\text{Gr}_H^H\mathcal{YD}$, and is in fact a braided Hopf algebra in $\text{Gr}_H^H\mathcal{YD}$; see the proof of [17, Lemma 5.2.8]. We have

$$g \cdot d_n = q^n d_n, \quad \delta(d_n) = g^n \otimes d_n \quad (0 \leq n < N)$$

in S . Notice $z^N = 0$ in S . Let $m (\leq N)$ be the minimal positive integer such that $z^m = 0$ in S . Then, $S(n) = kz^n$ if $n < m$. By the q -binomial formula [2, p. 446] in S , and also in R

$$\Delta(z^n) = \sum_{i=0}^n \binom{n}{i}_q z^i \otimes z^{n-i}, \quad (7)$$

we see that $\binom{m}{1}_q = 0$. It follows that $q^m = 1$, since $q \neq 1$ by Lemma 3.3.

To see $m = N$, suppose on the contrary that $m < N$. Then the ideal $(z) \subset S$ generated by z is a braided Hopf ideal such that $d_m \notin (z)$. In the quotient braided Hopf algebra $\bar{S} := S/(z)$ in $\text{Gr}_H^H\mathcal{YD}$, the natural image \bar{d}_m of d_m is a non-zero primitive such that

$$g^m \cdot \bar{d}_m = q^{m^2} \bar{d}_m = \bar{d}_m, \quad \delta(\bar{d}_m) = g^m \otimes \bar{d}_m.$$

This contradicts Lemma 3.3, since the dual result of [21, Theorem 2.2] proves that $\dim \bar{S} \mid N$, and so $\text{ch } k \nmid \dim \bar{S}$. Therefore, $m = N$. It follows that $q^N = 1$, and z generates S and hence R .

Moreover, $q^n \neq 1$ if $0 < n < N$. Indeed, if q were a primitive n th root of 1 with $0 < n < N$, then we would have by (7) a primitive, z^n , in R which contradicts Lemma 3.3. Then it follows by the same reason that $z^N = 0$ in R . Now we see $R = R_q(H, g, \chi)$. \square

Remark 3.5. Suppose $\text{ch } k = p > 0$. One sees that $R = k[z]/(z^p - z)$ in ${}^k_k\mathcal{YD}$ gives an example which shows that Proposition 3.4 does not necessarily hold true without the assumption that $\text{ch } k \nmid \dim R$.

Corollary 3.6. Let A be a finite-dimensional Hopf algebra with the Chevalley property. Write $H = A_0$. Suppose that $\text{ch } k$ does not divide the index $\dim A / \dim H$ of H in A . Then the diagram R of A is of the form $R_q(H, g, \chi)$ if and only if $\dim A_1 / \dim H = 2$.

Proof. We have $\dim R_0 = 1$. Let $Q = A/AH^+$ as before. Then, $\dim A / \dim H = \dim Q = \dim R$. Moreover, $\dim A_1 / \dim H = 2$ if and only if $P(Q) (\simeq P(R))$ is 1-dimensional. The corollary now follows from Proposition 3.4. \square

4. Solution to the three problems for $\widehat{\mathcal{D}}_{2n}, \widehat{\mathcal{T}}_{4m}, \mathcal{A}_{4m}$ and \mathcal{B}_{4m}

Throughout in this section we suppose that $\text{ch } k \neq 2$ and $\sqrt{-1} \in k$. We will often use the notation in [15].

Recall the three problems raised in the Introduction. We will solve them for the finite-dimensional cosemisimple Hopf algebras

$$\widehat{\mathcal{D}}_{2n} \quad (n \geq 3), \quad \widehat{\mathcal{T}}_{4m} \quad (m \geq 2), \quad \mathcal{A}_{4m} \quad (m \geq 3), \quad \mathcal{B}_{4m} \quad (m \geq 2).$$

For the second problem we will put on k some additional assumption under which H is necessarily semisimple. Then by Propositions 2.3 and 2.5, the first two problems reduce to the classification and the selfduality of YD data for possible q . We know from [15, Proposition 3.6] that the group $G(H)$ of grouplikes in H is isomorphic to $\mathbb{Z}_2, \mathbb{Z}_2 \oplus \mathbb{Z}_2$ or \mathbb{Z}_4 , where \mathbb{Z}_r denotes the cyclic group of order r . Therefore the possibility of q is restricted to the case $q = -1$ or $\pm\sqrt{-1}$ if $G(H) \simeq \mathbb{Z}_4$, and to the case $q = -1$ otherwise.

Let $K = k\langle a \rangle$ denote the group Hopf algebra generated by a grouplike a of order 2. This is identified, through a unique isomorphism, to the dual Hopf algebra H^* , in which we let e_0, e_1 denote the dual basis of $1, a$, so that $e_0 = (1/2)(1 + a), e_1 = (1/2)(1 - a)$. The Hopf algebras listed in (1) all include K as a central Hopf subalgebra. To describe them below, we will give generators together with relations as algebra over K , and define by their values the coalgebra structure Δ, ε and the antipode S .

To recall first the definitions of $\widehat{\mathcal{D}}_{2n}, \widehat{\mathcal{T}}_{4m}$, let $n \geq 3$ be an integer. By [15, Definition 3.1], $\widehat{\mathcal{D}}_{2n}$ is the commutative Hopf algebra including K which is generated by an element x over K , and is defined by the relation

$$x^n = 1$$

together with the structure

$$\Delta(x) = x \otimes e_0x + x^{-1} \otimes e_1x, \quad \varepsilon(x) = 1, \quad S(x) = e_0x^{-1} + e_1x. \quad (8)$$

If n is even, we write $n = 2m$ with $m \geq 2$, and \widehat{T}_{4m} is then the commutative Hopf algebra defined in the same way as above except that (8) is replaced by

$$\Delta(x) = x \otimes e_0x + ax^{-1} \otimes e_1x, \quad \varepsilon(x) = 1, \quad S(x) = e_0x^{-1} - e_1x.$$

Thus, $\widehat{\mathcal{D}}_{4m} = \widehat{T}_{4m}$ as algebras. If k contains a primitive n th root of 1, then

$$\widehat{\mathcal{D}}_{2n} \simeq (kD_{2n})^*, \quad \widehat{T}_{4m} \simeq (kT_{4m})^*,$$

where D_{2n} is the dihedral group of order $2n$, and T_{4m} is the dicyclic (or binary dihedral) group of order $4m$. In particular, $T_8 = Q_8$, the quaternion group.

By [15, Proposition 3.6], we have

$$G(\widehat{\mathcal{D}}_{2n}) = \begin{cases} \langle a \rangle \simeq \mathbb{Z}_2 & \text{if } n \text{ is odd,} \\ \langle a, x^m \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } n = 2m \text{ is even,} \end{cases}$$

$$G(\widehat{T}_{4m}) = \begin{cases} \langle (e_0 + \sqrt{-1}e_1)x^m \rangle \simeq \mathbb{Z}_4 & \text{if } m \text{ is odd,} \\ \langle a, x^m \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } m \text{ is even.} \end{cases}$$

The Hopf algebras cannot be selfdual since they are commutative, but not cocommutative. Hence the selfduality problem is out of the question for them.

Theorem 4.1. *Suppose $H = \widehat{\mathcal{D}}_{2n}$ ($n \geq 3$) and $q = -1$.*

- (1) *If n is odd or $n \equiv 0 \pmod{4}$, there exists no YD datum.*
- (2) *Suppose $n = 2m \equiv 2 \pmod{4}$.*
 - (a) *There exist precisely two YD data*

$$(x^m, \chi_1), \quad (ax^m, \chi_1),$$

where χ_1 is the algebra map $H \rightarrow k$ defined by

$$\chi_1(a) = 1, \quad \chi_1(x) = -1. \quad (9)$$

These two data are isomorphic to each other.

- (b) *Let $R_q = R_q(H, g, \chi_1)$, where $g = x^m$ or ax^m . Then $R_q \# H$ cannot be lifted non-trivially in the sense that if a Hopf algebra A with coradical H has R_q as its diagram, then $A \simeq R_q \# H$.*

Proof. To prove (1) and (2)(a), suppose (g, χ) is a YD datum. For Condition (2) in Definition 2.1, we may suppose h is an element of the standard generators. But, a can

be excluded since it is a central grouplike. Since H is now commutative, g can be canceled from Condition (2), which is thus equivalent to

$$\chi(e_0x)x + \chi(e_1x)x^{-1} = \chi(x)e_0x + \chi(x)^{-1}e_1x.$$

Since x and x^{-1} are K -linearly independent, it follows that $\chi(e_1x) = 0$, $\chi(e_0x) = \chi(x) = \chi(x)^{-1}$. This implies that $\chi(a) = 1$, $\chi(x) = -1$. It is now easy to see that the possible YD data are given as above. The two YD data are isomorphic to each other through the automorphism of H defined by $a \leftrightarrow a$, $x \leftrightarrow ax$.

To prove (2)(b), suppose a Hopf algebra A with coradical H has the diagram $R_q(H, g, \chi_1)$, where $g = x^m$. Then we have a $(1, g)$ -primitive z as in Proposition 3.1(2). In particular, $gzg^{-1} = -z$. Since g and a commute, kz , an eigenspace of the g -conjugation on $P_{1,g}(A)$ (see (6)), is stable under the a -conjugation. It follows that $aza^{-1} = z$, since $aya^{-1} = y$. By computing squares in $\Delta(z) = g \otimes z + z \otimes 1$, we see that z^2 is a primitive in H , and hence $z^2 = 0$.

Since $xy + yx = 0$ we have that $xz + zx \in H$, so $xzx^{-1} + z \in H$. But H is commutative, so $xzx^{-1} + z = x^{-1}(xzx^{-1} + z)x = z + x^{-1}zx$, showing that $x^{-1}zx = xzx^{-1}$. Using this, an easy computation shows that $\Delta(xzx^{-1}) = g \otimes xzx^{-1} + xzx^{-1} \otimes 1$. Hence $xzx^{-1} + z \in P_{1,g}(H) = k(g - 1)$. Write $xzx^{-1} + z = \alpha(g - 1)$ for some $\alpha \in k$. Then $xzx^{-1} = \alpha z(g - 1) = -\alpha(g + 1)z$ and $xzx^{-1}z = \alpha(g - 1)z$, and then if we square the relation $xzx^{-1} + z = \alpha(g - 1)$ we find $-2\alpha z = 2\alpha^2(1 - g)$, showing that $\alpha = 0$. We conclude that $xz = -zx$, which ends the proof. \square

Theorem 4.2. *Suppose $H = \widehat{T}_{4m}$ ($m \geq 2$).*

- (1) *If m is even and $q = -1$, or if m is odd and $q = \pm\sqrt{-1}$, then there exists no YD datum.*
- (2) *Suppose m is odd and $q = -1$.*
 - (a) *There exist precisely two YD data*

$$((e_0 + \sqrt{-1}e_1)x^m, \chi_1), \quad ((e_0 - \sqrt{-1}e_1)x^m, \chi_1),$$

where χ_1 is the same algebra map as above, defined by (9). These two data are isomorphic to each other.

- (b) *Let $g = (e_0 + \sqrt{-1}e_1)x^m$. For any scalar $\alpha \in k$ denote by $A(\alpha)$ the Hopf algebra with generators a, x, z such that a, x satisfy the same algebra and coalgebra relations as in \widehat{T}_{4m} , and*

$$az = za, \quad xz = -zx, \quad z^2 = \alpha(a - 1),$$

$$\Delta(z) = g \otimes z + z \otimes 1, \quad \varepsilon(z) = 0.$$

If a Hopf algebra A with coradical H has the diagram $R_q(H, g, \chi_1)$, then there exists $\alpha \in k$ such that $A \simeq A(\alpha)$. Moreover $A(\alpha) \simeq A(\alpha')$ if and only if either $\alpha = \alpha' = 0$ or $\alpha, \alpha' \neq 0$ and α'/α is a square in k . Thus the non-trivial liftings are classified by the factor group $k^/(k^*)^2$.*

Proof. Parts (1) and (2)(a) can be proved in the same way as the corresponding parts of Theorem 4.1. For (2)(b), we first note that it is easy to see directly that $A(\alpha)$ is a well defined Hopf algebra. Let A be a Hopf algebra with coradical H and diagram R_q . As in Theorem 4.1(2)(b), there exists $z \in P_{1,g}(A)$, whose projection in $\text{gr } A$ is the element $y \in R_q$, such that $gz = -zg$, $az = za$, and also $xzx^{-1} + z \in P_{1,g}(H)$. Then $xzx^{-1} + z = \gamma(g - 1)$ for some $\gamma \in k$. On the other hand $\Delta(z^2) = g^2 \otimes z^2 + z^2 \otimes 1$ and $z^2 \in H$, showing that $z^2 = \alpha(a - 1)$ for some $\alpha \in k$. If we use these relations when we square $xzx^{-1} + z = \gamma(g - 1)$, we find $-2\gamma z = \gamma^2(a - 2g + 1)$, showing that $\gamma = 0$. Thus we have $A \simeq A(\alpha)$.

For the last statement, suppose in general that A and z are as in Proposition 3.1. Since $A_1 = H \oplus zH$ as showed in the proof of Proposition 3.1(2), we see that if $w \in A - H$ is a $(1, g')$ -primitive for some grouplike g' in H , then $g' = g$ and $w \in kz \oplus k(1 - g)$; see (6). If in addition $gwg^{-1} = qw$, then $w \in kz$. If $f: A(\alpha) \rightarrow A(\alpha')$ is a Hopf algebra isomorphism, we can apply the result to $w = f(z)$ and $g' = f(g)$, to see that $f(z) = \beta z'$ for some non-zero scalar β . By applying f to the relation $z^2 = \alpha(a - 1)$ we find that $\alpha = \alpha' \beta^2$. For the other way around one can construct directly an isomorphism. \square

To recall the definitions of $\mathcal{A}_{4m}, \mathcal{B}_{4m}$, let $m \geq 2$ be an integer. By [15, Definition 3.3], \mathcal{A}_{4m} is the Hopf algebra including K as a central Hopf subalgebra which is generated by two elements s_+, s_- over K , and is defined by the relations

$$s_{\pm}^2 = 1, \quad (s_+s_-)^m = 1 \tag{10}$$

together with the structure

$$\Delta(s_{\pm}) = s_{\pm} \otimes e_0s_{\pm} + s_{\mp} \otimes e_1s_{\pm}, \quad \varepsilon(s_{\pm}) = 1, \quad S(s_{\pm}) = e_0s_{\pm} + e_1s_{\mp}.$$

\mathcal{B}_{4m} is the Hopf algebra defined in the same way as above except that (10) is replaced by

$$s_{\pm}^2 = 1, \quad (s_+s_-)^m = a.$$

Thus, $\mathcal{A}_{4m} = \mathcal{B}_{4m}$ as coalgebras. By [15, Remark 3.4], $\mathcal{A}_8 \simeq \widehat{\mathcal{D}}_8 \simeq (kD_8)^*$. If k is algebraically closed, \mathcal{B}_8 is the unique (co)semisimple Hopf algebra of dimension 8, due to Kac and Paljutkin [11], which is neither commutative nor cocommutative; see also Section 5.

In both \mathcal{A}_{4m} and \mathcal{B}_{4m} we write

$$s_+(i) = \underset{\longleftarrow i \longrightarrow}{s_+s_-s_+s_- \cdots}, \quad s_-(i) = \underset{\longleftarrow i \longrightarrow}{s_-s_+s_-s_+ \cdots},$$

for any $i > 0$. By [15, Proposition 3.6], we have

$$G(\mathcal{A}_{4m}) = \langle a, s_+(m) \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2,$$

$$G(\mathcal{B}_{4m}) = \begin{cases} \langle (e_0 + \sqrt{-1}e_1)s_+(m) \rangle \simeq \mathbb{Z}_4 & \text{if } m \text{ is odd,} \\ \langle a, (e_0 + \sqrt{-1}e_1)s_+(m) \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 & \text{if } m \text{ is even.} \end{cases}$$

By the proof of [15, Proposition 3.6], for $H = \mathcal{A}_{4m}$ or $H = \mathcal{B}_{4m}$ we have a decomposition

$$H = kG(H) \oplus \left(\bigoplus_{i=1}^{m-1} C_i \right) \tag{11}$$

as a direct sum of subcoalgebras, where for any $1 \leq i \leq m - 1$ we have that $C_i := Ks_+(i) \oplus Ks_-(i)$ is a 4-dimensional simple subcoalgebra with comatrix basis $\begin{pmatrix} e_0s_+(i) & e_1s_-(i) \\ e_1s_+(i) & e_0s_-(i) \end{pmatrix}$. We also recall from [15] that

$$\{a^i(s_-)^j(s_+s_-)^r \mid 0 \leq i, j \leq 1, 0 \leq r \leq m - 1\} \tag{12}$$

is a basis of H . In particular H is free as a left K -module.

Theorem 4.3. *Suppose $H = \mathcal{A}_{4m}$ ($m \geq 3$) and $q = -1$.*

- (1) *If m is even, there exists no YD datum.*
- (2) *Suppose m is odd.*
 - (a) *There exist precisely two YD data*

$$(s_+(m), \chi_2), \quad (as_+(m), \chi_3),$$

where χ_2 and χ_3 are defined by

$$\chi_2(a) = -1, \quad \chi_2(s_{\pm}) = 1, \quad \chi_3(a) = \chi_3(s_{\pm}) = -1.$$

These two data are isomorphic to each other.

- (b) *If k contains a primitive m th root of 1, then the two YD data are both selfdual.*
- (c) *Let $R_q = R_q(H, s_+(m), \chi_2)$. The graded Hopf algebras $R_q \# H$ arising from the YD data cannot be lifted non-trivially.*

Proof. To prove (1) and (2)(a), suppose (g, χ) is a YD datum. Suppose g is central in H , so that g can be canceled from (2) in Definition 2.1. Take $h = s_+$ in (2). Then we have

$$\chi(e_0s_+)s_+ + \chi(e_1s_+)s_- = \chi(s_+)e_0s_+ + \chi(s_-)e_1s_+,$$

which implies that $\chi(a) = 1$, $\chi(s_+) = \chi(s_-) = -1$, since s_+ and s_- are K -linearly independent. We see $g \neq a$, since $\chi(g) = -1$.

- (1) Suppose m is even. Then any grouplike is central. But, $\chi(s_+(m)) = \chi(as_+(m)) = 1$. This proves Part (1).
- (2) Suppose m is odd.
 - (a) We see $g = s_+(m)$ or $as_+(m)$. Take $h = s_{\pm}$ in (2). Since $s_{\pm}g = gs_{\mp}$, we then have

$$\chi(e_0s_{\pm})s_{\pm} + \chi(e_1s_{\pm})s_{\mp} = \chi(s_{\pm})e_0s_{\mp} + \chi(s_{\mp})e_1s_{\mp},$$

which implies that $\chi(a) = -1$, $\chi(s_+) = \chi(s_-) = \pm 1$. We see that the two pairs given above exhaust all possible YD data. The two data are isomorphic to each other through the automorphism of H given by $a \leftrightarrow a$, $s_{\pm} \leftrightarrow as_{\pm}$.

- (b) This together with the corresponding part of the next theorem will be proved later in the end of this section.
- (c) Denote $g = s_+(m)$. By Proposition 3.1 there exists $z \in A$ such that $P_{1,g}(A) = k(g-1) + kz$ with the natural image of z in grA being y and $gz = -zg$. As in Theorem 4.1(2)(b) we see that $az = -za$. Note that $ze_0 = e_1z$ and $ze_1 = e_0z$.

Since $hz \equiv \sum \chi(h_1)zh_2 \pmod{H}$, we have that $s_+z + zs_+, s_-z + zs_- \in H$. Denote $t_+ = s_+z + zs_+, t_- = s_-z + zs_-$. Using the relations $s_-g = gs_+ = s_+(m-1)$ and $s_+g = gs_- = s_-(m-1)$, direct computations show that

$$\begin{aligned}\Delta(t_+) &= s_-(m-1) \otimes e_0t_+ + s_+(m-1) \otimes e_1t_+ + t_+ \otimes e_0s_+ + t_- \otimes e_1s_+, \\ \Delta(t_-) &= s_-(m-1) \otimes e_1t_- + s_+(m-1) \otimes e_0t_- + t_- \otimes e_0s_- + t_+ \otimes e_1s_-.\end{aligned}$$

Taking into account (11), denote

$$\begin{pmatrix} e_{11} & e_{12} \\ e_{21} & e_{22} \end{pmatrix} = \begin{pmatrix} e_{0s_+} & e_{1s_-} \\ e_{1s_+} & e_{0s_-} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} e_{0s_+(m-1)} & e_{1s_-(m-1)} \\ e_{1s_+(m-1)} & e_{0s_-(m-1)} \end{pmatrix},$$

the comatrix basis in C_1 and C_{m-1} . Note that

$$\begin{aligned}e_0e_{ii} &= e_{ii}, & e_0f_{ii} &= f_{ii}, & e_1e_{ii} &= e_1f_{ii} = 0 & \text{for any } i, \\ e_0e_{ij} &= e_0f_{ij} = 0, & e_1e_{ij} &= e_{ij}, & e_1f_{ij} &= f_{ij} & \text{for any } i \neq j.\end{aligned}$$

The comultiplication formulas for t_+ and t_- become

$$\Delta(t_+) = (f_{12} + f_{22}) \otimes e_0t_+ + (f_{11} + f_{21}) \otimes e_1t_+ + t_+ \otimes e_{11} + t_- \otimes e_{21}, \quad (13)$$

$$\Delta(t_-) = (f_{12} + f_{22}) \otimes e_1t_- + (f_{11} + f_{21}) \otimes e_0t_- + t_- \otimes e_{22} + t_+ \otimes e_{12}. \quad (14)$$

These show that $t_+, t_- \in C_1 \oplus C_{m-1}$, so

$$t_+ = \sum \alpha_{ij}e_{ij} + \sum \beta_{ij}f_{ij}, \quad t_- = \sum \alpha'_{ij}e_{ij} + \sum \beta'_{ij}f_{ij}.$$

By looking at the coefficients of $e_{21} \otimes e_{12}, e_{12} \otimes e_{22}, f_{12} \otimes f_{21}, f_{22} \otimes f_{21}$ in the comultiplication formula for t_+ , we see that $\alpha_{22} = \alpha_{12} = \beta_{11} = \beta_{21} = 0$, so

$$t_+ = \alpha_{11}e_{11} + \alpha_{21}e_{21} + \beta_{12}f_{12} + \beta_{22}f_{22}.$$

Similarly, by looking at the coefficients of $e_{11} \otimes e_{11}, e_{21} \otimes e_{11}, f_{11} \otimes f_{12}, f_{22} \otimes f_{22}$ in the comultiplication formula for t_- , we find that $\alpha_{11} = \alpha_{21} = \beta_{12} = \beta_{22} = 0$, so

$$t_- = \alpha_{12}e_{12} + \alpha_{22}e_{22} + \beta_{11}f_{11} + \beta_{21}f_{21}.$$

Now going again to the comultiplication of t_+ with these formulas and identifying the corresponding coefficients, we find that

$$t_+ = \alpha e_{11} + \beta e_{21} - \alpha f_{12} - \alpha f_{22}, \tag{15}$$

$$t_- = \alpha e_{12} + \beta e_{22} - \beta f_{11} - \beta f_{21} \tag{16}$$

for some scalars α, β . These show that

$$s_+z = -zs_+ + (\alpha e_0 + \beta e_1)s_+ - \alpha s_-(m - 1),$$

$$s_-z = -zs_- + (\beta e_0 + \alpha e_1)s_- - \beta s_+(m - 1).$$

The previous two relations show that

$$s_-s_+z = zs_-s_+ + (\alpha - \beta)as_-s_+ + \beta s_+(m) - \alpha s_+(m - 2).$$

Applying repeatedly this, we obtain that

$$(s_-s_+)^h z = z(s_-s_+)^h + h(\alpha - \beta)a(s_-s_+)^h + \beta s_+(m + 2h - 2) - \alpha s_+(m + 2h - 4) + \dots - \alpha s_+(m - 2h)$$

where $m = 2h + 1$. If we multiply by s_+ to the left, we obtain after computations that

$$gz = -zg + (\alpha e_0 + \beta e_1)s_+(2h + 1) - \alpha s_+(2) + h(\alpha - \beta)as_+(2h + 1) + \beta s_+(4) - \alpha s_+(6) + \dots + \beta s_+(4h) - \alpha 1.$$

Note that we used the relations $s_-(4h - 2) = s_+(4)$, $s_-(4h - 4) = s_+(6)$, etc. We also have $s_+(2h + 1) = s_-(2h + 1)$.

Since $gz = -zg$, we have that

$$0 = \frac{\alpha + \beta}{2}s_-(2h + 1) + \frac{\alpha - \beta}{2}as_-(2h + 1) + h(\alpha - \beta)as_-(2h + 1) - \alpha 1 - \alpha s_+(2) + \beta s_+(4) - \alpha s_+(6) + \dots + \beta s_+(4h)$$

and we see that $\alpha = \beta = 0$ by using (12). This shows that $s_+z = -zs_+$, $s_-z = -zs_-$, and also $z^2 = 0$ follows from $\Delta(z^2) = z^2 \otimes 1 + 1 \otimes z^2$. We conclude that $A \simeq \text{gr}A$. \square

Theorem 4.4. *Suppose $H = \mathcal{B}_{4m}$ ($m \geq 2$).*

- (1) *Suppose $q = -1$. If m is odd or $m \equiv 0 \pmod{4}$, there exists no YD datum.*
- (2) *Suppose $q = -1$ and $m \equiv 2 \pmod{4}$.*

(a) There exist precisely four YD data

$$(g_{\pm}, \chi_{\pm}), \quad (g_{\pm}, \chi_{\mp}),$$

where $g_{\pm} = (e_0 \pm \sqrt{-1}e_1)s_+(m)$, and χ_{\pm} are defined by

$$\chi_{\pm}(a) = 1, \quad \chi_{\pm}(s_+) = \pm 1, \quad \chi_{\pm}(s_-) = \mp 1.$$

We have

$$(g_+, \chi_+) \simeq (g_-, \chi_-) \not\simeq (g_+, \chi_-) \simeq (g_-, \chi_+).$$

(b) If k contains a primitive $4m$ th root of 1, then the four YD data are all selfdual.

(c) The graded Hopf algebras $R_q \# H$ arising from the YD data cannot be lifted non-trivially.

(3) Suppose m is odd and $q = \sqrt{-1}$ (respectively $q = -\sqrt{-1}$).

(a) There exist precisely two YD data which are given by

$$(g_{\pm}, \psi_{\pm}) \quad (\text{respectively } (g_{\pm}, \psi_{\mp})) \quad \text{if } m \equiv 1 \pmod{4},$$

$$(g_{\pm}, \psi_{\mp}) \quad (\text{respectively } (g_{\pm}, \psi_{\pm})) \quad \text{if } m \equiv 3 \pmod{4},$$

where $g_{\pm} = (e_0 \pm \sqrt{-1}e_1)s_+(m)$ as above, and ψ_{\pm} are defined by

$$\psi_{\pm}(a) = -1, \quad \psi_{\pm}(s_+) = \pm 1, \quad \psi_{\pm}(s_-) = \mp 1.$$

The two data are isomorphic to each other.

(b) If k contains a primitive m th root of 1, then the two YD data are both selfdual.

(c) The graded Hopf algebras $R_q \# H$ arising from the YD data cannot be lifted non-trivially.

Proof. To prove (1) and (2)(a), suppose (g, χ) is a YD datum for $q = -1$.

(1) Suppose m is odd. Suppose first $\chi(a) = 1$. Then, $g = g_+$ or $g_- (= g_+^{-1})$. Since $\chi(s_+s_-)^m = \chi(a) = 1$, it follows that $\chi(s_{\pm}) = -1$. Take $h = s_+$, $g = g_{\pm}$ in (2) in Definition 2.1. Then we have $-s_+g_{\pm} = -g_{\pm}s_+$, whence $s_+s_+(m) = s_+(m)s_+$. This implies $s_- = as_+$, a contradiction. Suppose next $\chi(a) = -1$. Then $g = a$, and $\chi(s_{\pm}) = \pm 1$ or ∓ 1 . Take $h = s_+$, $g = a$ in (2). Then we have $s_- = as_+$, a contradiction.

Suppose m is even. Then, $\chi(a) = \chi(s_+s_-)^m = 1$, and so $g = g_+$ or $g_- (= ag_+)$. But, if $m \equiv 0 \pmod{4}$, $\chi(s_+(m)) = 1$ and hence $\chi(g_{\pm}) = 1$. Therefore Part (1) follows.

(2) Suppose $m \equiv 2 \pmod{4}$.

(a) To have $\chi(g_{\pm}) = -1$, it must hold that $\chi(s_{\pm}) = \pm 1$ or ∓ 1 . In any case, Condition (2) holds true for $h = s_{\pm}$, $g = g_{\pm}$, since then $s_{\pm}g = ags_{\pm}$. This proves that all possible YD data are given by the four pairs above.

The automorphism f_0 of H defined by

$$f_0(a) = a, \quad f_0(s_{\pm}) = s_{\mp}$$

gives rise to isomorphisms $(g_+, \chi_+) \simeq (g_-, \chi_-)$, $(g_+, \chi_-) \simeq (g_-, \chi_+)$.

We remark that H fits into a Hopf algebra extension (see [15, Proposition 3.11]),

$$K = k\langle a \rangle \hookrightarrow H \twoheadrightarrow kD_{2m}. \tag{17}$$

To see $(g_+, \chi_+) \not\simeq (g_{\pm}, \chi_{\mp})$, let f be an automorphism of H . Since a is a unique non-trivial central grouplike, we see that

$$f(a) = a, \tag{18}$$

and f induces an automorphism \bar{f} of kD_{2m} . Hence there exists an odd integer $0 < r < m$ prime to m such that

$$\bar{f}(s_{\pm}) = s_{\pm}(r) \quad \text{or} \quad s_{\mp}(r),$$

where D_{2m} is supposed to be generated by s_+ and s_- , and defined by the same relations as (10). By replacing f with $f_0 \circ f$, we may suppose $\bar{f}(s_{\pm}) = s_{\pm}(r)$, so that $e_0 f(s_{\pm}) = e_0 s_{\pm}(r)$. Since $\begin{pmatrix} e_0 f(s_+) & e_1 f(s_-) \\ e_1 f(s_+) & e_0 f(s_-) \end{pmatrix}$ forms a comatrix basis, we have

$$0 = \Delta(e_0 f(s_+)) - \Delta(e_0 s_+(r)) = e_1 f(s_-) \otimes e_1 f(s_+) - e_1 s_-(r) \otimes e_1 s_+(r),$$

whence $e_1 f(s_{\pm}) = \alpha^{\pm 1} e_1 s_{\pm}(r)$ for some $0 \neq \alpha \in k$. Since $e_1 = e_1 f(s_+)^2 = \alpha^2 e_1 s_+(r)^2 = \alpha^2 e_1$, we have $\alpha = 1$ or -1 , so that

$$f(s_{\pm}) = s_{\pm}(r) \quad \text{or} \quad a s_{\pm}(r). \tag{19}$$

Write $r = 2t + 1$. Then, $f(s(m)) = f(s_+ s_-)^{m/2} = a^t s(m)$, and hence

$$f(g_+) = \begin{cases} g_+ & \text{if } t \text{ is even,} \\ g_- & \text{if } t \text{ is odd.} \end{cases}$$

Since in addition

$$\chi_+(f(s_{\pm})) = \begin{cases} \chi_+(s_{\pm}) & \text{if } t \text{ is even,} \\ \chi_-(s_{\pm}) & \text{if } t \text{ is odd,} \end{cases}$$

we conclude that $(g_+, \chi_+) \not\simeq (g_{\pm}, \chi_{\mp})$.

Note. We have proved that the automorphism group of \mathcal{B}_{4m} (for any $m \geq 2$) consists of the f defined by (18), (19) for each odd $0 < r < m$ prime to m , together with $f_0 \circ f$.

(c) With arguments as in Theorem 4.3(2)(c) we see that there is $z \in P_{1,g}(A)$ with the natural image of z in $\text{gr}A$ being y , and such that $gz = -zg$, $az = za$, $s_+z - azs_+$, $s_-z + azs_- \in H$. Denote $t_+ = s_+z - azs_+$ and $t_- = s_-z + azs_-$. H decomposes as a direct sum of coalgebras with comatrix bases given by the same formulas as in case of \mathcal{A}_{4m} .

In the case where $m > 2$ we keep the same notation as in Theorem 4.3(2)(c) for these. In this case we have that

$$\begin{aligned} \Delta(t_+) &= (f_{22} + \sqrt{-1}f_{12}) \otimes e_0t_+ + t_+ \otimes e_{11} + (f_{11} - \sqrt{-1}f_{21}) \otimes e_1t_+ \\ &\quad + t_- \otimes e_{21}, \end{aligned} \quad (20)$$

$$\begin{aligned} \Delta(t_-) &= (f_{11} - \sqrt{-1}f_{21}) \otimes e_0t_- + t_- \otimes e_{22} + (f_{22} + \sqrt{-1}f_{12}) \otimes e_1t_- \\ &\quad + t_+ \otimes e_{12}. \end{aligned} \quad (21)$$

Note that $f'_{11} = f_{11}$, $f'_{12} = \sqrt{-1}f_{12}$, $f'_{21} = -\sqrt{-1}f_{21}$, $f'_{22} = f_{22}$ is a comatrix basis for C_{m-1} , satisfying similar relations at multiplication with e_0, e_1 . We see that formulas (20), (21) become exactly (13), (14) with f_{ij} substituted by f'_{ij} , so by using (15), (16) we see directly that

$$t_+ = \alpha e_{11} + \beta e_{21} - \sqrt{-1}\alpha f_{12} - \alpha f_{22}, \quad (22)$$

$$t_- = \alpha e_{12} + \beta e_{22} - \beta f_{11} + \sqrt{-1}\beta f_{21} \quad (23)$$

showing that

$$\begin{aligned} s_+z &= azs_+ + (\alpha e_0 + \beta e_1)s_+ - \alpha(e_0 + \sqrt{-1}e_1)s_-(m-1), \\ s_-z &= -azs_- + (\beta e_0 + \alpha e_1)s_- - \beta(e_0 - \sqrt{-1}e_1)s_+(m-1). \end{aligned}$$

Using these two relations we find after a few computations that

$$s_+s_-z = -zs_+s_- + ((\beta - \alpha)e_0 + (\alpha + \beta)e_1)s_+s_- + (\alpha - \beta)(e_0 - \sqrt{-1}e_1)s_-(m-2).$$

Applying two times the above relation we get $(s_+s_-)^2z = z(s_+s_-)^2$.

Write $m = 4h + 2$, $h \geq 1$. We have that $gz = -zg$, where $g = (e_0 + \sqrt{-1}e_1)s_+(m) = (e_0 + \sqrt{-1}e_1)(s_+s_-)^{2h+1}$. Since $e_0 + \sqrt{-1}e_1$ is central and invertible, this implies that $(s_+s_-)^{2h+1}z = -z(s_+s_-)^{2h+1}$. But

$$\begin{aligned} (s_+s_-)^{2h+1}z &= s_+s_-z(s_+s_-)^{2h} \\ &= -z(s_+s_-)^{2h+1} + ((\beta - \alpha)e_0 + (\alpha + \beta)e_1)(s_+s_-)^{2h+1} \\ &\quad + (\alpha - \beta)(e_0 - \sqrt{-1}e_1). \end{aligned}$$

Using $(s_+s_-)^{2h+1}z = -z(s_+s_-)^{2h+1}$, this gives

$$(\beta - \alpha)(s_+s_-)^{2h+1} + \frac{(1 - \sqrt{-1})(\alpha - \beta)}{2}1 + \frac{(1 + \sqrt{-1})(\alpha - \beta)}{2}a = 0$$

which implies $\alpha = \beta = 0$ by (12).

In the case where $m = 2$, similar computations show that

$$\begin{aligned} \Delta(t_+) &= (e_{22} + \sqrt{-1}e_{12}) \otimes e_0t_+ + t_+ \otimes e_{11} + (e_{11} - \sqrt{-1}e_{21}) \otimes e_1t_+ + t_- \otimes e_{21}, \\ \Delta(t_-) &= (e_{11} - \sqrt{-1}e_{21}) \otimes e_0t_- + t_- \otimes e_{22} + (e_{22} + \sqrt{-1}e_{12}) \otimes e_1t_- + t_+ \otimes e_{12}. \end{aligned}$$

These show that t_+ and t_- belong to the unique simple comatrix coalgebra of dimension 4 of H , and then by direct computations one gets

$$\begin{aligned} t_+ &= (\alpha e_0 + \beta e_1)s_+ - \alpha(e_0 + \sqrt{-1}e_1)s_-, \\ t_- &= \beta(-e_0 + \sqrt{-1}e_1)s_+ + (\beta e_0 + \alpha e_1)s_- \end{aligned}$$

for some scalars α, β . Now using the relation $zg = -gz$, one obtains $\alpha = \beta = 0$ as in the case $m > 2$. This shows that $t_+ = t_- = 0$, so indeed any lifting is trivial.

(3) Suppose m is odd and $q = \pm\sqrt{-1}$.

(a) Suppose (g, ψ) is a YD datum. To have $\psi(g) = \pm\sqrt{-1}$, g must equal g_+ or g_- , so that $\psi(a) = \psi(g_{\pm}^2) = -1$. Since m is odd, we see $s_+(m)s_{\pm} = as_{\mp}s_+(m)$, and so $ge_0s_{\pm} = e_0s_{\mp}g, ge_1s_{\pm} = -e_1s_{\mp}g$. Take $h = s_{\pm}$ in (2). Then we have

$$\psi(s_{\pm})s_{\mp} = \psi(s_{\pm})e_0s_{\mp} - \psi(s_{\mp})e_1s_{\mp},$$

and so $\psi(s_+) = -\psi(s_-)$. We see now that the two pairs given above exhaust all possible YD data. The two data are isomorphic to each other through f_0 .

(c) We prove for $q = \sqrt{-1}, m = 4h + 1 (h \geq 1)$, and the YD datum (g_+, ψ_+) . The other cases are similar. Write $g = g_+$. Let A be a Hopf algebra with coradical H and diagram R_q . Again as in Theorem 4.3(2)(c) there exists $z \in P_{1,g}(A) - H$ whose natural image in $\text{gr } A \simeq R_q \# H$ is y , and $gz = \sqrt{-1}zg$ and $az = -za$. Denote $t_+ = s_+z + azs_+, t_- = s_-z - azs_-$. Then direct computations show that

$$\Delta(t_{\pm}) = s_{\pm}g \otimes e_0t_{\pm} + s_{\mp}g \otimes e_1t_{\pm} + t_{\pm} \otimes e_0s_{\pm} + t_{\mp} \otimes e_1s_{\pm}.$$

In terms of the comatrix bases these write exactly as (20), (21), therefore the solution for t_+, t_- is given by (22), (23). Thus

$$\begin{aligned} s_+z &= -azs_+(\alpha e_0 + \beta e_1)s_+ - \alpha(e_0 + \sqrt{-1}e_1)s_-(m - 1), \\ s_-z &= azs_-(\beta e_0 + \alpha e_1)s_- - \beta(e_0 - \sqrt{-1}e_1)s_+(m - 1). \end{aligned}$$

Using these as in part (2), an inductive argument shows that

$$(s_-s_+)^{2h}z = z(s_-s_+)^{2h} + (e_0 + \sqrt{-1}e_1)S$$

where

$$\begin{aligned} S &= \sum_{i=0}^{h-1} (-\beta s_+(8h - 1 - 8i) + \alpha s_+(8h - 3 - 8i) + \beta s_+(8h - 5 - 8i) \\ &\quad - \alpha s_+(8h - 7 - 8i)). \end{aligned}$$

Then this gives

$$s_+(m)z = -azs_+(m) + (\alpha e_0 + \beta e_1)s_+(m) - \alpha(e_0 + \sqrt{-1}e_1)s_+(m + 1) + (e_0 + \sqrt{-1}e_1)T \tag{24}$$

where

$$T = \sum_{i=0}^{h-1} (-\beta s_-(8h - 2 - 8i) + \alpha s_-(8h - 4 - 8i) + \beta s_-(8h - 6 - 8i) - \alpha s_-(8h - 8 - 8i))$$

(with the convention that $s_-(0) = 1$). The relation $gz = \sqrt{-1}zg$ implies that $s_+(m)z = -azs_+(m)$, and then if we use the relations $s_+(m) = as_-(s_+s_-)^{2h}$, $s_+(m + 1) = (s_+s_-)^{2h+1}$, $s_-(j) = a(s_+s_-)^{m-j/2}$ for even j , and the basis of \mathcal{B}_{4m} from (12), the relation (24) shows that $\alpha = \beta = 0$, and we are done. \square

For the postponed proofs of selfduality we will use some interesting results by S. Suzuki. Suzuki [22] constructed cosemisimple Hopf algebras,

$$A_{NL}^{v\lambda} \quad (N \geq 1, L \geq 2; v, \lambda \in \{+, -\}),$$

of dimension $4NL$, and determined all braidings on each of the Hopf algebras. Braidings (or coquasitriangular structures) on a bialgebra H are those bilinear forms on it which are in one-to-one correspondence with the braidings on the tensor category of right H -comodules.

By definition, $A_{NL}^{v\lambda}$ is generated by the elements x_{ij} of a comatrix basis $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. Suppose $N = 1, v = +$. Then the defining relations for $A_{1L}^{+\lambda}$ is given by

$$\begin{aligned} x_{11}^2 &= x_{22}^2, & x_{12}^2 &= x_{21}^2, & x_{11}^2 + x_{12}^2 &= 1, \\ x_{ij}x_{kl} &= 0 & \text{whenever } i + j + k + l & \text{ is odd,} \\ \underleftarrow{L} x_{11} x_{22} x_{11} \cdots &= \underleftarrow{L} x_{22} x_{11} x_{22} \cdots, & \underleftarrow{L} x_{21} x_{12} x_{21} \cdots &= \underleftarrow{L} \lambda x_{12} x_{21} x_{12} \cdots. \end{aligned}$$

Suppose $L = m$. We see $\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} e_0^{s_+} & e_1^{s_-} \\ e_1^{s_+} & e_0^{s_-} \end{pmatrix}$ gives isomorphisms

$$A_{1m}^{++} \simeq \mathcal{A}_{4m}, \quad A_{1m}^{+-} \simeq \mathcal{B}_{4m}.$$

It follows by [22, Proposition 3.10] that given a scalar ξ such that

$$\xi^{2m} = 1 \quad (\text{respectively } \xi^{2m} = -1), \tag{25}$$

we have a braiding σ_ξ^+ on \mathcal{A}_{4m} (respectively σ_ξ^- on \mathcal{B}_{4m}) which evaluates:

	e_0s_+	e_1s_-	e_1s_+	e_0s_-
e_0s_+	0	0	0	0
e_1s_-	0	ξ	ξ^{-1}	0
e_1s_+	0	ξ^{-1}	ξ	0
e_0s_-	0	0	0	0

By [22, Propositions 2.9 and 3.10], σ_ξ^+ is symmetric (or it corresponds to a symmetric braiding on the comodule category) only when $\xi^2 = 1$, while σ_ξ^- cannot be symmetric. (σ_ξ^+ and σ_ξ^- are denoted by $\tilde{\sigma}_{\xi, \xi^{-1}}$ in [22].)

Proposition 4.5. (1) *Suppose m is odd and ξ is a primitive m th root of 1, so that $\xi^{2m} = 1$; see (25). Then the braiding σ_ξ^+ on \mathcal{A}_{4m} is non-degenerate, whence it is minimal in the following sense: it does not arise from any braiding on a non-trivial quotient Hopf algebra (this is equivalent to that the dual quasitriangular structure on \mathcal{A}_{4m}^* is minimal in Radford’s sense [20]).*

(2) *Suppose ξ is a primitive $4m$ th root of 1, so that $\xi^{2m} = -1$; see (25). Then the braiding σ_ξ^- on \mathcal{B}_{4m} is non-degenerate, whence it is minimal.*

Proof. We prove first Part (2).

(2) Write $H = \mathcal{B}_{4m}$. Let $\varphi: H \xrightarrow{\sim} H^{\text{op}}$ denote the isomorphism defined by $\varphi(a) = a$, $\varphi(s_\pm) = s_\pm$; see [15, p. 203, line 2]. Define $\langle \cdot, \cdot \rangle = \sigma_\xi^- \circ (\text{id} \otimes \varphi)$. Then, $\langle \cdot, \cdot \rangle: H \otimes H \rightarrow k$ is a (symmetric) Hopf pairing which evaluates:

	a	s_+	s_-
a	1	-1	-1
s_+	-1	ξ	ξ^{-1}
s_-	-1	ξ^{-1}	ξ

It suffices to prove that $\langle \cdot, \cdot \rangle$ is non-degenerate, or equivalently the induced the Hopf algebra map

$$\kappa: H \rightarrow H^*, \quad \kappa(h) = \langle h, \cdot \rangle$$

is an isomorphism.

Set $w = s_+s_-$ in H . Let $J = k[a, w] \subset H$ denote the commutative subalgebra generated by a and w . As is easily seen, J is a normal Hopf subalgebra of dimension $2m$ such that the quotient Hopf algebra H/J^+H is generated by the natural image \bar{s}_+ of s_+ , which is a grouplike of order 2. Since $\langle w, a \rangle = \langle a, a \rangle = 1$, it follows that $\kappa(J) \subset (kD_{2m})^*$, where we regard $(kD_{2m})^* \subset H^*$ through the natural injection which arises from the quotient map in (17). Since $\langle s_+, a \rangle = -1$, the Hopf algebra map $k\langle \bar{s}_+ \rangle \rightarrow (k\langle a \rangle)^*$ induced from κ is an isomorphism. So, it suffices to prove that $\kappa|_J: J \rightarrow (kD_{2m})^*$ is an isomorphism.

We have a split extension of groups,

$$\langle s_+s_- \rangle \hookrightarrow D_{2m} = \langle s_+, s_- \rangle \twoheadrightarrow \langle s_+ \rangle.$$

Since $\langle a, s_+s_- \rangle = 1$, $\kappa|_J$ induces $k\langle a \rangle \rightarrow (k\langle s_+ \rangle)^* \subset (kD_{2m})^*$, which is an isomorphism since $\langle a, s_+ \rangle = -1$. So, it suffices to prove that

$$\mu: \bar{J} := J/(k\langle a \rangle)^+ J \rightarrow (k\langle s_+s_- \rangle)^*$$

induced from $\kappa|_J$ is an isomorphism. Notice that \bar{J} is generated by the natural image \bar{w} of w , which is a grouplike of order m . We compute that $\langle w, s_\pm \rangle = \xi^{\mp 2}$, $\langle w, w \rangle = \xi^4$. If ξ is a primitive $4m$ th root of 1, then ξ^4 is a primitive m th root of 1, so that μ and hence κ are isomorphisms.

(1) This is proved in the same way just as above. Notice that if m is odd and ξ is a primitive m th root of 1, then ξ^4 is still such a root. \square

We have proved in fact that \mathcal{A}_{4m} is selfdual if m is odd and k contains a primitive m th root of 1, while \mathcal{B}_{4m} is selfdual if k contains a primitive $4m$ th root of 1. We remark that if m is even, \mathcal{A}_{4m} is not selfdual since then $\#G(\mathcal{A}_{4m}^*) = 8 \neq \#G(\mathcal{A}_{4m})$; in fact an algebra map $\mathcal{A}_{4m} \rightarrow k$ can evaluate 1 or -1 independently at a , s_+ and s_- .

Proof of Theorem 4.4(2)(b). To prove the selfduality of the YD data (g_\pm, χ_\pm) , (g_\pm, χ_\mp) for $q = -1$ when $H = \mathcal{B}_{4m}$ and $m \equiv 2 \pmod{4}$, we compute some values of the Hopf pairing given above:

$$\begin{aligned} \langle g_\pm, a \rangle &= 1, \\ \langle g_\pm, s_+ \rangle &= \langle e_0 \pm \sqrt{-1}e_1, s_+ \rangle \langle w, s_+ \rangle^{m/2} = \pm \sqrt{-1} \xi^{-m}, \\ \langle g_\pm, s_- \rangle &= \langle e_0 \pm \sqrt{-1}e_1, s_- \rangle \langle w, s_- \rangle^{m/2} = \pm \sqrt{-1} \xi^m. \end{aligned}$$

Suppose ξ is a primitive $4m$ th root of 1. Then, $\xi^m = \pm \sqrt{-1}$. If $\xi^m = \sqrt{-1}$, then we see $\kappa(g_\pm) = \chi_\pm = g_\pm \circ \kappa$, so that κ gives an isomorphism $(H, g_\pm, \chi_\pm) \simeq (H^*, \chi_\pm, g_\pm)$. If $\xi^m = -\sqrt{-1}$, then we see $\kappa(g_\pm) = \chi_\mp = g_\pm \circ \kappa$, so that κ gives an isomorphism $(H, g_\pm, \chi_\mp) \simeq (H^*, \chi_\mp, g_\pm)$. \square

The corresponding parts (b) in Theorem 4.3(2) and Theorem 4.4(3) can be proved in the same way. But, these results follow immediately from the selfduality of the Hopf algebras \mathcal{A}_{4m} , \mathcal{B}_{4m} , since each YD datum in question is unique (up to isomorphism).

5. Hopf algebras of dimension 16 with the Chevalley property

The aim of this section is to complete the classification of Hopf algebras of dimension 16 with the Chevalley property over an algebraically closed field of characteristic zero. Pointed Hopf algebras of dimension 16 are classified in [10], and semisimple Hopf algebras of dimension 16 are classified in [12]. Then by the Nichols–Zoeller theorem and the fact that a semisimple Hopf algebra of dimension 2 or 4 is a group algebra, the only case that we have to investigate is when the coradical is a semisimple Hopf algebra of dimension 8 which is not a group algebra. We study this last case over an algebraically closed field k

of characteristic $\neq 2$. For such a field there exist 3 isomorphism types of semisimple Hopf algebras of dimension 8 non-isomorphic to group algebras (see [14]): two commutative Hopf algebras, duals of group algebras, namely $(kD_4)^*$ and $(kQ)^*$, where D_4 is the dihedral group with 8 elements and Q is the group of quaternions, and a non-commutative non-cocommutative Hopf algebra H_8 which is denoted by \mathcal{A} in [14]. It is interesting to make the remark that $(kD_4)^*$ and $(kQ)^*$ are isomorphic as algebras (both are isomorphic to a direct product of 8 copies of k), and also they are isomorphic as coalgebras (since the irreducible representations of D_4 and Q have the same dimensions), but they are not isomorphic as Hopf algebras. It is proved in [15,24] that the Hopf algebras $(kD_4)^*$, $(kQ)^*$ and H_8 can be distinguished by the fact that their categories of corepresentations (i.e. comodules) are not monoidally equivalent. The Hopf algebra H_8 is presented by generators c, b, x with relations

$$c^2 = 1, \quad b^2 = 1, \quad x^2 = \frac{1}{2}(1 + c + b - cb),$$

$$cb = bc, \quad xc = bx, \quad xb = cx$$

and it has the comultiplication, the counit and the antipode defined by

$$\Delta(c) = c \otimes c, \quad \varepsilon(c) = 1, \quad S(c) = c,$$

$$\Delta(b) = b \otimes b, \quad \varepsilon(b) = 1, \quad S(b) = b,$$

$$\Delta(x) = \frac{1}{2}(x \otimes x + bx \otimes x + x \otimes cx - bx \otimes cx), \quad \varepsilon(x) = 1, \quad S(x) = x.$$

Since \mathcal{B}_8 is a non-commutative non-cocommutative semisimple Hopf algebra of dimension 8 we have $\mathcal{B}_8 \simeq H_8$. The presentation of H_8 with generators a, s_+, s_- satisfying the relations of \mathcal{B}_8 appears implicitly in the proof of [14, Remark 2.14 (1)]. It is helpful to see explicitly an isomorphism between H_8 and \mathcal{B}_8 . Thus if we have the presentation of \mathcal{B}_8 with generators a, s_+, s_- , then some elements c, b, x satisfying the relations from the presentation of H_8 are given by

$$c = (e_0 + \sqrt{-1}e_1)_{s_+s_-}, \quad b = (e_0 - \sqrt{-1}e_1)_{s_+s_-},$$

$$x = \frac{1 + \sqrt{-1}}{2}e_{0s_+} + \frac{1}{\sqrt{2}}e_{1s_-} + \frac{1}{\sqrt{2}}e_{1s_+} + \frac{1 - \sqrt{-1}}{2}e_{0s_-}.$$

These formulas can be obtained by direct (tedious) computations, by using the fact that x lies in the unique comatrix simple coalgebra of dimension 4 of H_8 , therefore x is a linear combination of $e_{0s_+}, e_{1s_-}, e_{1s_+}, e_{0s_-}$, and then by using the fact that $S(x) = x$ and the comultiplication formula for x .

For the other way around (the inverse of the isomorphism), some elements a, s_+, s_- in H_8 are given by

$$a = cb, \quad s_{\pm} = \left(\frac{1 \mp \sqrt{-1}}{4} + \frac{1}{2\sqrt{2}} \right) x + \left(\frac{1 \pm \sqrt{-1}}{4} - \frac{\sqrt{-1}}{2\sqrt{2}} \right) cx \\ + \left(\frac{1 \pm \sqrt{-1}}{4} + \frac{\sqrt{-1}}{2\sqrt{2}} \right) bx + \left(\frac{1 \mp \sqrt{-1}}{4} + \frac{1}{2\sqrt{2}} \right) cbx.$$

By using the results of Section 4 we obtain the following.

Theorem 5.1. *Let k be an algebraically closed field of characteristic $\neq 2$. Then there exist precisely two isomorphism types of Hopf algebras of dimension 16 with the coradical a non-cocommutative Hopf subalgebra of dimension 8. These Hopf algebras have coradical $H_8 \simeq \mathcal{B}_8$, and they are selfdual.*

The two new Hopf algebras are biproducts over \mathcal{B}_8 as showed in Theorem 4.4. Therefore they can also be presented as biproducts over H_8 , with generators c, b, x, y , relations of H_8 , and

$$y^2 = 0, \quad yc = -cy, \quad yb = -by, \quad yx = \pm\sqrt{-1}cxy, \\ \Delta(y) = c \otimes y + y \otimes 1, \quad \varepsilon(y) = 0.$$

The construction of these two Hopf algebras can be also performed by a method similar to the one in [9], by taking certain Ore extensions of H_8 , defining Hopf algebra structures on them, which extend the Hopf algebra structure of H_8 , and then factoring by a certain Hopf ideal.

Little is known for Hopf algebras of dimension 16 with the coradical being not a Hopf subalgebra (over an algebraically closed field of characteristic zero). It is known that such Hopf algebras exist; see [7]. Some steps for classifying these are done in [8], where certain cases for the coradical are discarded.

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