Solutions to Selected Math 366 Final Exam Practice Problems **Spring 2009**

1. Each of the following statements has one of the forms

$$\sim p \qquad p \wedge q \qquad p \vee q \qquad p \rightarrow q \qquad p \leftrightarrow q$$

Find the appropriate form and indicate what each statement variable in your choice represents.

(a) If Archibald passes the first exam, then he will not drop the course.

Solution. The statement has form

$$p \rightarrow q$$

where
$p =$ "Archibald passes the first exam."
$q =$ "Archibald will not drop the course."

2. Use truth tables to verify each of the following logical equivalences.

(a) $p \vee (\sim p \wedge q) \equiv p \vee q$

Solution.

p	q	$\sim p$	$\sim p \wedge q$	$p \vee (\sim p \wedge q)$	$p \vee q$
T	T	F	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	F	T	F	F	F

Since $p \vee (\sim p \wedge q)$ and $p \vee q$ have the same truth table, $p \vee (\sim p \wedge q) \equiv p \vee q$.

3. Determine which of the following argument forms are valid and which are not. Justify your answers. If the form is valid, verify that it is by two methods: truth tables and step by step derivations using theorem 1.1.1 and table 1.3.1 from the text.

(b) $p \rightarrow (q \rightarrow r)$
$\sim r$
p
$\therefore \sim q$

Solution.
Premises:

$p \rightarrow (q \rightarrow r)$

$\sim r$

p

1. $p \rightarrow (q \rightarrow r)$ premise
p premise
$\therefore$ $q \rightarrow r$ by MP

2. $q \to r$ from 1
$\sim r$ premise
$\therefore \sim q$ by MT

5. Find a Boolean expression which has the following I/O table.

P	Q	R	*ouput*
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	0

The recognizers for the rows with output 1 are $P \wedge Q \wedge R$, $P \wedge \sim Q \wedge \sim R$ and $\sim P \wedge \sim Q \wedge R$. Therefore,

$$(P \wedge Q \wedge R) \vee (P \wedge \sim Q \wedge \sim R) \vee (\sim P \wedge \sim Q \wedge R)$$

has the above I/O table.

6. Each of the expressions below has one of the forms

$$\forall x \in D, P(x) \qquad \exists x \in D \; s.t. \; P(x)$$

Determine the appropriate form and indicate the interpretation of the domain D and the predicate $P(x)$.
(a) Everyone in the class who works hard will pass.
Solution. The form is

$$\forall x \in D, P(x)$$

where
D = the domain of people in the class who work hard
$P(x)$ = "x will pass."

7. Find a counterexample for each of the following universal statements.

(a) $\forall x \in \mathbf{R}, x^3 \neq -x$

Solution. 0 is a counterexample.

8. Give the first sentence of a direct proof of each of the following statements. Also indicate what remains to be proved.

(b) If an integer is prime and different from 2 then it is odd.

Solution. The first two sentences of the proof are

Assume n is an integer such that n is prime and $n \neq 2$. We will show that n is odd.

The second sentence is what remains to be proved.

9.Constructive Proof of an Existential Statement.

(f) Prove that there exist sets A and B such that $A - B \neq A \cup B$.

Proof. Let $A = \emptyset$ and $B = \{0\}$. Clearly, A and B are sets. Also

$$A - B \;=\; \emptyset - \{0\} \;=\; \emptyset$$

and

$$A \cup B \;=\; \emptyset \cup \{0\} \;=\; \{0\}$$

We see that $0 \notin A - B$ and $0 \in A \cup B$. Therefore, $A - B \;\neq\; A \cup B$.

We have shown that there are sets A and B such that $A - B \;\neq\; A \cup B$. *QED*

10. Direct Proof of a Universal Statement.

(d) Prove that for any sets A, B, and C, if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$.

Proof. Assume that A, B and C are sets such that $A \subseteq B$ and $A \subseteq C$. We will show that $A \subseteq B \cap C$.

Assume $x \in A$. We will show that $x \in B \cap C$. Since $x \in A$ and $A \subset B$, $x \in B$. Since $x \in A$ and $A \subseteq C$, $x \in C$. Since $x \in B$ and $x \in C$, $x \in B \cap C$ by definition.

Since $x \in A$ was arbitrary, we have shown that for all x, if $x \in A$ then $x \in B \cap C$. By definition, $A \subseteq B \cap C$.

Since A, B and C were arbitrary sets such that $A \subseteq B$ and $A \subseteq C$, for all sets A, B, and C, if $A \subseteq B$ and $A \subseteq C$ then $A \subseteq B \cap C$. *QED*

11. Proof by Cases.

(c) Prove that for any integer n, $n^2 + n$ is even.

Proof. Assume n is an arbitrary integer. We will show that $n^2 + n$ is even.

By the Quotient-Remainder Theorem, there are integers q and r such that $n \;=\; 2q \;+\; r$ and $0 \leq r < 2$. Since r is an integer and $0 \leq r < 2$, $r = 0$ or $r = 1$. We will argue by cases.

Case 1: Assume $r = 0$.

Let $k = 2q^2 + 2$. By the axiom SPI, k is an integer. Also

$$\begin{aligned} n^2 + n \;&=\; (2q + 0)^2 + (2q + 0) \\ &=\; 4q^2 + 2q \\ &=\; 2(2q^2 + q) \\ &=\; 2k \end{aligned}$$

We have shown there is an integer k such that $n^2 + n = 2k$. By definition, $n^2 + n$ is even.

Case 2: Assume $r = 1$.

Let $k = 2q^2 + 4q + 1$. By the axiom SPI, k is an integer. Also

$$\begin{aligned} n^2 + n \;&=\; (2q + 1)^2 + (2q + 1) \\ &=\; 4q^2 + 6q + 2 \\ &=\; 2(2q^2 + 4q + 1) \end{aligned}$$

$$= 2k$$

We have shown that there is an integer k such that $n^2 + n = 2k$. By definition, $n^2 + n$ is even.
In every case, n is even.
Since n was an arbitrary integer, for every integer n, $n^2 + n$ is even. *QED*

12. Mathematical Induction.

(a) Prove that for any integer n, if $n \geq 0$ then 4 divides $5^n - 1$.

Proof. We will use mathematical induction. Let $P(n)$ be the property of integers n such that

$$P(n) \quad \text{iff} \quad 4 \text{ divides } 5^n - 1$$

(basis step) We will prove $P(0)$ i.e. 4 divides $5^0 - 1$.
Let $k = 1$. k is an integer. We see that $5^0 - 1 = 4 = 4k$. We have shown there is an integer k such that $5^0 - 1 = 4k$. By definition, 4 divides $5^0 - 1$.
(inductive step) Assume $k \in \mathbf{Z}$, $k \geq 0$, and $P(k)$ i.e. 4 divides $5^k - 1$. We will show that $P(k+1)$ i.e. 4 divides $5^{k+1} - 1$.
By the inductive hypothesis, there is an integer q such that $5^k - 1 = 4q$. We see that $5^k = 4q + 1$. Therefore, $5^{k+1} = 20q + 5$. Let $m = 5q + 1$. By SPI, m is an integer.

$$5^{k+1} - 1 = (20q + 5) - 1$$

$$= 20q + 4$$

$$= 4(5q + 1)$$

$$= 4m$$

We have shown there is an integer m such that $5^{k+1} - 1 = 4m$ By definition, 4 divides $5^{k+1} - 1$.
By mathematical induction, for every integer n, if $n \geq 0$ then 4 divides $5^k - 1$. *QED*

13. Strong Mathematical Induction.

(a) Suppose $c_0, c_1, c_2, \ldots$ is a sequence defined as follows:

$$c_0 = 0, c_1 = 1,$$

$$c_k = 2c_{k-1} - c_{k-2} + 2 \text{ for all integers } k \geq 2.$$

Prove that $c_n = n^2$ for all integers $n \geq 0$.

Proof. We will use strong mathematical induction. Let $P(n)$ be the property of integers n such that

$$P(n) \quad \text{iff} \quad c_n = n^2$$

(basis step) We will prove $P(0)$ and $P(1)$ i.e. $c_0 = 0^2$ and $c_1 = 1^2$.
$c_0 = 0 = 0^2$ and $c_1 = 1 = 1^2$.
(inductive step) Assume $k \in \mathbf{Z}$, $k > 1$, and $P(i)$ holds for all integers i with $0 \leq i < k$ i.e. $c_i = i^2$ for all integers i with $0 \leq i < k$. We will show that $P(k)$ i.e. $c_k = k^2$.

$$c_k = 2c_{k-1} - c_{k-2} + 2$$

$$= 2(k-1)^2 - (k-2)^2 + 2 \qquad \text{(by the inductive hypothesis)}$$

$$= 2(k^2 - 2k + 1) - (k^2 - 4k + 4) + 2$$

$$= k^2$$

By strong mathematical induction, for every integer n, if $n \geq 0$ then $c_n = n^2$. *QED*

14. Proof by Contradiction or Contraposition.

(a) Prove that there is not a largest odd integer.

Proof. We will argue by contradiction and assume there is a largest odd integer. So, there is some integer n such that n is the largest odd integer. Fix such n.

Since n is odd, there is an integer k such that $n = 2k + 1$. By SPI, $n + 2$ is an integer. Let $m = k + 1$. By SPI, m is an integer. Also, $n + 2 = (2k + 1) + 2 = 2(k + 1) + 1 = 2m + 1$. We have shown there is an integer m such that $n + 2 = 2m + 1$. By definition, $n + 2$ is odd. Since $n + 2$ is odd and $n + 2 \not\leq n$, n is not the largest odd integer – contradiction. *QED*

(b) For any integer n, if n^2 is odd then n is odd.

Proof. We will argue by contraposition. Assume n is an integer such that n is not odd. We will show that n^2 is not odd.

Since n is not odd, n is even by a theorem in the text. Since n is even, there is an integer k such that $n = 2k$. Let $m = 2k^2$. By SPI, m is an integer. Also, $n^2 = (2k)^2 = 4k^2 = 2(2k^2) = 2m$. We have shown that there is an integer m such that $n^2 = 2m$. By definition, n^2 is even. By a theorem in the text, n^2 is not odd.

Since n was an arbitrary integer such that n is not odd, for any integer n, if n is not odd, then n^2 is not odd. Therefore, for any integer n, if n^2 is odd then n is odd. *QED*

15. Computations with Sets.

(a) Let $A = \{a, c, d\}$ and $B = \{b, c, f\}$. Compute $A \cup B$, $A \cap B$, $A - B$ and $A \times B$ using "bracket" notation.

Solution.

$$A \cup B = \{a, c, d, b, c, f\}$$

$$A \cap B = \{c\}$$

$$A - B = \{a, d\}$$

$$A \times B = \{(a, b), (a, c), (a, f), (c, b), (c, c), (c, f), (d, b), (d, c), (d, f)\}$$

16. More Proofs with Sets.

(d) Prove or disprove: For any sets A, B, and C, $A \cup (B \cap C) = (A \cup B) \cap C$.

We will disprove the statement i.e. prove the negation: There are sets A, B, and C such that $A \cup (B \cap C) \neq (A \cup B) \cap C$.

Proof. Let $A = \{0\}$, $B = \emptyset$ and $C = \emptyset$.

$$A \cup (B \cap C) = \{0\} \cup (\emptyset \cap \emptyset)$$

$$= \{0\}$$

and

$$\begin{aligned}(A \cup B) \cap C &= (\{0\} \cup \emptyset) \cap \emptyset \\ &= \emptyset\end{aligned}$$

Therefore, $0 \in A \cup (B \cap C)$ but also $0 \notin (A \cup B) \cap C$. This implies that $A \cup (B \cap C) \neq (A \cup B) \cap C$.

We have shown that there are sets A, B and C such that $A \cup (B \cap C) \neq (A \cup B) \cap C$. *QED*

19. 1-1 functions.

Determine whether each of the following functions is 1-1. Provide a proof of your answer.

(c) The function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = -2x + 1$.

We will show that g is a 1-1 function from $\mathbb{R}$ to $\mathbb{R}$.

Proof. Assume that $x_1, x_2 \in \mathbb{R}$ such that $g(x_1) = g(x_2)$. We will show that $x_1 = x_2$.

Since $g(x_1) = g(x_2)$, $-2x_1 + 1 = -2x_2 + 1$. Subtracting 1 from both sides we have $-2x_1 = -2x_2$. Dividing both sides by -2 we have $x_1 = x_2$.

Since $x_1, x_2 \in \mathbb{R}$ were arbitrary such that $g(x_1) = g(x_2)$, for all $x_1, x_2 \in \mathbb{R}$, if $g(x_1) = g(x_2)$ then $x_1 = x_2$. By definition, g is a 1-1 function from $\mathbb{R}$ to $\mathbb{R}$. *QED*

20. Onto functions.

Determine whether each of the following functions is onto. Provide a proof of your answer.

(f) The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$.

We will prove that f is a function from $\mathbb{R}$ onto $\mathbb{R}$.

Proof. Assume $y \in \mathbb{R}$.

Let $x = \sqrt[3]{y}$. Clearly, $x \in \mathbb{R}$. Also

$$\begin{aligned}f(x) &= f(\sqrt[3]{y}) \\ &= (\sqrt[3]{y})^3 \\ &= y\end{aligned}$$

We have shown there exists $x \in \mathbb{R}$ such that $f(x) = y$.

Since $y \in \mathbb{R}$ was arbitrary, for all $y \in \mathbb{R}$ there exists $x \in \mathbb{R}$ such that $f(x) = y$. By definition, f is a function from $\mathbb{R}$ onto $\mathbb{R}$. *QED*