

MATH 649 HW 2

1. Show that a subset of ω is definable in $(\omega, 0, S, <)$ iff it is finite or its complement is finite. Hint: Use the fact that the theory of the structure admits quantifier elimination.
2. Problem 4 of section 3.2. Hint: Use the Positive Quantifier Elimination Theorem in problem 4 below.
3. Assume that \mathcal{A} and \mathcal{B} are models of a theory T which admits QE. Show that if \mathcal{A} is a substructure of \mathcal{B} then \mathcal{A} is an elementary substructure of \mathcal{B} .
4. A formula is *positive* if it contains no occurrences of \neg (equivalently, if it is in the set generated from the atomic formulas by \wedge, \vee, \forall and \exists). A theory T *admits positive QE* if every formula is equivalent under T to a positive quantifier free formula. Prove the following *Positive Quantifier Elimination Theorem*:

If T is a theory such that

- (a) The negation of any atomic formula is equivalent to a positive quantifier free formula in T .
- (b) For any finite sequence of atomic formulas $\alpha_1, \dots, \alpha_n$ and variable x , $\exists x(\alpha_1 \wedge \dots \wedge \alpha_n)$ is equivalent to a quantifier free formula in T .

then T admits positive elimination of quantifiers.

Hint: First show that any quantifier free formula is equivalent to a positive quantifier free formula in T by the induction principle. Use the induction principle again for the general case. You may use the fact that any positive quantifier free formula is logically equivalent to a positive quantifier free formula in disjunctive normal form.