# On intertwining operators and recursions

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*Dedicated to James Lepowsky on the occasion of his sixtieth birthday*

Abstract. S. Capparelli, J. Lepowsky and A. Milas initiated a new approach of getting Rogers-Ramanujan-type recursions by studying the principal subspaces, introduced by B. Feigin and A. Stoyanovsky, of the standard $\widehat{\mathfrak{sl}(2)}$-modules. We extend their approach to the level 1 standard modules for $\widehat{\mathfrak{sl}(3)}$. We obtain exact sequences which lead us to a complete set of recursions satisfied by the graded dimensions (characters) of the principal subspaces of these modules.

## 1. Introduction

Since vertex operator algebras were introduced mathematically by R. Borcherds and I. Frenkel-J. Lepowsky-A. Meurman, they have turned out to be extremely useful in many areas of mathematics: the theory of finite groups, modular functions, combinatorics, integrable systems, string theory, and conformal field theory. The theory of vertex operator algebras ([**B**], [**FLM**], [**FHL**]; cf. [**LL**]) provides, among many other things, constructions, both early and contemporary, of a great variety of representations of twisted and untwisted affine Lie algebras. The affine Lie algebras of type $A$, $D$ and $E$, twisted by the principal automorphism of the underlying finite-dimensional simple Lie algebra, were realized by means of certain twisted vertex operators ([**LW1**], [**KKLW**]; see also [**L**]). In [**FK**] and [**S**], the untwisted affine algebras of types $A^{(1)}$, $D^{(1)}$ and $E^{(1)}$ were represented using untwisted vertex operators. These approaches were generalized for all the standard $\widehat{\mathfrak{sl}(2)}$-modules in [**LW2**]–[**LW4**], where the $\mathcal{Z}$-algebra program was initiated, and thus a representation-theoretic proof of the Rogers-Ramanujan identities was given. In the past years other classical and also new combinatorial identities have been obtained (see e.g. [**LP2**], [**Ca**], [**MP**], [**Mis1**] and [**Mis2**]).

Certain specializations of the solution of the Rogers-Ramanujan recursion lead to the sum sides of the Rogers-Ramanujan identities (cf. [**A**]). A vertex-algebraic interpretation of this recursion was given in [**CapLM1**]. By studying the principal subspaces, introduced by B. Feigin and A. Stoyanovsky in [**FS1**]–[**FS2**], of the level 1 standard $\widehat{\mathfrak{sl}(2)}$-modules, S.Capparelli, J. Lepowsky and A. Milas recovered the Rogers-Ramanujan recursion and the graded dimensions (the generating functions

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of the dimensions of the homogeneous spaces) of these subspaces. In [**CapLM2**] they proved that the graded dimensions of the principal subspaces of the level $k > 1$ standard $\widehat{\mathfrak{sl}(2)}$-modules satisfy the Rogers-Selberg recursions. As a continuation of this program, we have recently obtained in [**C1**] a complete set of recursions ($q$-difference equations) satisfied by the graded dimensions of the principal subspaces of the level 1 standard modules for $\widehat{\mathfrak{sl}(l+1)}$ with $l \geq 2$. Similar results regarding the principal subspaces of higher-level $\widehat{\mathfrak{sl}(3)}$-modules have been obtained in [**C2**]. One of the main features of this approach is to uncover new structures even when it is hard to find combinatorial identities.

The present paper is an announcement of the author's results about the principal subspaces of the level 1 standard $\widehat{\mathfrak{sl}(3)}$-modules. We refer to [**C1**] for details, background and the proofs of the results in the present paper.

The principal subspaces of the standard $\widehat{\mathfrak{sl}(l+1)}$-modules for $l \geq 1$ were introduced and studied by B. Feigin and A. Stoyanovsky in [**FS1**] and [**FS2**]. These are the subspaces generated by the affinization of the nilpotent subalgebra of $\mathfrak{sl}(l+1)$, denoted by $\mathfrak{n}$, consisting of the strictly upper-triangular matrices. B. Feigin and A. Stoyanovsky found formulas for the principal subspaces of the standard modules for $\widehat{\mathfrak{sl}(2)}$, which led to new proofs of the Rogers-Ramanujan and Gordon identities. G. Georgiev also computed the graded dimensions of the principal subspaces of certain standard $\widehat{\mathfrak{sl}(l+1)}$-modules, by constructing combinatorial bases. In a recent paper, M. Primc introduces Feigin-Stoyanovsky-type principal subspaces and constructs bases of them. In [**ARS**] E. Ardonne, R. Kedem and M. Stone use principal subspaces to give a formula for the $q$-characters of arbitrary standard $\widehat{\mathfrak{sl}(l+1)}$-modules.

The principal subspaces of the level 1 standard modules $L(\Lambda_i)$ for $\widehat{\mathfrak{sl}(3)}$ are denoted by $W(\Lambda_i)$ for $i = 0, 1, 2$. These are defined as $W(\Lambda_i) = U(\bar{\mathfrak{n}}) \cdot v_{\Lambda_i}$, where $\bar{\mathfrak{n}} = \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}]$ and $\Lambda_i$ and $v_{\Lambda_i}$ are the highest weights and highest weight vectors, respectively, of $L(\Lambda_i)$ (cf. [**FS1**] and [**FS2**]). We identify $W(\Lambda_i)$ with the quotient space $U(\bar{\mathfrak{n}})/I_{\Lambda_i}$, where $I_{\Lambda_i}$ is the annihilator of the highest weight vector $v_{\Lambda_i}$ in $U(\bar{\mathfrak{n}})$. In [**FS2**] a result was announced describing these ideals (at least for the principal subspace of the vacuum representation $L(\Lambda_0)$). In this paper we prove the completeness of the list of relations for the principal subspaces $W(\Lambda_i)$ for $i = 0, 1, 2$ given in [**FS2**], and thus we give a presentation of these principal subspaces (see Theorems 3.1 and 3.6). The description of the left ideals $I_{\Lambda_i}$ combined with intertwining vertex operators enables us to construct exact sequences of maps among principal subspaces (see Theorem 4.2).

Our main goal is to find a complete set of recursions ($q$-difference equations), whose solutions are the graded dimensions of the principal subspaces of the level 1 standard $\widehat{\mathfrak{sl}(3)}$-modules. These recursions are consequences of the exact sequences mentioned above. We denote by $\chi_0(x_1, x_2; q)$ the graded dimension with respect to certain natural "weight" and "charge" gradings. We have obtained the following result:

THEOREM 1.1. *The graded dimension $\chi_0(x_1, x_2; q)$ of the principal subspace $W(\Lambda_0)$ satisfies the recursions*

$$\chi_0(x_1, x_2; q) = \chi_0(x_1 q, x_2; q) + x_1 q \chi_0(x_1 q^2, x_2 q^{-1}; q)$$

*and*

$$\chi_0(x_1, x_2; q) = \chi_0(x_1, x_2 q; q) + x_2 q \chi_0(x_1 q^{-1}, x_2 q^2; q).$$

By solving these recursions we recover the graded dimensions of the principal subspaces of the level 1 standard $\widehat{\mathfrak{sl}(3)}$-modules previously obtained by Georgiev using a different approach.

As we have mentioned, the results discussed in this paper are generalized to the principal subspaces of level 1 standard $\widehat{\mathfrak{sl}(l+1)}$-modules in [**C1**], and to the principal subspaces of certain higher-level standard $\widehat{\mathfrak{sl}(3)}$-modules in [**C2**]. Related questions in new directions are beginning to be investigated in joint work with J. Lepowsky and A. Milas.

I am very pleased to dedicate this paper to my advisor, Professor James Lepowsky, on the occasion of his sixtieth birthday. I am grateful to him for suggesting this area of research, his enthusiasm and guidance.

## 2. Preliminaries

In this section we recall the vertex operator construction of the level 1 standard $\widehat{\mathfrak{sl}(3)}$-modules ([**FK**], [**S**]) and the associated vertex operator algebra and module structures ([**B**], [**FLM**]). We also review the construction of certain intertwining operators among these standard modules [**DL**]. We work in the setting of [**FLM**] and [**DL**], to which we refer for more details. See also [**LL**].

We work with the Lie algebra $\mathfrak{sl}(3)$, which has a basis

$$\{x_{\alpha_1},\ x_{\alpha_2},\ x_{\alpha_1+\alpha_2},\ h_{\alpha_1},\ h_{\alpha_2},\ x_{-\alpha_1},\ x_{-\alpha_2},\ x_{-\alpha_1-\alpha_2}\}$$

and the standard bracket relations

$$[x_{\pm\alpha_1}, x_{\pm\alpha_2}] = x_{\pm(\alpha_1+\alpha_2)},\ \ [x_{\alpha_1}, x_{-\alpha_1}] = h_{\alpha_1},\ \ [x_{\alpha_2}, x_{-\alpha_2}] = h_{\alpha_2},$$
$$[x_{\alpha_1}, x_{-\alpha_2}] = 0,\ \ [x_{\alpha_2}, x_{-\alpha_1}] = 0,$$
$$[h_{\alpha_1}, x_{\pm\alpha_1}] = \pm 2x_{\pm\alpha_1},\ \ [h_{\alpha_2}, x_{\pm\alpha_2}] = \pm 2x_{\pm\alpha_2},$$
$$[x_{\pm\alpha_2}, h_{\alpha_1}] = \pm x_{\pm\alpha_2},\ \ [x_{\pm\alpha_1}, h_{\alpha_2}] = \pm x_{\pm\alpha_1}.$$

Denote by $\mathfrak{h}$ the Cartan subalgebra $\mathbb{C}h_{\alpha_1} \oplus \mathbb{C}h_{\alpha_2}$ of $\mathfrak{sl}(3)$. The standard symmetric invariant nondegenerate bilinear form $\langle x, y\rangle = \mathrm{tr}(xy)$ defined for any $x$ and $y$ in $\mathfrak{sl}(3)$ allows us to identify $\mathfrak{h}$ with $\mathfrak{h}^*$. Take $\alpha_1$, $\alpha_2$ and $\alpha_1+\alpha_2$ to be the (positive) roots corresponding to the vectors $x_{\alpha_1}$, $x_{\alpha_2}$ and $x_{\alpha_1+\alpha_2}$. In particular, $\langle\alpha,\alpha\rangle = 2$ for any root $\alpha$. Under our identification we have $h_{\alpha_1} = \alpha_1$ and $h_{\alpha_2} = \alpha_2$. We denote by $Q = \mathbb{Z}\alpha_1 + \mathbb{Z}\alpha_2$ the root lattice and by $P = \mathbb{Z}\lambda_1 + \mathbb{Z}\lambda_2$ the weight lattice. Here $\lambda_1$ and $\lambda_2$ are the fundamental weights ($\langle\lambda_i, \alpha_j\rangle = \delta_{i,j}$ for $i, j = 1, 2$).

It is known that there exists a central extension of $P$ by a finite cyclic group which we denote by $A$, satisfying the following condition:

$$\epsilon(\alpha,\beta)\epsilon(\beta,\alpha)^{-1} = (-1)^{\langle\alpha,\beta\rangle} \ \text{ for } \ \alpha,\beta \in Q,$$

where

$$\epsilon : P \times P \longrightarrow A \tag{2.1}$$

is a 2-cocycle corresponding to this extension. Set

$$c(\lambda,\mu) = \epsilon(\lambda,\mu)\epsilon(\mu,\lambda)^{-1} \ \text{ for } \ \lambda,\mu \in P, \tag{2.2}$$

and this is the commutator map of this central extension (cf. [**FLM**] and [**DL**]).

We shall often be working with the positive nilpotent subalgebra of $\mathfrak{sl}(3)$,

$$\mathfrak{n} = \mathbb{C}x_{\alpha_1} \oplus \mathbb{C}x_{\alpha_2} \oplus \mathbb{C}x_{\alpha_1+\alpha_2},$$

which we may think of as consisting of the strictly upper triangular matrices.

Now we consider the untwisted affine Lie algebra associated to $\mathfrak{sl}(3)$,

$$\widehat{\mathfrak{sl}(3)} = \mathfrak{sl}(3) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

where $c$ is a nonzero central element and

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m\langle x, y\rangle \delta_{m+n,0}c$$

for any $x, y \in \mathfrak{sl}(3)$ and $m, n \in \mathbb{Z}$. By adjoining the degree operator $d$ such that $[d, x \otimes t^m] = mx \otimes t^m$ and $[d, c] = 0$ we obtain the affine Kac-Moody algebra $\widetilde{\mathfrak{sl}(3)} = \widehat{\mathfrak{sl}(3)} \oplus \mathbb{C}d$ (cf. [**K**]). Consider the following subalgebras of $\widehat{\mathfrak{sl}(3)}$:

$$\bar{\mathfrak{n}} = \mathfrak{n} \otimes \mathbb{C}[t, t^{-1}],$$

$$\bar{\mathfrak{n}}^+ = \mathfrak{n} \otimes \mathbb{C}[t],$$

$$\widehat{\mathfrak{h}} = \mathfrak{h} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c,$$

and

$$\widehat{\mathfrak{h}}_{\mathbb{Z}} = \coprod_{m\in\mathbb{Z}\setminus 0} \mathfrak{h} \otimes t^m \oplus \mathbb{C}c.$$

The latter is a Heisenberg subalgebra of $\widehat{\mathfrak{sl}(3)}$, in the sense that its commutator subalgebra is equal to its center, which is one-dimensional. The form $\langle\cdot,\cdot\rangle$ on $\mathfrak{h}$ extends to $\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ ($\langle c, c\rangle = 0$, $\langle d, d\rangle = 0$ and $\langle c, d\rangle = 1$). The simple roots of the untwisted affine algebra $\widehat{\mathfrak{sl}(3)}$ are $\alpha_0, \alpha_1, \alpha_2 \in (\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d)^*$, which we identify with $\mathfrak{h} \oplus \mathbb{C}c \oplus \mathbb{C}d$ using the form. The fundamental weights of $\widehat{\mathfrak{sl}(3)}$ are $\Lambda_0, \Lambda_1, \Lambda_2$, where $\langle \Lambda_i, \alpha_j\rangle = \delta_{i,j}$, $\langle \Lambda_i, c\rangle = 1$ for $i, j = 0, 1, 2$ and $\langle \Lambda_0, d\rangle = 0$, $\langle \Lambda_1, d\rangle = \langle \Lambda_2, d\rangle = -1/3$. (The numbers $0$, $-1/3$ and $-1/3$ are the negatives of the conformal weights of the highest weight vectors.) We use the notation $L(\Lambda_0)$, $L(\Lambda_1)$, $L(\Lambda_2)$ for the standard $\widehat{\mathfrak{sl}(3)}$-modules of level 1. These are also called the integrable highest weight modules of level 1.

From now on we will write $x(m)$ for the action of $x \otimes t^m$ on any $\widehat{\mathfrak{sl}(3)}$-module, where $x \in \mathfrak{sl}(3)$ and $m \in \mathbb{Z}$. It will be clear from the context whether $x(m)$ is an operator or an element of $U(\widehat{\mathfrak{sl}(3)})$.

We form the induced $\widehat{\mathfrak{h}}$-module

$$M(1) = U(\widehat{\mathfrak{h}}) \otimes_{U(\mathfrak{h}\otimes\mathbb{C}[t]\oplus\mathbb{C}c)} \mathbb{C},$$

such that $\mathfrak{h}\otimes\mathbb{C}[t]$ acts trivially and $c$ acts as identity on the one-dimensional module $\mathbb{C}$. Denote by $\mathbb{C}[Q]$ and $\mathbb{C}[P]$ the group algebras of the lattices $Q$ and $P$ introduced above, with bases $\{e^\alpha | \alpha \in Q\}$ and $\{e^\lambda | \lambda \in P\}$, respectively. Consider the following vector spaces:

$$V_P = M(1) \otimes \mathbb{C}[P],$$

$$V_Q = M(1) \otimes \mathbb{C}[Q]$$

and

$$V_Q e^{\lambda_i} = M(1) \otimes \mathbb{C}[Q]e^{\lambda_i} \text{ for } i = 1, 2.$$

Then

$$V_P = V_Q \oplus V_Q e^{\lambda_1} \oplus V_Q e^{\lambda_2}.$$

For every $\lambda \in P$ set

$$Y(e^{\lambda}, x) = E^{-}(-\lambda, x)E^{+}(-\lambda, x)\epsilon_{\lambda}e^{\lambda}x^{\lambda}, \tag{2.3}$$

an End $V_P$-valued formal Laurent series in $x^{1/3}$, where $E^{\pm}(-\lambda, x)$ are defined by:

$$E^{\pm}(-\lambda, x) = \exp\left(\sum_{\pm n \geq 1} \frac{-\lambda(n)}{n} x^{-n}\right) \tag{2.4}$$

and

$$\epsilon_{\lambda}e^{\lambda}(v \otimes e^{\mu}) = \epsilon(\lambda, \lambda + \mu)v \otimes e^{\lambda+\mu}, \tag{2.5}$$

$$x^{\lambda}(v \otimes e^{\mu}) = x^{\langle \lambda, \mu \rangle} v \otimes e^{\mu} \tag{2.6}$$

for all $v \in M(1)$, $\lambda, \mu \in P$. We shall often write $e^{\lambda}$ instead of $1 \otimes e^{\lambda}$ and $h(m)$ instead of $h(m) \otimes 1$ ($h \in \mathfrak{h}$, $\lambda \in P$, $m \in \mathbb{Z}$). In this paper, by $e^{\lambda}$ we mean either an operator or a vector of $V_P$ and this will be clear from the context. More generally, for a generic homogeneous vector in $V_P$ set

$$Y\left(\prod_{i=1}^{r} h_i(-m_i) \otimes e^{\lambda}, x\right) = : \prod_{i=1}^{r} \left(\frac{1}{(m_i - 1)} \left(\frac{d}{dx}\right)^{m_i - 1} h_i(x)\right) Y(e^{\lambda}, x) :, \tag{2.7}$$

where $h_i \in \mathfrak{h}$, $m_i \in \mathbb{Z}_+$ and $\lambda \in P$. Recall that $h_i(x) = \sum_{m \in \mathbb{Z}} h_i(m)x^{-m-1}$ and $:\cdot:$ stands for the normal ordering operation. The formula (2.7) determines a well-defined linear map from $V_P$ to the vector space of End $V_P$-valued formal Laurent series in $x^{1/3}$.

There is a natural $\widehat{\mathfrak{sl}(3)}$-module structure on $V_P$, result which is stated below.

THEOREM 2.1. [**FK**], [**S**]; *cf.* [**FLM**] *The vector space $V_P$ is an $\widehat{\mathfrak{sl}(3)}$-module of level 1. The action $x_{\alpha}(m)$ of $x_{\alpha} \otimes t^m$ for a root $\alpha$ and $m \in \mathbb{Z}$ is given by the coefficient of $x^{-m-1}$ in $Y(e^{\alpha}, x)$. Moreover, the direct summands $V_Q$, $V_Q e^{\lambda_1}$ and $V_Q e^{\lambda_2}$ of $V_P$ are the standard $\widehat{\mathfrak{sl}(3)}$-modules of level 1 with highest weights $\Lambda_0$, $\Lambda_1$ and $\Lambda_2$ and highest weight vectors $v_{\Lambda_0} = 1 \otimes 1$, $v_{\Lambda_1} = 1 \otimes e^{\lambda_1}$, $v_{\Lambda_2} = 1 \otimes e^{\lambda_2}$.*

Moreover, the vector spaces $V_Q$, $V_P$ and $V_Q e^{\lambda_i}$, $i = 1, 2$ are naturally endowed with vertex operator algebra and module structures. This result is due to Borcherds and Frenkel-Lepowsky-Meurman.

THEOREM 2.2. [**B**], [**FLM**] *The formulas (2.3) and (2.7) give a vertex operator algebra structure on $V_Q$ and a $V_Q$-module structure on $V_P$. The vertex operator algebra $V_Q$ is simple and $V_Q e^{\lambda_1}$, and $V_Q e^{\lambda_2}$ are irreducible $V_Q$-modules.*

From now on we will identify the standard modules as follows:

$$V_Q \simeq L(\Lambda_0), \quad V_Q e^{\lambda_1} \simeq L(\Lambda_1) \text{ and } V_Q e^{\lambda_2} \simeq L(\Lambda_2).$$

We now discuss intertwining operators and fusion rules for the vertex operator algebra $L(\Lambda_0)$ and its modules $L(\Lambda_0), L(\Lambda_1)$ and $L(\Lambda_2)$. See Chapter 12 of [**DL**] for a detailed exposition. In order to construct intertwining operators one needs to introduce two operators $e^{i\pi\lambda}$ and $c(\cdot\,, \lambda)$ on $V_P$ as follows:

$$e^{i\pi\lambda}(v \otimes e^{\beta}) = e^{i\pi\langle \lambda, \beta \rangle} v \otimes e^{\beta},$$

$$c(\cdot\,, \lambda)(v \otimes e^{\beta}) = c(\beta, \lambda)v \otimes e^{\beta},$$

where $v \in M(1)$ and $\beta, \lambda \in P$ (recall the commutator map (2.2) introduced above).

Let $k, r, s = 0, 1, 2$. Then

$$\begin{aligned} \mathcal{Y}(\cdot\,, x) : L(\Lambda_r) &\longrightarrow \text{Hom}\,(L(\Lambda_s), L(\Lambda_p))\{x\} \\ w &\mapsto \mathcal{Y}(w, x) = Y(w, x)e^{i\pi\lambda_r}c(\cdot\,, \lambda_r). \end{aligned} \tag{2.8}$$

defines an intertwining operator of type

$$\left( \begin{array}{c} L(\Lambda_p) \\ L(\Lambda_r) \quad L(\Lambda_s) \end{array} \right),$$

which is nonzero if and only if $p \equiv r + s \bmod 3$ (cf. [**DL**]).

In this work we will use intertwining operators of type

$$\left( \begin{array}{c} L(\Lambda_j) \\ L(\Lambda_j) \quad L(\Lambda_0) \end{array} \right), \tag{2.9}$$

where $j = 1, 2$. It is known that the dimension of the space of intertwining operators of type (2.9) is one (cf. [**DL**]). Also the operator $\mathcal{Y}(\cdot, x)$ of type (2.9) involves only integral powers of $x$ (see Remark 5.4.2 of [**FHL**]). In particular, we have the linear maps

$$\mathcal{Y}(e^{\lambda_j}, x) : L(\Lambda_0) \longrightarrow L(\Lambda_j)[[x, x^{-1}]] \tag{2.10}$$

for $j = 1, 2$.

## 3. Principal subspaces

The principal subspaces of the standard $\widehat{\mathfrak{sl}(3)}$-modules were introduced and studied by Feigin and Stoyanovsky in [**FS1**]-[**FS2**] and later by Georgiev in [**G1**]. These are defined as follows:

$$W(\Lambda_i) = U(\bar{\mathfrak{n}}) \cdot v_{\Lambda_i}, \tag{3.1}$$

for $i = 0, 1, 2$. Here $U(\bar{\mathfrak{n}})$ is the universal enveloping algebra of $\bar{\mathfrak{n}}$ and $v_{\Lambda_i}$ is a highest weight vector of $L(\Lambda_i)$.

Consider the natural surjective maps

$$\begin{aligned} f_{\Lambda_i} : U(\bar{\mathfrak{n}}) &\longrightarrow W(\Lambda_i) \\ a &\mapsto a \cdot v_{\Lambda_i} \end{aligned}$$

where $i = 0, 1, 2$. Denote by $I_{\Lambda_i}$ the annihilator of the highest weight vector $v_{\Lambda_i}$ in $U(\bar{\mathfrak{n}})$:

$$I_{\Lambda_i} = \text{Ker}\, f_{\Lambda_i}, \tag{3.2}$$

a left ideal of $U(\bar{\mathfrak{n}})$. The aim of this section is to give a precise description of the ideals $I_{\Lambda_i}$. This is equivalent to finding a complete set of relations for $W(\Lambda_i)$, that is, to giving a presentation of these principal subspaces. This question was raised and discussed partially in [**FS2**]. The presentation of the principal subspaces will be a key ingredient to obtain recursions satisfied by the graded dimensions of the principal subspaces.

We consider the following formal infinite sums:

$$R_t^{[1]} = \sum_{m_1+m_2=t} x_{\alpha_1}(m_1)x_{\alpha_1}(m_2) \tag{3.3}$$

and

$$R_t^{[2]} = \sum_{m_1+m_2=t} x_{\alpha_2}(m_1)x_{\alpha_2}(m_2) \tag{3.4}$$

for any $t \in \mathbb{Z}$. For each integer $t$, $R_t^{[1]}$ and $R_t^{[2]}$ act naturally on any highest weight $\widehat{\mathfrak{sl}(3)}$-module, in particular, on $L(\Lambda_i)$ for $i = 0, 1, 2$. In order to describe the ideals $I_{\Lambda_i}$ it will be convenient to truncate $R_t^{[1]}$ and $R_t^{[2]}$ as follows:

$$R_{t;m}^{[j]} = \sum_{\substack{m_1 + m_2 = t \\ m_1, m_2 \leq m}} x_{\alpha_1}(m_1) x_{\alpha_1}(m_2) \tag{3.5}$$

where $j = 1, 2$ and $m$ is a (possibly positive) integer. We shall often be viewing (3.5) as elements of $U(\bar{\mathfrak{n}})$ rather than as endomorphisms of modules. It will be clear from the context when expressions such as (3.5) are understood as elements of the enveloping algebra or as operators. We denote by $J$ the two-sided ideal of $U(\bar{\mathfrak{n}})$ generated by the elements $R_{t;m}^{[1]}$ and $R_{t;m}^{[2]}$ for all $t, m \in \mathbb{Z}$.

The description of $I_{\Lambda_0}$ is given by the following theorem, which was initially announced in [**FS2**]. The result regarding the ideals $I_{\Lambda_1}$ and $I_{\Lambda_2}$ is analogous and will be given at the end of this section.

THEOREM 3.1. *The annihilator of the highest weight vector of $L(\Lambda_0)$ in $U(\bar{\mathfrak{n}})$ has the following structure:*

$$I_{\Lambda_0} = J + U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^+. \tag{3.6}$$

As one can see, the inclusion $J + U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^+ \subset I_{\Lambda_0}$ follows from the highest weight vector property and from the fact that $Y(e^{\alpha_1}, x)^2$ and $Y(e^{\alpha_2}, x)^2$ are well-defined operators which equal zero on the standard module $L(\Lambda_0)$, and in particular on $W(\Lambda_0)$. The opposite inclusion requires certain results which will be stated below. The proofs are given in [**C1**].

REMARK 3.2. The action of the Virasoro algebra operator $L(0)$ on the vector space $V_P$ gives a grading of $V_P$ by "conformal weights" (cf. [**LL**]). Since

$$\text{wt } h(-m) = m$$

for $h \in \mathfrak{h}$ and $m \in \mathbb{Z}$ and

$$\text{wt } e^{\lambda} = \frac{1}{2}\langle \lambda, \lambda \rangle$$

for $\lambda \in P$, this is a $\frac{1}{6}\mathbb{Z}$-grading. In particular,

$$\text{wt } v_{\Lambda_0} = 0, \ \ \text{wt } v_{\Lambda_1} = \frac{1}{3} \ \text{ and } \ \text{wt } v_{\Lambda_2} = \frac{1}{3}.$$

We shall restrict this grading to the standard modules $L(\Lambda_i)$ and to the principal subspaces $W(\Lambda_i)$, $i = 0, 1, 2$. For any root $\alpha$ and for any integer $m$ one has that $x_\alpha(m)$, viewed as either an element of the enveloping algebra or as an operator, has weight $-m$:

$$\text{wt } x_\alpha(m) = -m.$$

For any $i = 0, 1, 2$ we have

$$L(0) I_{\Lambda_i} \subset I_{\Lambda_i}. \tag{3.7}$$

Also, for $j = 1, 2$ and $t, m \in \mathbb{Z}$ we have

$$L(0) R_{t;m}^{[j]} = -t R_{t;m}^{[j]}. \tag{3.8}$$

In particular, the ideal $J$ of $U(\bar{\mathfrak{n}})$ is $L(0)$-stable.

Let us set $\mathfrak{B} = \{x_{\alpha_2}, x_{\alpha_1}, x_{\alpha_1+\alpha_2}\}$, a basis of the Lie subalgebra $\mathfrak{n}$. We order the elements of $\mathfrak{B}$ as follows:

$$x_{\alpha_2} \prec x_{\alpha_1} \prec x_{\alpha_1+\alpha_2}.$$

Set $\bar{\mathfrak{B}} = \{x(m) \mid x \in B,\ m \in \mathbb{Z}\}$, so that $\bar{\mathfrak{B}}$ is a basis of the Lie algebra $\bar{\mathfrak{n}}$. We choose the following total order $\preceq$ on $\bar{\mathfrak{B}}$:

$$x_1(m_1) \preceq x_2(m_2) \text{ iff } x_1 \prec x_2 \text{ or } x_1 = x_2 \text{ and } m_1 \leq m_2.$$

Then by the Poincaré-Birkhoff-Witt theorem we get a basis of the universal enveloping algebra $U(\bar{\mathfrak{n}})$:

$$x_{\alpha_2}(m_1) \cdots x_{\alpha_2}(m_s) x_{\alpha_1}(n_1) \cdots x_{\alpha_1}(n_r) x_{\alpha_1+\alpha_2}(l_1) \cdots x_{\alpha_1+\alpha_2}(l_p),$$

with $m_1 \leq \cdots \leq m_s$, $n_1 \leq \cdots \leq n_r$ and $l_1 \leq \cdots \leq l_p$. We shall refer to the expressions $x_{\gamma_1}(m_1) \ldots x_{\gamma_r}(m_r)$, where $\gamma_1, \ldots, \gamma_r \in \{\alpha_1, \alpha_2, \alpha_1+\alpha_2\}$ and $m_1, \ldots, m_r \in \mathbb{Z}$, as the (noncommutative) *monomials* in $U(\bar{\mathfrak{n}})$.

Remark 3.3. Consider a homogeneous basis element of $U(\bar{\mathfrak{n}})$ of the form $x_{\alpha_2}(n_1) \cdots x_{\alpha_2}(n_s) x_{\alpha_1}(m_1) \cdots x_{\alpha_1}(m_r) x_{\alpha_1+\alpha_2}(l_1) \cdots x_{\alpha_1+\alpha_2}(l_p)$ such that $m_r, l_p \leq -1$. Then by using the brackets, this monomial can be written as a linear combination of elements $x_{\alpha_2}(m_{1,2}) \cdots x_{\alpha_2}(m_{r_2,2}) x_{\alpha_1}(m_{1,1}) \cdots x_{\alpha_1}(m_{r_1,1})$ of the same weight, such that $m_{1,2} \leq \cdots \leq m_{r_2,2}$, $m_{1,1} \leq \cdots \leq m_{r_1,1} \leq -1$, and elements of the left ideal $U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^+$. This result was proved and exploited in [**G1**].

The computation

$$\begin{aligned}
&x_{\alpha_2}(m_{r_2,2}) x_{\alpha_1}(m_{1,1}) \cdots x_{\alpha_1}(m_{r_1,1}) \\
&-\sum_{t=1}^{r_1} x_{\alpha_2}(m_{r_2,2}-1) x_{\alpha_1}(m_{1,1}) \cdots x_{\alpha_1}(m_{t,1}+1) \cdots x_{\alpha_1}(m_{r_1,1}) \\
&+\sum_{\substack{s,t=1\\ r\neq s}}^{r_1} x_{\alpha_2}(m_{r_2,2}-2) x_{\alpha_1}(m_{1,1}) \cdots x_{\alpha_1}(m_{s,1}+1) \cdots x_{\alpha_1}(m_{t,1}+1) \cdots x_{\alpha_1}(m_{r_1,1}) \\
&\qquad\vdots \\
&+(-1)^{r_1} x_{\alpha_2}(m_{r_2,2}-r_1) x_{\alpha_1}(m_{1,1}+1) \cdots x_{\alpha_1}(m_{r_1,1}+1) \in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^+
\end{aligned}$$

leads to the following:

Lemma 3.4. *Let $r_1, r_2 \geq 1$ and $m_{1,2}, \ldots, m_{r_2,2}, m_{1,1}, \ldots, m_{r_1,1}$ be integers such that $m_{1,2} \leq \cdots \leq m_{r_2,2}$ and $m_{1,1} \leq \cdots \leq m_{r_1,1} \leq -1$. Assume that $m_{r_2,2} \geq r_1$. Then*

$$x_{\alpha_2}(m_{1,2}) \cdots x_{\alpha_2}(m_{r_2,2}) x_{\alpha_1}(m_{1,1}) \cdots x_{\alpha_1}(m_{r_1,1}) \tag{3.9}$$

*is a linear combination of monomials of the same weight as (3.9),*

$$x_{\alpha_2}(m'_{1,2}) \cdots x_{\alpha_2}(m'_{r_2,2}) x_{\alpha_1}(m'_{1,1}) \cdots x_{\alpha_1}(m'_{r_1,1}), \tag{3.10}$$

*with*

$$m'_{1,2} \leq \cdots \leq m'_{r_2,2} \leq r_1 - 1,\ m'_{1,1} \leq \cdots \leq m'_{r_1,1} \leq -1, \tag{3.11}$$

*and monomials of $U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^+$. In particular, any homogeneous element of $U(\bar{\mathfrak{n}})$ can be written as a sum of two homogeneous elements, one such that the corresponding sequences of integers satisfy (3.11) and the other an element of $U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^+$.*

Using the previous lemma, one can deduce that the principal subspace $W(\Lambda_0)$ of $L(\Lambda_0)$ is spanned by the elements

$$x_{\alpha_2}(m_{1,2})\cdots x_{\alpha_2}(m_{r_2,2})x_{\alpha_1}(m_{1,1})\cdots x_{\alpha_1}(m_{r_1,1})\cdot 1 \tag{3.12}$$

such that

$$m_{1,2}\leq\cdots\leq m_{r_2,2}\leq r_1-1 \ \text{ and } \ m_{1,1}\leq\cdots\leq m_{r_1,1}\leq -1. \tag{3.13}$$

Let $\alpha=\alpha_1$ or $\alpha_2$ and $m_1,\dots,m_r\in\mathbb{Z}$ for $r\geq 2$. We say that a monomial $x_\alpha(m_1)\cdots x_\alpha(m_r)$ satisfies the *difference two condition* if the corresponding sequence $m_1,\dots,m_r$ satisfies the two difference condition, i.e.,

$$m_l-m_{l-1}\geq 2 \quad \text{for any } \ l, \ \ 2\leq l\leq r.$$

Now by using the elements $R^{[1]}_{t;m}$ and $R^{[2]}_{t;m}$ (recall (3.5)) and certain linear maps associated with the intertwining operators (2.10) we obtain the following result:

Lemma 3.5. *Let $r_1,r_2\geq 1$ and let $m_{1,2},\dots,m_{r_2,2},m_{1,1},\dots,m_{r_1,1}$ be integers such that $m_{1,2}\leq\cdots\leq m_{r_2,1}\leq r_1-1$ and $m_{1,1}\leq\cdots\leq m_{r_1,1}\leq -1$. Then*

$$M=x_{\alpha_2}(m_{1,2})\cdots x_{\alpha_2}(m_{r_2,2})x_{\alpha_1}(m_{1,1})\cdots x_{\alpha_1}(m_{r_1,1})$$

*can be expressed as*

$$M=M_1+M_2+M_3,$$

*where $M_1$ is a linear combination of monomials satisfying the difference two condition, $M_2\in J$, $M_3\in U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^+$, and such that* $\mathrm{wt}\ \mathrm{M}_1=\mathrm{wt}\ \mathrm{M}_2=\mathrm{wt}\ \mathrm{M}_3=\mathrm{wt}\ \mathrm{M}$. *Furthermore, any linear combination of monomials satisfying the difference two condition (such as $M_1$) that belongs to $I_{\Lambda_0}$ must be zero.*

Now Lemmas 3.4 and 3.5 yield the inclusion

$$I_{\Lambda_0}\subset J+U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^+,$$

and thus Theorem 3.1. The desired result regarding the presentation of the principal subspaces $W(\Lambda_1)$ and $W(\Lambda_2)$ is:

Theorem 3.6. *The annihilators of highest weight vectors of $L(\Lambda_1)$ and $L(\Lambda_2)$ in $U(\bar{\mathfrak{n}})$ are described as follows:*

$$I_{\Lambda_1}=J+U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^++U(\bar{\mathfrak{n}})x_{\alpha_1}(-1) \tag{3.14}$$

*and*

$$I_{\Lambda_2}=J+U(\bar{\mathfrak{n}})\bar{\mathfrak{n}}^++U(\bar{\mathfrak{n}})x_{\alpha_2}(-1). \tag{3.15}$$

Combining Theorems 3.1 and 3.6 we obtain the discrepancy between the ideals $I_{\Lambda_0}$ and $I_{\Lambda_i}$ for $i=1,2$; this is important in the proof of the exactness of our sequences of maps among principal subspaces:

Corollary 3.7. *We have*

$$I_{\Lambda_1}=I_{\Lambda_0}+U(\bar{\mathfrak{n}})x_{\alpha_1}(-1) \tag{3.16}$$

*and*

$$I_{\Lambda_2}=I_{\Lambda_0}+U(\bar{\mathfrak{n}})x_{\alpha_2}(-1). \tag{3.17}$$

## 4. Main result

In this section, by setting up two exact sequences for the principal subspaces of the level 1 standard $\widehat{\mathfrak{sl}(3)}$-modules (see Theorem 4.2), we are able to find a complete set of Rogers-Ramanujan-type recursions (see Theorem 4.3), which characterize the graded dimensions of the principal subspace $W(\Lambda_0)$.

Recall from Remark 3.2 the grading of $V_P$ by "conformal weights". The vector space $V_P$ has gradings by "charge" given by the eigenvalues of the operators $\lambda_1$ and $\lambda_2$ (thought as $\lambda_1(0)$ and $\lambda_2(0)$), and these are compatible with the weight grading. We shall consider these gradings restricted to the principal subspaces $W(\Lambda_0)$, $W(\Lambda_1)$ and $W(\Lambda_2)$.

Now for $i = 0, 1, 2$ we consider the graded dimensions of $W(\Lambda_i)$ (the generating functions of the dimensions of the homogeneous subspaces), where we use the formal variables $x_1, x_2$ and $q$:

$$\chi_i(x_1, x_2; q) = \dim_*(W(\Lambda_i), x_1, x_2; q) = tr|_{W(\Lambda_i)} x_1^{\Lambda_1} x_2^{\Lambda_2} q^{L(0)}.$$

The linear maps

$$e^{\lambda_1} : V_P \longrightarrow V_P \ \text{ and } \ e^{\lambda_2} : V_P \longrightarrow V_P$$

are clearly isomorphisms with $e^{-\lambda_1}$ and $e^{-\lambda_2}$ as inverses. The restrictions of these maps to $W(\Lambda_0)$ are the linear maps

$$e^{\lambda_1} : W(\Lambda_0) \longrightarrow W(\Lambda_1) \ \text{ and } \ e^{\lambda_2} : W(\Lambda_0) \longrightarrow W(\Lambda_2). \tag{4.1}$$

One can view (4.1) as essentially the "constant factors" of $\mathcal{Y}(e^{\lambda_j}, x)$ for $j = 1, 2$. By using the maps (4.1) we obtain relations between the graded dimensions of the principal subspaces of the level 1 standard $\widehat{\mathfrak{sl}(3)}$-modules.

Proposition 4.1. *We have*

$$\chi_1(x_1, x_2; q) = x_1^{2/3} x_2^{1/3} q^{1/3} \chi_0(x_1 q, x_2; q) \tag{4.2}$$

*and*

$$\chi_2(x_1, x_2; q) = x_1^{1/3} x_2^{2/3} q^{1/3} \chi_0(x_1, x_2 q; q). \tag{4.3}$$

Now we consider the weights $\lambda^j = \alpha_j - \lambda_j \in P$ for $j = 1, 2$ and the linear isomorphisms $e^{\lambda^j} : V_P \longrightarrow V_P$. The restrictions of $e^{\lambda^j}$ to $W(\Lambda_j)$ are the linear maps

$$e^{\lambda^1} : W(\Lambda_1) \longrightarrow W(\Lambda_0) \ \text{ and } \ e^{\lambda^2} : W(\Lambda_2) \longrightarrow W(\Lambda_0). \tag{4.4}$$

Then it follows that

$$\begin{aligned} &e^{\lambda^j}(x_{\alpha_i}(m_{1,i}) \cdots x_{\alpha_i}(m_{r_i,i}) \cdot v_{\Lambda_j}) \\ &\quad = x_{\alpha_i}(m_{1,i} - \langle \lambda^j, \alpha_i \rangle) \cdots x_{\alpha_i}(m_{r_i,i} - \langle \lambda^j, \alpha_i \rangle) x_{\alpha_j}(-1) \cdot v_{\Lambda_0} \end{aligned}$$

for any $r_j > 0$, $m_{1,i}, \ldots, m_{r_i,i} \in \mathbb{Z}$ and $i, j = 1, 2$. These maps are injective.

Recall from the end of Section 2 the intertwining operators

$$\mathcal{Y}(e^{\lambda_j}, x) : L(\Lambda_0) \longrightarrow L(\Lambda_j)[[x, x^{-1}]], \tag{4.5}$$

where $j = 1, 2$. As in [**G1**] and [**CapLM1**]–[**CapLM2**] we consider the constant terms (the coefficients of $x^0$) of these intertwining operators and denote them by $\mathcal{Y}_c(e^{\lambda_j}, x)$. These are linear maps between principal spaces

$$\mathcal{Y}_c(e^{\lambda_1}, x) : W(\Lambda_0) \longrightarrow W(\Lambda_1) \ \text{ and } \ \mathcal{Y}_c(e^{\lambda_2}, x) : W(\Lambda_0) \longrightarrow W(\Lambda_2). \tag{4.6}$$

The map $\mathcal{Y}_c(e^{\lambda_j}, x)$ has the property that it sends the highest weight vector $v_{\Lambda_0}$ to $v_{\Lambda_j}$ for $j = 1, 2$ (after certain normalization of $\mathcal{Y}(e^{\lambda_j}, x)$ if necessary):

$$\mathcal{Y}_c(e^{\lambda_j}, x)(v_{\Lambda_0}) = v_{\Lambda_j}.$$

Also, by the Jacobi identity in the definition of the notion of intertwining operator one sees in particular that $\mathcal{Y}_c(e^{\lambda_j}, x)$ commutes with the action of $U(\bar{\mathfrak{n}})$:

$$[\mathcal{Y}_c(e^{\lambda_j}, x), U(\bar{\mathfrak{n}})] = 0.$$

Thus the maps (4.6) are surjective.

The main result of this paper is the following theorem, which gives exact sequences of maps among principal subspaces. To prove this theorem we use properties of intertwining operators and Corollary 3.7.

THEOREM 4.2. *Recall the linear maps $e^{\lambda^1}$, $e^{\lambda^2}$, $\mathcal{Y}_c(e^{\lambda_1}, x)$ and $\mathcal{Y}_c(e^{\lambda_2}, x)$ introduced above (see (4.4) and (4.6)). The sequences of maps between principal subspaces*

$$0 \longrightarrow W(\Lambda_1) \xrightarrow{e^{\lambda^1}} W(\Lambda_0) \xrightarrow{\mathcal{Y}_c(e^{\lambda_1},x)} W(\Lambda_1) \longrightarrow 0 \tag{4.7}$$

*and*

$$0 \longrightarrow W(\Lambda_2) \xrightarrow{e^{\lambda^2}} W(\Lambda_0) \xrightarrow{\mathcal{Y}_c(e^{\lambda_2},x)} W(\Lambda_2) \longrightarrow 0 \tag{4.8}$$

*are exact.*

As a consequence of this theorem we obtain recursions satisfied by the graded dimension $\chi_0(x_1, x_2; q)$ of the principal subspace $W(\Lambda_0)$.

THEOREM 4.3. *We have*

$$\chi_0(x_1, x_2; q) = \chi_0(x_1 q, x_2; q) + x_1 q \chi_0(x_1 q^2, x_2 q^{-1}; q) \tag{4.9}$$

*and*

$$\chi_0(x_1, x_2; q) = \chi_0(x_1, x_2 q; q) + x_2 q \chi_0(x_1 q^{-1}, x_2 q^2; q). \tag{4.10}$$

Now by solving the recursions (4.9), (4.10) and by using Proposition 4.1 we obtain the graded dimensions of $W(\Lambda_0)$, $W(\Lambda_1)$ and $W(\Lambda_2)$.

COROLLARY 4.4. *Using the notation*

$$(q)_m = (1-q)(1-q^2)\cdots(1-q^m) \quad \textit{for} \quad m \geq 0$$

*we have*

$$\chi_0(x_1, x_2; q) = \sum_{r_1, r_2 \geq 0} \frac{q^{r_1^2 + r_2^2 - r_1 r_2}}{(q)_{r_1} (q)_{r_2}} x_1^{r_1} x_2^{r_2}, \tag{4.11}$$

$$\chi_1(x_1, x_2; q) = x_1^{2/3} x_2^{1/3} q^{1/3} \sum_{r_1, r_2 \geq 0} \frac{q^{r_1^2 + r_2^2 - r_1 r_2 + r_1}}{(q)_{r_1} (q)_{r_2}} x_1^{r_1} x_2^{r_2} \tag{4.12}$$

*and*

$$\chi_2(x_1, x_2; q) = x_1^{1/3} x_2^{2/3} q^{1/3} \sum_{r_1, r_2 \geq 0} \frac{q^{r_1^2 + r_2^2 - r_1 r_2 + r_2}}{(q)_{r_1} (q)_{r_2}} x_1^{r_1} x_2^{r_2}. \tag{4.13}$$

Thus we have recovered the formulas for the graded dimensions of $W(\Lambda_i)$ for $i = 0, 1, 2$ previously obtained in [**G1**] by a different method.

Remark 4.5. In [**C2**] we have obtained recursions that characterize the graded dimensions of certain principal subspaces of higher-level standard $\widehat{\mathfrak{sl}(3)}$-modules, and these combined with the known graded dimensions of principal subspaces with highest weights $i\Lambda_0 + (k-i)\Lambda_j$, where $0 \leq i \leq k$ and $j = 1, 2$, enable us to derive the graded dimensions of the principal subspaces with highest weights of type $i\Lambda_1 + (k-i)\Lambda_2$ for $1 \leq i \leq k$.

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