1. (d) There exist two real numbers which are not equal to each other and whose product is equal to their sum.

Proof. Let $x = 3$ and $y = \frac{3}{2}$.

Clearly, $x, y \in \mathbb{R}$ and $x \neq y$. Moreover,

$$\begin{aligned} xy &= 3 \cdot \tfrac{3}{2} \\ &= \tfrac{9}{2} \end{aligned}$$

and

$$\begin{aligned} x + y &= 3 + \tfrac{3}{2} \\ &= \tfrac{9}{2} \end{aligned}$$

Therefore, $xy = x + y$.

We have shown there are $x, y \in \mathbb{R}$ such that $x \neq y$ and $xy = x + y$. **QED**

2. (b) For any integers a, b, c and d, if a divides b and c divides d then ac divides bd.

Proof. Assume a, b, c and d are integers such that a divides b and c divides d. We will show ac divides bd.

Since a divides b, there is $n \in \mathbb{Z}$ such that $b = an$. Since c divides d, there is $m \in \mathbb{Z}$ such that $d = cm$. Let $k = nm$. Since products of integers are integers, k is an integer. Moreover,

$$\begin{aligned} bd &= (an)(cm) \\ &= (ac)(nm) \\ &= (ac)k \end{aligned}$$

We have shown there is an integer k such that $bd = (ac)k$. By definition, ac divides bd.

Since a, b, c and d were arbitrary integers such that a divides b and c divides d, for all integers a, b, c and d, if a divides b and c divides d then ac divides bd. **QED**

3.(b) For any integer n, $n^2 + n$ is even.

Proof. Assume $n \in \mathbb{Z}$. We will show that $n^2 + n$ is even.

By the Quotient-Remainder Theorem, there are $q, r \in \mathbb{Z}$ such that $n = 2q + r$ and $0 \leq r < 2$. Since r is an integer and $0 \leq r < 2$, $r = 0$ or $r = 1$. We will argue by cases.

Case 1: Assume $r = 0$.

Let $k = 2q^2 + q$. Since sums and products of integers are integers, $k \in \mathbb{Z}$. Moreover,

$$\begin{aligned} n^2 + n &= (2q + r)^2 + (2q + r) \\ &= (2q + 0)^2 + (2q + 0) \\ &= 4q^2 + 2q \\ &= 2(2q^2 + q) \\ &= 2k \end{aligned}$$

We have shown there is an integer k such that $n^2 + n = 2k$. By definition, $n^2 + n$ is even.

Case 2: Assume $r = 1$.

Let $m = 2q^2 + 3q + 1$. Since sums and products of integers are integers, $m \in \mathbb{Z}$. Moreover,

$$\begin{aligned} n^2 + n &= (2q + r)^2 + (2q + r) \\ &= (2q + 1)^2 + (2q + 1) \\ &= 4q^2 + 6q + 2 \\ &= 2(2q^2 + 3q + 1) \\ &= 2m \end{aligned}$$

We have shown there is an integer m such that $n^2 + n = 2m$. By definition, $n^2 + n$ is even.

In every case, $n^2 + n$ is even.

Since $n \in \mathbb{Z}$ was arbitrary, for all $n \in \mathbb{Z}$, $n^2 + n$ is even. **QED**

4.(b) For every $n \in \mathbb{Z}$, if $n \geq 1$ then 4 divides $6^n - 2^n$.

Proof. We will use mathematical induction. Let $P(n)$ be the property of integers n such that

$$P(n) \text{ iff } 4 \text{ divides } 6^n - 2^n$$

(basis step) We will show $P(1)$ i.e. 4 divides $6^1 - 2^1$.

Let $x = 1$. Clearly, $x \in \mathbb{Z}$. Moreover,

$$\begin{aligned} 6^1 - 2^1 &= 4 \\ &= 4 \cdot 1 \\ &= 4x \end{aligned}$$

We have shown there is $x \in \mathbb{Z}$ such that $6^1 - 2^1 = 4x$. By definition, $6^1 - 2^1$ is divisible by 4.

(inductive step) Assume $k \in \mathbb{Z}$ such that $k \geq 1$ and $P(k)$ i.e. 4 divides $6^k - 2^k$. We will show $P(k+1)$ i.e. 4 divides $6^{k+1} - 2^{k+1}$.

Since 4 divides $6^k - 2^k$, there is $y \in \mathbb{Z}$ such that $6^k - 2^k = 4y$. Let $m = 6y + 2^k$. Since sums, products and positive powers of integers are integers, $m \in \mathbb{Z}$. Moreover,

$$\begin{aligned} 6^{k+1} - 2^{k+1} &= 6 \cdot 6^k - 2^{k+1} \\ &= 6 \cdot (4y + 2^k) - 2^{k+1} \\ &= 24y + 6 \cdot 2^k - 2^{k+1} \\ &= 24y + 6 \cdot 2^k - 2 \cdot 2^k \\ &= 24y + 4 \cdot 2^k \\ &= 4 \cdot (6y + 2^k) \\ &= 4m \end{aligned}$$

We have shown there is $m \in \mathbb{Z}$ such that $6^{k+1} - 2^{k+1} = 4m$. By definition, 4 divides $6^{k+1} - 2^{k+1}$.

By the principle of mathematical induction, for every integer n, if $n \geq 1$ then 4 divides $6^k - 2^k$.
QED

5.(a) Suppose $c_0, c_1, c_2, \ldots$ is a sequence defined as follows:

$c_0 = 0, c_1 = 1,$

$c_k = 2c_{k-1} - c_{k-2} + 2$ for all integers $k \geq 2$.

Prove that $c_n = n^2$ for all integers $n \geq 0$.

Proof. We will use strong mathematical induction. Let $P(n)$ be the property of integers n such that

$$P(n) \quad \text{iff} \quad c_n = n^2$$

(basis step) We will prove $P(0)$ and $P(1)$ i.e. $c_0 = 0^2$ and $c_1 = 1^2$.
$c_0 = 0 = 0^2$ and $c_1 = 1 = 1^2$.

(inductive step) Assume $k \in \mathbf{Z}$, $k > 1$, and $P(i)$ holds for all integers i with $0 \leq i < k$ i.e. $c_i = i^2$ for all integers i with $0 \leq i < k$. We will show that $P(k)$ i.e. $c_k = k^2$.

$$\begin{aligned} c_k &= 2c_{k-1} - c_{k-2} + 2 \\ &= 2(k-1)^2 - (k-2)^2 + 2 \qquad \text{(by the inductive hypothesis)} \\ &= 2(k^2 - 2k + 1) - (k^2 - 4k + 4) + 2 \\ &= k^2 \end{aligned}$$

By strong mathematical induction, for every integer n, if $n \geq 0$ then $c_n = n^2$. *QED*

6.(a) There is no smallest real number x such that $1 < x < 2$.

Proof. We will argue by contradiction. Assume there is a smallest real number x such that $1 < x < 2$ i.e. there is a real number x such that $1 < x < 2$ and for all real numbers y, if $1 < y < 2$ then $x \leq y$. Let $z = \frac{1}{2}(1 + x)$. Since sums and products of real numbers are real numbers, $z \in \mathbb{R}$. Moreover,

$$\begin{aligned} 1 &= \tfrac{1}{2} \cdot (1 + 1) \\ &< \tfrac{1}{2}(1 + x) \\ &= z \end{aligned}$$

and

$$\begin{aligned} z &= \tfrac{1}{2}(1 + x) \\ &< \tfrac{1}{2}(x + x) \\ &= x \end{aligned}$$

Since $x < 2$ and $z < x$, $z < 2$.

We have shown there is $z \in \mathbb{R}$ such that $1 < z < 2$ and $z < x$. Since for all real numbers y, if $1 < y < 2$ then $x \leq y$, we must have $x \leq z$. Therefore, $z \not< x$ – a contradiction.

By the method of arguing by contradiction, there is no smallest real number x wuch that $1 < x < 2$. **QED**

7.(a) Let $A = \{a, c, d\}$ and $B = \{b, c, f\}$ be subsets of the universal set $U = \{a, b, c, d, e, f, g\}$.
Solution:

$$A \cup B = \{a, b, c, d, f\}$$

$$A \cap B = \{c\}$$

$$A - B = \{a, d\}$$

$$A^c = \{b, e, f, g\}$$

$$A \times B = \{(a, b), (a, c), (a, f), (c, b), (c, c), (c, f), (d, b), (d, c), (d, f)\}$$