SET THEORY AXIOMS AND BASIC THEOREMS

for Math 366

I. AXIOMS

1. (Extensionality Axiom) For any sets A and B, if $\forall x(x \in A \text{ iff } x \in B)$ then $A = B$.
2. (Definition of $\subseteq$) For all sets A and B,

$$A \subseteq B \text{ iff } \forall x(x \in A \to x \in B)$$

3. (Definition of the Emptyset)
   (a) $\emptyset$ is a set.
   (b) For all x, $x \notin \emptyset$.
4. (Definition of Intersection)
   (a) For all sets A and B, $A \cap B$ is a set.
   (b) For all sets A and B, for all x,

$$x \in A \cap B \text{ iff } x \in A \text{ and } x \in B$$

5. (Definition of Union)
   (a) For all sets A and B, $A \cup B$ is a set.
   (b) For all sets A and B, for all x,

$$x \in A \cup B \text{ iff } x \in A \text{ or } x \in B$$

6. (Definition of Relative Complement)
   (a) For all sets A and B, $A - B$ is a set.
   (b) For all sets A and B, for all x,

$$x \in A - B \text{ iff } x \in A \text{ and } x \notin B$$

7. (Definition of Power Set)
   (a) For any set A, $\mathsf{P}(A)$ is a set.
   (b) For any set A, for all x,
   $$x \in \mathsf{P}(A) \text{ iff } x \subseteq A$$
8. (Axioms for Bracket Notation I)
   (a) For all $x_1, \ldots, x_n$, $\{x_1, \ldots, x_n\}$ is a set.
   (b) For all $x_1, \ldots, x_n$ and all y,
   $$y \in \{x_1, \ldots, x_n\} \text{ iff } y = x_1 \text{ or } y = x_2 \text{ or } \ldots \text{ or } y = x_n$$
9. (Axioms for Bracket Notation II) Assume $\mathsf{P}(x)$ is a property of arbitrary objects x.
   (a) For any set A, $\{x \in A \,|\, \mathsf{P}(x)\}$ is a set.
   (b) For any set A, for all y,
   $$y \in \{x \in A \,|\, \mathsf{P}(x)\} \text{ iff } y \in A \text{ and } \mathsf{P}(y)$$
10. (Pairing Axiom) For all x, y, u and v,
    $$(x, y) = (u, v) \text{ iff } x = y \text{ and } u = v$$
11. (Definition of Cartesian Products)
    (a) For all sets A and B, $A \times B$ is a set.
    (b) For all sets A and B, for all x,
    $$x \in A \times B \text{ iff } \exists a \in A \, \exists b \in B \, s.t. \; x = (a, b)$$

II. BASIC THEOREMS

1. For all sets A and B and all x, if $A \subseteq B$ and $x \in A$ then $x \in B$.
2. For all sets A and B and all x, if $A \subseteq B$ and $x \notin B$ then $x \notin A$.
3. For all sets A and B, if $A \subseteq B$ and $B \subseteq A$ then $A = B$.
4. For any set A, if $x \notin A$ for all x then $A = \emptyset$.
5. For any set A, if $A \neq \emptyset$ then there exists x such that $x \in A$.