Math 366 Practice Problems for Final Exam **Spring 2009**

1. Each of the following statements has one of the forms

$$\sim p \qquad p \wedge q \qquad p \vee q \qquad p \to q \qquad p \leftrightarrow q$$

Find the appropriate form and indicate what each statement variable in your choice represents.

(a) If Archibald passes the first exam, then he will not drop the course.

(b) The moon is not made of green cheese.

(c) Harry got out of bed and brushed his teeth.

2. Use truth tables to verify each of the following logical equivalences.

(a) $p \vee (\sim p \wedge q) \;\equiv\; p \vee q$

(b) $p \leftrightarrow (p \wedge q) \;\equiv\; p \to q$

(c) $p \to (q \vee r) \;\equiv\; (p \wedge \sim q) \to r$

3. Determine which of the following argument forms are valid and which are not. Justify your answers. If the form is valid, verify that it is by two methods: truth tables and step by step derivations using theorem 1.1.1 and table 1.3.1 from the text.

(a) $p \vee q$

$\therefore\ p$

(b) $p \to (q \to r)$

$\sim r$

p

$\therefore\ \sim q$

(c) $\sim q$

$\therefore\ \sim (p \wedge q)$

(d) $\sim p \to q$

$\sim q$

$\therefore\ p$

(e) p

q

$\therefore (p \wedge q) \vee r$

(f) $\sim p \wedge q$

$p \vee r$

$\therefore r$

4. Use truth tables to show that the following argument form is valid.

$p \rightarrow (q \rightarrow r)$

$\sim r$

p

$\sim q$

5. Make up an I/O table for three inputs P, Q, and R. Find a Boolean expression for your I/O table.

6. Each of the expressions below has one of the forms

$$\forall x \in D, P(x) \qquad \exists x \in D \; s.t. \; P(x)$$

Determine the appropriate form and indicate the interpretation of the domain D and the predicate $P(x)$.
(a) Everyone in the class who works hard will pass.
(b) Someone is sleeping.

7. Find a counterexample for each of the following universal statements.
(a) $\forall x \in \mathbf{R}, x^3 \neq -x$
(b) Everyone who lives in Columbus has been to Madagascar.

8. Give the first sentence of a direct proof of each of the following statements. Also indicate what remains to be proved.
(a) The product of two rational numbers is rational.
(b) If an integer is prime and different from 2 then it is odd.

9.Constructive Proof of an Existential Statement.
(a) Prove that there is an even integer n and an integer m such that $n = 3m + 1$.
(b) Prove that there exists a rational number q such that $9\,q^2 = 4$.
(c) Prove that there exist two real numbers whose product is less than their sum.
(d) Prove that there exist two real numbers which are not equal such that $xy = x + y$.
(e) Prove that there exist sets A and B such that $A - B = A \cup B$.
(f) Prove that there exist sets A and B such that $A - B \neq A \cup B$.

10. Direct Proof of a Universal Statement.
(a) Prove that if n is an integer which is divisible by 6 then n is divisible by 3.
(b) Prove that for any integers a, b, c, and d, if a divides b and c divides d then $a \cdot c$ divides $b \cdot d$.
(c) Prove that the square of every even integer is divisible by 4.

(d) Prove that for any sets A, B, and C, if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.
(e) Prove that for any sets A, B, C and D, if $A \subseteq B$ and $C \subseteq D$ then $A \times C \subseteq B \times D$.

11. Proof by Cases.
(a) Prove that for every integer n, n and $n+2$ have the same parity (i.e. either n and $n+2$ are both even or n and $n+2$ are both odd).
(b) Prove that for any integer n, $n^2 + n$ is even.
(c) Prove that for any integer n, if 3 divides $2n$ then 3 divides n. Hint: Use the Quotient-Remainder Theorem with divisor 3 i.e. use the fact that for any integer k there is an integer q such that either $k = 3q$, $k = 3q + 1$ or $k = 3q + 2$.
(d) Prove that for any integers n and m, $n^2 + 3m \neq 2$. Hint: Use the Quotient-Remainder Theorem with divisor 3.
(e) Prove that for any sets A, B, and C, if $A \subseteq C$ then $A \cup (B \cap C) \subseteq C$.

12. Mathematical Induction.
(a) Prove that for any integer n, if $n \geq 0$ then 4 divides $5^n - 1$.
(b) Prove that for any integer n, if $n \geq 1$ then 4 divides $6^n - 2^n$.
(c) Show that $2n + 1 < 2^n$ for every integer n with $n \geq 3$.
(d) Using the fact that $2n + 1 < 2^n$ for every integer n with $n \geq 3$, show that for every integer n, if $n \geq 5$ then $n^2 < 2^n$.

13. Strong Mathematical Induction.
(a) Suppose $c_0, c_1, c_2, \ldots$ is a sequence defined as follows:

$$c_0 = 0, c_1 = 1,$$

$$c_k = 2c_{k-1} - c_{k-2} + 2 \text{ for all integers } k \geq 2.$$

Prove that $c_n = n^2$ for all integers $n \geq 0$.
(b) Suppose $c_0, c_1, c_2, \ldots$ is a sequence defined as follows:

$$c_0 = 2, c_1 = 5,$$

$$c_k = 5c_{k-1} - 6c_{k-2} \text{ for all integers } k \geq 2.$$

Prove that $c_n = 2^n + 3^n$ for all integers $n \geq 0$.

14. Proof by Contradiction or Contraposition.
(a) Prove that there is not a largest odd integer.
(b) For any integer n, if n^2 is odd then n is odd.

15. Computations with Sets.
(a) Let $A = \{a, c, d\}$ and $B = \{b, c, f\}$. Compute $A \cup B$, $A \cap B$, $A - B$ and $A \times B$ using "bracket" notation.
(b) Let $A = \{1, 3\}$ and $B = \{2, 3\}$. Compute $A \cup B$, $A \cap B$, $A - B$, and $A \times B$ using "bracket" notation.

16. More Proofs with Sets.
(a) Prove or disprove: For all sets A and B, if $A \subseteq B$ then $A \cap B = A$.
(b) Prove or disprove: For all sets A and B, $(A - B) \cup (B - A) = A \cup B$.
(c) Prove or disprove: For any sets A, B, and C, if $A \subseteq B$ then $A - (B \cap C) \subseteq A - C$.
(d) Prove or disprove: For any sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap C$.

17. Functions.
Choose finite sets A and B. Also choose a relation R from A to B.
(a) Find $dom(R)$ and $ran(R)$.
(b) Find R^{-1}.
(c) Is R a function from A to B? If so, find $R(x)$ for each $x \in A$.
(d) Is R a 1-1 function from A to B?
(e) Is R a function from A onto B?

18. Determining whether a relation is a function.
Determine whether the following relations are functions. Justify your answers with proofs.
(a) R is the binary relation on $\mathbb{R}$ determined by xRy iff $x^2 + y^2 = 2$.
(b) R is the binary relation on $\mathbb{R}$ determined by xRy iff $x = y^2$.
(c) R is the following relation from $\{0, 1, 2\}$ to $\{0, 1, 2, 3\}$: $\{(0, 0), (1, 2), (2, 1)\}$

19. 1-1 functions.
Determine whether each of the following functions is 1-1. Provide a proof of your answer.
(a) $f : \{0, 1, 2\} \to \{a, b, c, d\}$ where $f(0) = d$, $f(1) = b$, and $f(2) = c$.
(b) Make up a function $f : X \to Y$ where $X = \{a, b, c, d\}$ and $Y = \{0, 1, 2\}$ by drawing it's arrow diagram. Is it 1-1?
(c) The function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = -2x + 1$.
(d) The function $h : \mathbb{R} \to [0, \infty)$ given by $h(x) = x^2$.
(e) The function $h : \mathbb{Z} \to \mathbb{Z}$ given by $h(n) = n^3$.
(f) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.
(g) The function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = x^3 - x$.
(h) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$.
(i) The function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = sin\, x$.

20. Onto functions.
Determine whether each of the functions in problem 19 is onto. Provide a proof of your answer.
Hint: Part (g) is tricky. You may want to use the intermediate value theorem from calculus.