MATH 649 HW 2

1. Show that a subset of ω is definable in $(\omega, 0, \mathsf{S}, <)$ iff it is finite or its complement is finite. Hint: Use the fact that the theory of the structure admits quantifier elimination.

2. Problem 4 of section 3.2.

3. Assume that $\mathcal{A}$ and $\mathcal{B}$ are models of a theory T which admits QE. Show that if $\mathcal{A}$ is a substructure of $\mathcal{B}$ then $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$

4. A formula is *positive* if it contains no occurrences of $\neg$ (equivalently, if it is in the set generated from the atomic formulas by $\wedge$, $\vee$, $\forall$ and $\exists$). A theory T *admits positive QE* if every formula is equivalent under T to a positive quantifier free formula. Prove the following *Positive Qunatifier Elimination Theorem*:

> If T is a theory such that
>
> (a) The negation of any atomic formula is equivalent to a positive quantifier free formula in T.
>
> (b) For any finite sequence of atomic formulas $\alpha_1, \ldots, \alpha_n$ and variable x, $\exists x(\alpha_1 \wedge \cdots \wedge \alpha_n)$ is equivalent to a quantifier free formula in T.
>
> then T admits positive elmination of quantifiers.

Hint: First show that any quantifier free formula is equivalent to a positive quantifier free formula in T by the induction principle. Use the induction principle again for the general case. You may use the fact that any positive quantifier free formula is logically equivalent to a positive quantifier free formula in disjunctive normal form.