Recurrence and Rate of Growth

We will develop techniques for finding the rate of growth of a given function $T(n)$ with nonegative real values which is given by a recurrence relation i.e. where $T(n)$ is given in terms of the values $T(m)$ where $m < n$. We will concentrate on recurrence relations which have one of the following forms

$$T(n) = aT(n-b) + f(n) \quad (n \geq b)$$

$$T(n) = aT\left(\left\lfloor \frac{n}{b} \right\rfloor\right) + f(n) \quad (n \geq b)$$

where a and b are positive integers and $b > 1$ in the second relation.

Consider the special case

$$T(n) = aT(n-1) + f(n)$$

for $n > 0$. This equation can be represented visually by the tree in Figure 1. Notice that the top node has a descendants.

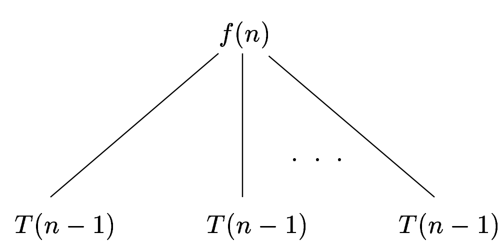

Figure 1: First level of recursion tree for $T(n) = aT(n-1) + f(n)$.

We can expand $T(n-1)$ to get

$$T(n) = f(n) + af(n-1) + a^2T(n-2)$$

which is illustrated in Figure 2.

Continuing we eventually get

$$T(n) = f(n) + af(n-1) + \cdots + a^k f(n-k) + \cdots + a^{n-1}f(1) + a^nT(0)$$

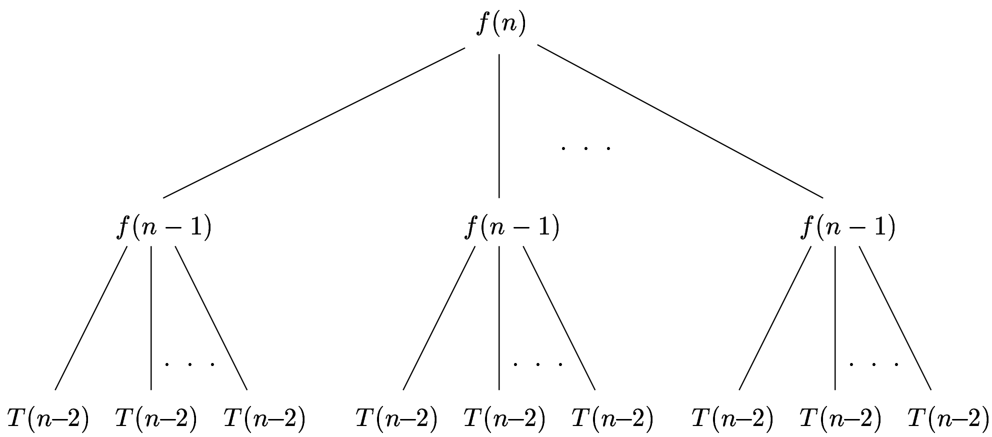

Figure 2: Second level of recursion tree for $T(n) = aT(n-1) + f(n)$.

This can be seen clearly by considering the *recursion tree* in Figure 3. Once again, notice that each node other than those on the bottom level has a descendants on the next level below.

Example 1. Suppose $T(n)$ satisfies

$$T(n) = T(n-1) + n$$

for $n > 0$. The recursion tree in Figure 4 shows that

$$T(n) = n + (n-1) + \cdots + 1 + T(0)$$

which is in $O(n^2)$.

Let's return to the general recurrence

$$T(n) = aT(n-b) + f(n) \quad (n \geq b)$$

First, consider the case when n is divisible by b. Say, $n = q \cdot b$. The line of reasoning above establishes that

$$T(n) = f(n) + af(n-b) + \cdots + a^k f(n - k \cdot b) + \cdots + a^{q-1} f(b) + a^q T(0)$$

In fact, we can establish a formula for the general case. If $n = q \cdot b + r$ where $0 \leq r < b$ then

$$T(n) = f(n) + af(n-b) + \cdots + a^k f(n-k \cdot b) + \cdots + a^{q-1} f(n-(q-1) \cdot b) + a^q T(r)$$

Notice that $r = n - q \cdot b$.

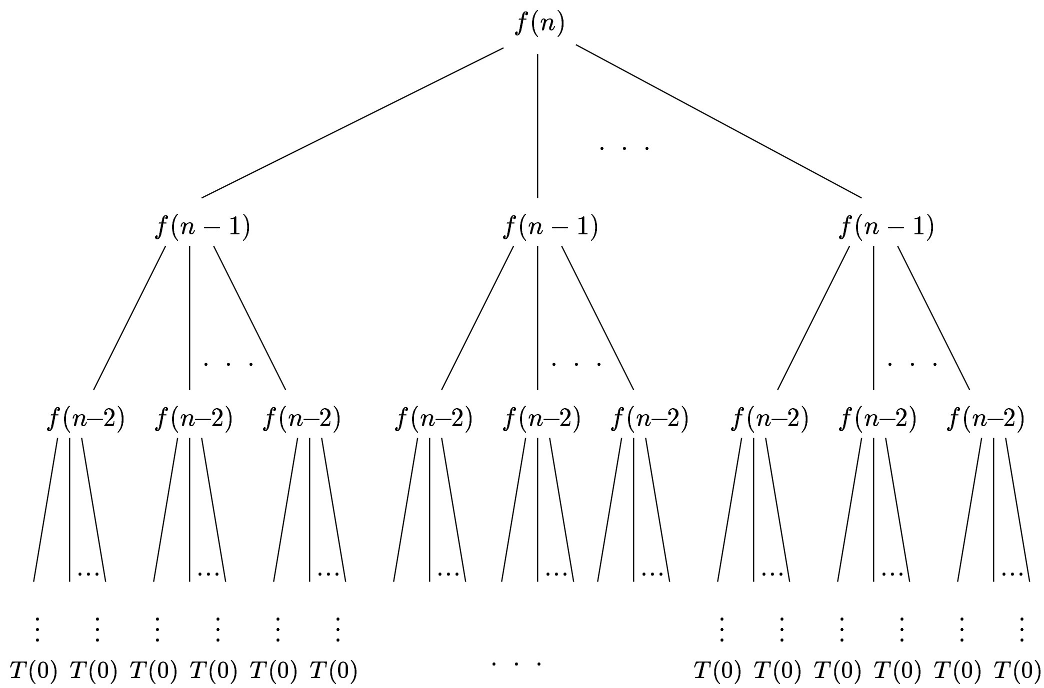

Figure 3: Recursion tree for $T(n) = aT(n-1) + f(n)$.

Example 2. Suppose $T(n)$ satisfies

$$T(n) = 4T(n-2) + n^3$$

for all $n > 1$. Assume that n is divisible by 2 so that $n = 2 \cdot k$ for some natural number k. The recursion tree in Figure 5 shows that

$$\begin{aligned} T(2 \cdot k) &= 8 \cdot k^3 + 4 \cdot 8 \cdot (k-1)^3 + 4^2 \cdot 8 \cdot (k-2)^3 + \cdots + 4^{k-1} \cdot 8 \cdot 1^3 + 4^k \cdot T(0) \\ &= 8 \cdot 4^k (\tfrac{k^3}{4^k} + \tfrac{(k-1)^3}{4^{k-1}} + \cdots + \tfrac{1^3}{4^1}) + 4^k \cdot T(0) \\ &= (8(\tfrac{k^3}{4^k} + \tfrac{(k-1)^3}{4^{k-1}} + \cdots + \tfrac{1^3}{4^1}) + T(0))4^k \end{aligned}$$

Since the infinite sum $\sum_{i=1}^{\infty} \frac{i^3}{4^i}$ converges (e.g. by the ratio test), the function given by the final expression, recalling that $k = \frac{n}{2}$, is in $\Theta(2^n)$. This seems to indicate that $T(n)$ is in $\Theta(2^n)$. However, the equation only holds for even integers. We can find a similar expression for $T(n)$ which is in $\Theta(2^n)$ when n has the form $2k + 1$. Since every natural number has one of the two forms ($2k$ or $2k + 1$), we conclude that $T(n)$ is $\Theta(2^n)$ (see exercise 1).

We now turn our attention to recurrences of the second sort

$$T(n) = aT\left(\left\lfloor \frac{n}{b} \right\rfloor\right) + f(n) \quad (n \geq b)$$

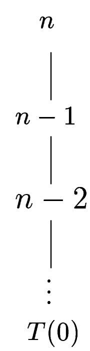

Figure 4: Recursion tree for $T(n) = T(n-1) + n$.

We again try to estimate the order of growth using a recursion tree. The reader should sketch the tree for the case $n = b^k$ for some natural number k (so $k = \log_b n$). It will look much like diagram 3. The recursion tree shows that

$$T(n) \;=\; f(b^k) + a \cdot f(b^{k-1}) + \cdots + a^i \cdot f(b^{k-i}) + \cdots + a^{k-1} \cdot f(b) + a^k \cdot T(1)$$

or, equivalently,

$$T(n) \;=\; f(n) + a \cdot f\left(\frac{n}{b}\right) + \cdots + a^i \cdot f\left(\frac{n}{b^i}\right) + \cdots + a^{k-1} \cdot f\left(\frac{n}{b^{k-1}}\right) + a^k \cdot T(1)$$

In fact, for general n we have the following expression for $T(n)$

$$f(n) + a \cdot f\left(\left\lfloor\frac{n}{b}\right\rfloor\right) + \cdots + a^i \cdot f\left(\left\lfloor\frac{n}{b^i}\right\rfloor\right) + \cdots + a^{k-1} \cdot f\left(\left\lfloor\frac{n}{b^{k-1}}\right\rfloor\right) + a^k \cdot T\left(\left\lfloor\frac{n}{b^k}\right\rfloor\right)$$

where $k = \lfloor \log_b n \rfloor$. In the following examples, we will estimate the order of growth of the given function using a recursion tree.

Example 3. Suppose $T(n)$ satisfies

$$T(n) \;=\; 2T\left(\left\lfloor\frac{n}{2}\right\rfloor\right) + c \cdot n$$

where $c > 0$. Consider the case $n = 2^k$ for some natural number k. The recursion tree in diagram 6 shows that

$$T(n) \;=\; c \cdot n + 2 \cdot c \cdot \frac{n}{2} + 2^2 \cdot c \cdot \frac{n}{2^2} + \cdots + 2^{k-1} \cdot c \cdot \frac{n}{2^{k-1}} + 2^k \cdot T(1)$$

implying that $T(n) = k \cdot n \cdot c + 2^k \cdot T(1) = n(k \cdot c + T(1))$. Since $k = \lg n$

$$T(n) \;=\; n(c \cdot \lg n \;+\; T(1))$$

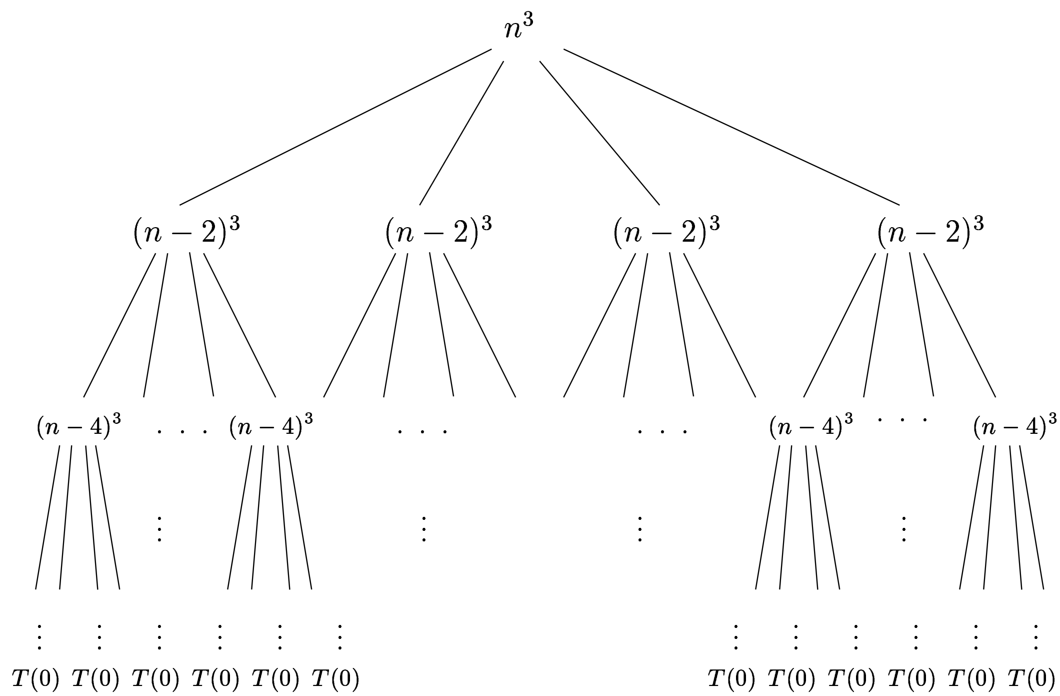

Figure 5: Recursion tree for $T(n) = 4T(n-2) + n^3$.

for n of the form 2^k. The final expression determines a function in $\Theta(n \lg n)$. So we estimate that $T(n)$ is in $\Theta(n \lg n)$.

Example 4. Suppose $T(n)$ satisfies

$$T(n) \ = \ 4T\Big(\Big\lfloor \frac{n}{2} \Big\rfloor\Big) + c \cdot n^2$$

where $c > 0$. Consider the case $n = 2^k$ for some natural number k. The recursion tree shows that

$$T(n) \ = \ c \cdot n^2 + 4 \cdot c \cdot \left(\frac{n}{2}\right)^2 + 4^2 \cdot c \cdot \left(\frac{n}{2^2}\right)^2 + \cdots + 4^{k-1} \cdot c \cdot \left(\frac{n}{2^{k-1}}\right)^2 + 4^k \cdot T(1)$$

implying that $T(n) = k \cdot n^2 \cdot c + 4^k \cdot T(1) = n^2(k \cdot c + T(1))$. Since $k = \lg n$

$$T(n) \ = \ n^2(c \cdot \lg n \ + \ T(1))$$

for n of the form 2^k. The final expression determines a function in $\Theta(n^2 \lg n)$. So we estimate that $T(n)$ is in $\Theta(n^2 \lg n)$.

Example 5. Suppose $T(n)$ satisfies

$$T(n) \;=\; 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + c \cdot n^2$$

where $c > 0$. Consider the case $n = 2^k$ for some natural number k. The recursion tree shows that

$$T(n) \;=\; c \cdot n^2 + 2 \cdot c \cdot \left(\frac{n}{2}\right)^2 + 2^2 \cdot c \cdot \left(\frac{n}{2^2}\right)^2 + \cdots + 2^{k-1} \cdot c \cdot \left(\frac{n}{2^{k-1}}\right)^2 + 2^k \cdot T(1)$$

implying that

$$T(n) \;=\; c\,n^2\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{k-1}}\right) \;+\; n \cdot T(1)$$

Since $\sum_{i=0}^{\infty} \frac{1}{2^i} = 2$

$$cn^2 \le T(n) \le 2cn^2 + nT(1)$$

for any n of the form 2^k. Therefore, we estimate that $T(n)$ is in $\Theta(n^2)$.

Example 6. Suppose $T(n)$ satisfies

$$T(n) \;=\; T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + c$$

where $c > 0$. Consider the case $n = 2^k$ for some natural number k. The recursion tree shows that

$$T(n) \;=\; c + c + \cdots + c + T(1)$$

where there are k occurences of c in the sum on the right hand side. Therefore, $T(n) = ck + T(1) = c \lg n \; + T(1)$. We estimate that $T(n)$ is in $\Theta(\lg n)$.

Example 7. Suppose $T(n)$ satisfies

$$T(n) \;=\; 9T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + c \cdot n$$

where $c > 0$. Consider the case $n = 3^k$ for some natural number k. The recursion tree shows that

$$T(n) \;=\; c \cdot n + 9 \cdot c \cdot \frac{n}{3} + 9^2 \cdot c \cdot \frac{n}{3^2} + \cdots + 9^{k-1} \cdot c \cdot \frac{n}{3^{k-1}} + 9^k \cdot T(1)$$

implying that

$$\begin{aligned} T(n) \;&=\; c\,n(1 + 3 + 3^2 + \cdots + 3^{k-1}) \;+\; n^2 \cdot T(1) \\ &=\; c\,n\left(\tfrac{3^k - 1}{3 - 1}\right) \;+\; n^2 \cdot T(1) \\ &=\; c\,n\left(\tfrac{n-1}{2}\right) \;+\; n^2 \cdot T(1) \end{aligned}$$

Since the final expression defines a function in $\Theta(n^2)$, we estimate that $T(n)$ is in $\Theta(n^2)$.

To prove our estimates were correct in the previous examples would be somewhat tedious. Instead we can use the following theorem.

Theorem *Assume $a \geq 1$ and $b > 1$ are integers, $f(n)$ is a given function with nonnegative real values and $T(n)$ is defined on the natural numbers to satisfy the recurrence*

$$T(n) \;=\; aT\left(\left\lfloor \frac{n}{b} \right\rfloor\right) + f(n)$$

for $n \geq b$.

1. *If $f(n)$ is in $O(n^{\log_b a\,-\,\epsilon})$ for some $\epsilon > 0$, then $T(n)$ is in $\Theta(n^{\log_b a})$.*

2. *If $f(n)$ is in $\Theta(n^{\log_b a})$, then $T(n)$ is in $\Theta(n^{\log_b a} \lg n)$.*

3. *If $f(n)$ is in $\Omega(n^{\log_b a\,+\,\epsilon})$ for some $\epsilon > 0$ and if $af\left(\left\lfloor \frac{n}{b} \right\rfloor\right) \leq cf(n)$ for some constant $c < 1$ and sufficiently large n, then $T(n)$ is in $\Theta(f(n))$.*

The proof of the theorem is beyond the scope of these notes.

Exercises.

1. Suppose q is a positive natural number and $f_r(n)$ is in $\Theta(g(n))$ for all natural numbers $0 \leq r < q$. Show that if $T(n) = f_r(n)$ whenever $n \, mod \, q$ is r then $T(n)$ is in $\Theta(g(n))$.

   For exercises 2-5, draw a recursion tree for the recurrence, find a sum which equals $T(n)$ and estimate the rate of growth of $T(n)$.

2. Suppose $T(n)$ satisfies

$$T(n) \;=\; T(n-2) + 5n^3 + 3n$$

   for $n \geq 2$.

3. Suppose $T(n)$ satisfies

$$T(n) \;=\; T(n-3) + 14n + 6$$

   for $n \geq 3$.

4. Suppose $T(n)$ satisfies

$$T(n) \;=\; T(n-4) + 3\log n$$

   for $n \geq 4$.

5. Suppose $T(n)$ satisfies

$$T(n) = 2T(n-1) + 3n$$

for $n \geq 1$.

For the remaining exercises, draw a recursion tree for the recurrence, find a sum which equals $T(n)$ when n has the form b^k (your sum should not involve k – rewrite it as $\log_b n$) and estimate the rate of growth of $T(n)$.

6. Suppose $T(n)$ satisfies

$$T(n) = T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + 5$$

for $n \geq 3$.

7. Suppose $T(n)$ satisfies

$$T(n) = T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + 4n$$

for $n \geq 3$.

8. Suppose $T(n)$ satisfies

$$T(n) = T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + \log n$$

for $n \geq 3$.

9. Suppose $T(n)$ satisfies

$$T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 3\log n$$

for $n \geq 2$.

10. Suppose $T(n)$ satisfies

$$T(n) = 2T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 2n + \log n$$

for $n \geq 2$.

11. Suppose $T(n)$ satisfies

$$T(n) = 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 2n^3 + n^2$$

for $n \geq 2$.

12. Suppose $T(n)$ satisfies

$$T(n) = 4T\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + 2n + 1$$

for $n \geq 2$.

13. Suppose $T(n)$ satisfies

$$T(n) = 9T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + 4n^2 + 1$$

for $n \geq 3$.

14. Suppose $T(n)$ satisfies

$$T(n) = 4T\left(\left\lfloor \frac{n}{3} \right\rfloor\right) + 4n^2 + 1$$

for $n \geq 3$.