Ordinal Arithmetic and Σ_1 Elementarity

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We will introduce a partial ordering $\preceq_1$ on the class of ordinals which will serve as a foundation to a new approach to ordinal notations for formal systems of set theory. $\preceq_1$ is defined by induction so that

$$\alpha \preceq_1 \beta \text{ iff } (\alpha, <, \preceq_1) \text{ is a } \Sigma_1\text{-elementary substructure of } (\beta, <, \preceq_1)$$

To be more precise, by induction on β we define the set of α such that $\alpha \preceq_1 \beta$. Note that we have taken some liberty in writing $(\alpha, <, \preceq_1)$ where we should have restricted the relations to α.

The original use of $\preceq_1$ was as a tool in confirming Reinhardt's conjecture that the Strong Mechanistic Thesis is consistent with Epistemic Arithmetic [1]. The purpose of this paper is to establish the properties of $\preceq_1$ needed for that result by giving an "effective" description of $\preceq_1$ in terms of ordinal arithmetic. In this way, $\preceq_1$ is reduced to familiar concepts from set theory. The role of $\preceq_1$ in constructing ordinal notations will be developed elsewhere.

1 Preliminaries

We will use lower case Greek letters to range over ordinals. ORD will denote the class of ordinals.

end is the operation defined on the ordinals as follows. Suppose α is an ordinal. If $\alpha = 0$ then $end(\alpha)$ is defined to be 0. Suppose α is nonzero. α can be written in a unique way as $\alpha_1 + \alpha_2 + \cdots + \alpha_n$ where each α_i is indecomposable (an ordinal is *indecomposable* if it has the form ω^η) and $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$. Define $end(\alpha)$ to be α_n.

Since the definition of a Σ_1 formula varies, we specify that a Σ_1 formula is a quantifier-free formula prefixed by a string of existential quantifications.

If $\mathcal{A}$ and $\mathcal{B}$ are structures for the same language we will write $\mathcal{A} \preceq_{\Sigma_1} \mathcal{B}$ to indicate that $\mathcal{A}$ is a Σ_1-elementary substructure of $\mathcal{B}$. For convenience, we will break with tradition by allowing stuctures whose universe is empty.

Definition 1.1 Suppose $\mathcal{A}$ and $\mathcal{B}$ are structures for the same language and $X \subseteq \mathcal{A} \cap \mathcal{B}$. Define $\mathcal{A} \cong_X \mathcal{B}$ iff there is an isomorphism of $\mathcal{A}$ and $\mathcal{B}$ which fixes all the elements of X.

The next proposition provides an equivalence that will allow us to avoid mentioning formulas.

Proposition 1.2 *Assume $\mathcal{A}$ and $\mathcal{B}$ are structures for a finite language without function symbols. $\mathcal{A}$ is a Σ_1-elementary substructure of $\mathcal{B}$ iff $\mathcal{A}$ is a substructure of $\mathcal{B}$ and whenever X is a finite subset of $|\mathcal{A}|$ and Y is a finite subset of $|\mathcal{B}| \setminus |\mathcal{A}|$ then there is a subset $\tilde{Y}$ of $|\mathcal{A}|$ such that*

$$X \cup Y \cong_X X \cup \tilde{Y}$$

Proof. Straightforward. □

Note that in the last line of the proposition we have used $X \cup Y$ and $X \cup \tilde{Y}$ to designated substructures of $\mathcal{B}$. We will continue to abuse notation in this way. Specifically, when X is a set of ordinals we will sometimes simply write X for the structure $(X, <, \preceq_1)$.

2 An Effective Characterization of $\preceq_1$

Some basic properties of $\preceq_1$ are collected in the following lemma.

Lemma 2.1

1. *$\preceq_1$ is a well-founded partial order in which the predecessors of any element are linearly ordered by $\preceq_1$.*
2. *If $\alpha \preceq_1 \beta$ and $\alpha \leq \gamma \leq \beta$ then $\alpha \preceq_1 \gamma$.*
3. *Suppose $\kappa \leq \alpha \leq \beta$. $[\kappa, \alpha) \preceq_{\Sigma_1} [\kappa, \beta)$ iff whenever $X \subseteq [\kappa, \alpha)$ and $Y \subseteq [\alpha, \beta)$ are finite then there is $\tilde{Y}$ contained in $[\kappa, \alpha)$ such that $X < \tilde{Y}$ and $X \cup Y \cong X \cup \tilde{Y}$.*
4. *$\alpha \preceq_1 \beta$ iff whenever $X \subseteq \alpha$ and $Y \subseteq [\alpha, \beta)$ are finite then there is $\tilde{Y}$ contained in α such that $X < \tilde{Y}$ and $X \cup Y \cong X \cup \tilde{Y}$.*
5. *If $\alpha < \beta$ and $\alpha \preceq_1 \beta$ then α is a limit ordinal.*

Proof. Straightforward. Use proposition 1.2 for part 3. Part 4 is the case $\kappa = 0$ of part 3. □

We will write $\alpha \preceq_1 \infty$ when $\alpha \preceq_1 \beta$ for all $\beta \geq \alpha$. Notice that for any ordinal α the collection of all β such that $\alpha \preceq_1 \beta$ is an interval of ordinals and either $\alpha \preceq_1 \infty$ or there is a largest β such that $\alpha \preceq_1 \beta$.

Definition 2.2 For α an ordinal, define *depth*(α) to be the largest δ such that $\alpha \preceq_1 \alpha + \delta$ when such δ exists and ∞ otherwise.

Notice that *depth*$(\alpha) = 0$ if α is not a limit ordinal. This follows from the last part of the previous lemma.

Definition 2.3 Define a function d on the class of ordinals by induction as follows. Let $d(0) = 0$ and $d(\alpha) = 0$ whenever α is a successor ordinal. If α is divisible by ε_0, i.e. α has the form $\varepsilon_0 \cdot \eta$ for some $\eta \neq 0$, define $d(\alpha) = \infty$. Suppose α is a nonzero limit ordinal which is not divisible by ε_0. Since α is not a successor ordinal, $end(\alpha)$ is ω^ρ for some nonzero ρ. Write ρ as $\rho_1 + \rho_2 + \cdots + \rho_k$ where $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_k$ and each ρ_i is indecomposable. Define $d(\alpha)$ by

$$d(\alpha) = \rho_1 + d(\rho_1) + \rho_2 + d(\rho_2) + \cdots + \rho_k + d(\rho_k)$$

To see that the definition of $d(\alpha)$ is a proper induction note that in the final case of the definition $end(\alpha)$ is less than ε_0 since α is not divisible by ε_0. Therefore, $\rho < \omega^\rho \leq \alpha$.

Note that $d(\alpha) = d(end(\alpha))$ so we could just as well define $d(\omega^\alpha)$ by induction on α and then extend d to all ordinals.

Lemma 2.4 *For all α and β, if $\beta \neq 0$ then*

$$d(\alpha + \beta) = d(\beta)$$

Proof. Since $d(\gamma) = d(end(\gamma))$ for all γ, $d(\alpha+\beta) = d(end(\alpha+\beta)) = d(end(\beta)) = d(\beta)$. □

The following theorem establishes that $depth(\alpha) = d(\alpha)$. The rest of this section will be devoted to its proof.

Theorem 2.5 *For all α and β, $\alpha \preceq_1 \beta$ iff $\alpha \leq \beta \leq \alpha + d(\alpha)$.*

The right hand side of the theorem has the obvious meaning when $d(\alpha) = \infty$.

There are ways of analyzing $\preceq_1$ without using ordinal arithmetic, but by doing so we lay bare the structure of $\preceq_1$ in familiar terms. The relation $\preceq_1$ and generalizations of it will be used elsewhere to calculate ordinals for various systems of set theory. There, another approach is crucial.

The ordinals which are $\preceq_1$-minimal will be useful in gaining a clearer understanding of $\preceq_1$ (α is $\preceq_1$-*minimal* if there is no $\beta \neq \alpha$ with $\beta \preceq_1 \alpha$). By the reflection principle, there are ordinals α with $depth(\alpha) = \infty$. The least such ordinal is the largest $\preceq_1$-minimal ordinal.

Definition 2.6 Let κ_α ($\alpha \leq \theta$) be the increasing enumeration of the $\preceq_1$-minimal ordinals. Define I_α to be the collection of β with $\kappa_\alpha \preceq_1 \beta$.

Lemma 2.7
1. *The enumeration $\alpha \mapsto \kappa_\alpha$ is strictly increasing and continuous.*
2. *If $\alpha < \beta \leq \theta$ then $I_\alpha < I_\beta$.*
3. *If $\alpha < \beta \leq \theta$, $\xi \in I_\alpha$, and $\eta \in I_\beta$ then $\xi \not\preceq_1 \eta$.*

4. *If* $\alpha < \theta$ *then* $I_\alpha = [\kappa_\alpha, \kappa_\alpha + depth(\kappa_\alpha)]$.

5. *If* $\alpha < \theta$ *then* $\kappa_{\alpha+1} = \kappa_\alpha + depth(\kappa_\alpha) + 1$.

6. $I_\theta = [\kappa_\theta, \infty)$.

7. *If* $\alpha \leq \theta$ *then* $\bigcup_{\xi<\alpha} I_\xi = \kappa_\alpha$.

8. $\bigcup_{\alpha\leq\theta} I_\alpha = ORD$.

Proof. Straightforward. □

The third and fourth parts of the preceding lemma show that the intervals I_α are the connectivity components of $\preceq_1$.

Lemma 2.8 *θ is a limit ordinal.*

Proof. Since $\kappa_0 = 0$, $depth(\kappa_0) = 0$. By the previous lemma, if $\alpha < \theta$ then $\kappa_{\alpha+1} = \kappa_\alpha + depth(\kappa_\alpha) + 1$. Since $\kappa_{\alpha+1}$ is a succesor ordinal, $depth(\kappa_{\alpha+1}) = 0$. But $depth(\kappa_\theta) = \infty$. So θ is a limit ordinal. □

Lemma 2.9 *If $\alpha \preceq_1 \alpha+1+\beta$ and $B, \tilde{B} \subseteq [\alpha+1, \alpha+1+\beta]$ then for all $A \subseteq [0, \alpha]$*

$$B \cong \tilde{B} \text{ iff } A \cup B \cong A \cup \tilde{B}$$

Proof. Assume α, β, B, and $\tilde{B}$ are as in the hypothesis of the lemma. Suppose in addition that $A \subseteq [0, \alpha]$ and $B \cong \tilde{B}$. Let $h : A \cup B \to A \cup \tilde{B}$ extend the isomorphism of B and $\tilde{B}$ and act as the identity on A. h clearly preserves $<$. To see that h preserves $\preceq_1$ simply note that for any $\xi \leq \alpha$ either $\xi \preceq_1 \eta$ for all $\eta \in [\alpha + 1, \alpha + 1 + \beta]$ or there is no $\eta \in [\alpha + 1, \alpha + 1 + \beta]$ such that $\xi \preceq_1 \eta$ (depending on whether $\xi \preceq_1 \alpha$ or not). This latter fact follows from parts 1 and 2 of lemma 2.1 and the assumption that $\alpha \preceq_1 \alpha + 1 + \beta$. □

Lemma 2.10 *If $\alpha \preceq_1 \alpha + 1 + \beta$ then for all $\xi, \eta \in [\alpha + 1, \alpha + 1 + \beta + 1]$*

$$\xi \preceq_1 \eta \text{ iff } [\alpha + 1, \xi) \preceq_{\Sigma_1} [\alpha + 1, \eta).$$

Proof. Assume $\alpha \preceq_1 \alpha + 1 + \beta$ and $\xi \leq \eta$ where $\xi, \eta \in [\alpha + 1, \alpha + 1 + \beta + 1]$.

($\Rightarrow$) Suppose $\xi \preceq_1 \eta$. Assume $X \subseteq [\alpha + 1, \xi)$ and $Y \subseteq [\xi, \eta)$. Let $X^+ = X \cup \{\alpha\}$. Since $\xi \preceq_1 \eta$, there is $\tilde{Y} \subseteq \xi$ such that $X^+ < \tilde{Y}$ and $X^+ \cup Y \cong X^+ \cup \tilde{Y}$. Since $\alpha \in X^+$, $\tilde{Y} \subseteq [\alpha + 1, \xi)$. And $X^+ \cup Y \cong X^+ \cup \tilde{Y}$ clearly implies that $X \cup Y \cong X \cup \tilde{Y}$.

($\Leftarrow$) Suppose $[\alpha + 1, \xi) \preceq_{\Sigma_1} [\alpha + 1, \eta)$. Assume $X \subseteq \xi$ and $Y \subseteq [\xi, \eta)$. Let $X' = X \cap [0, \alpha]$ and $X'' = X \cap [\alpha + 1, \xi)$. Since $[\alpha + 1, \xi) \preceq_{\Sigma_1} [\alpha + 1, \eta)$, there is $\tilde{Y} \subseteq [\alpha + 1, \xi)$ such that $X' < \tilde{Y}$ and $X' \cup Y \cong X' \cup \tilde{Y}$. Since $X'' \leq \alpha < \tilde{Y}$, $X < \tilde{Y}$. By the previous lemma, $X'' \cup X' \cup Y \cong X'' \cup X' \cup \tilde{Y}$ i.e. $X \cup Y \cong X \cup \tilde{Y}$.
□

Lemma 2.11 *If* $\alpha \preceq_1 \alpha + 1 + \beta$ *then* $[0, \beta + 1] \cong [\alpha + 1, \alpha + 1 + \beta + 1]$.

Proof. Let φ be the unique order isomorphism from $[\alpha + 1, \alpha + 1 + \beta + 1]$ onto $[0, \beta + 1]$ i.e. $\varphi(\alpha + 1 + \xi) = \xi$. Define $\preceq_1^*$ to be the projection of $\preceq_1$ under φ. We will show that $\preceq_1^*$ satisfies the inductive clause in the definition of $\preceq_1$ for ordinals $\leq \beta + 1$. This implies that $\preceq_1^*$ is the restriction of $\preceq_1$ to $[0, \beta + 1]$. Suppose $\xi, \eta \leq \beta + 1$.

$$\begin{array}{lll} \xi \preceq_1^* \eta & \text{iff} & \alpha + 1 + \xi \preceq_1 \alpha + 1 + \eta \\ & \text{iff} & [\alpha + 1, \alpha + 1 + \xi) \preceq_{\Sigma_1} [\alpha + 1, \alpha + 1 + \eta) \\ & \text{iff} & (\xi, <, \preceq_1^*) \preceq_{\Sigma_1} (\eta, <, \preceq_1^*) \end{array}$$

The first equivalence is by the definition of $\preceq_1^*$, the second by the previous lemma, and the last by appealing to the isomorphism φ. □

Lemma 2.12 *If* $\alpha \preceq_1 \alpha + \delta$ *and* $\beta \preceq_1 \beta + \delta$ *then* $[\alpha, \alpha + \delta] \cong [\beta, \beta + \delta]$.

Proof. For $\delta = 0$ the lemma is trivial. Suppose $\delta \neq 0$. By the previous lemma $[\alpha + 1, \alpha + \delta] \cong [\beta + 1, \beta + \delta]$. The lemma is now clear since $\alpha \preceq_1 \alpha + \xi$ and $\beta \preceq_1 \beta + \xi$ whenever $0 \leq \xi \leq \delta$. □

Lemma 2.13 *If* $\alpha \preceq_1 \infty$ *then* $ORD \cong [\alpha + 1, \infty)$.

Proof. Assume $\alpha \preceq_1 \infty$. By lemma 2.11, $[0, \beta + 1] \cong [\alpha + 1, \alpha + 1 + \beta + 1]$ for all β. The isomorphism must be $\xi \mapsto \alpha + 1 + \xi$ in each case. So $\xi \mapsto \alpha + 1 + \xi$ is an isomorophism of ORD and $[\alpha + 1, \infty)$. □

The condition that $\alpha \preceq_1 \alpha + 1 + \beta$ can be dropped in lemma 2.11, implying that the conclusion of the previous lemma holds for all α. Establishing this would allow a shorter but less illuminating conclusion to the proof of the theorem.

Lemma 2.14 *If* $0 < depth(\alpha)$ *there is a* $\lambda \leq \theta$ *such that*

$$[\alpha + 1, \alpha + depth(\alpha)] \cong \bigcup_{\xi \leq \lambda} I_\xi$$

and

$$depth(\alpha) = 1 + \kappa_\lambda + depth(\kappa_\lambda)$$

Proof. Assume $0 < depth(\alpha)$. If $depth(\alpha) = \infty$ the conclusion of the lemma holds with $\lambda = \theta$ by the previous lemma. So we may assume that $depth(\alpha) < \infty$.

Let β satisfy $1 + \beta = depth(\alpha)$. By lemma 2.11 $[0, \beta + 1] \cong [\alpha + 1, \alpha + 1 + \beta + 1]$. In particular, $[\alpha + 1, \alpha + depth(\alpha)] \cong [0, \beta]$. Let λ be maximal such that $\kappa_\lambda \leq \beta$. If we show that $\beta = \kappa_\lambda + depth(\kappa_\lambda)$ the conclusion of the lemma follows from the fact that $\bigcup_{\xi \leq \lambda} I_\xi = [0, \kappa_\lambda + depth(\kappa_\lambda)]$.

By choice of λ, $\beta < \kappa_{\lambda+1} = \kappa_\lambda + depth(\kappa_\lambda) + 1$. Equivalently, $\beta \leq \kappa_\lambda + depth(\kappa_\lambda)$.

In order to reach a contradiction, assume $\beta < \kappa_\lambda + depth(\kappa_\lambda)$. In this case $\kappa_\lambda \preceq_1 \beta + 1$ implying $\alpha + 1 + \kappa_\lambda \preceq_1 \alpha + 1 + \beta + 1$. Since $\kappa_\lambda \leq \beta$, $\alpha \preceq_1 \alpha + 1 + \kappa_\lambda$. But then $\alpha \preceq_1 \alpha + 1 + \beta + 1 = \alpha + depth(\alpha) + 1$ – contradiction. □

Using the previous lemma one can show that every $\xi < \kappa_\theta$ is either equal to some κ_α or is of the form $\eta + \kappa_\alpha$ where η is the largest ordinal less than ξ which is below ξ in $\preceq_1$. We will not need to establish this fact here, but it illustrates the key role of the κ_α.

Lemma 2.15 *If $\alpha \preceq_1 \alpha + 1 + \eta$ then $depth(\alpha + 1 + \eta) = depth(\eta)$.*

Proof. Let λ be given by the previous lemma. If $\eta \preceq_1 \eta + \tau$ then $\eta + \tau \in \bigcup_{\xi \leq \lambda} I_\xi$. And if $\alpha + 1 + \eta \preceq_1 \alpha + 1 + \eta + \tau$ Then $\alpha + 1 + \eta + \tau \leq \alpha + depth(\alpha)$. These facts along with the previous lemma imply that $\eta \preceq_1 \eta + \tau$ iff $\alpha + 1 + \eta \preceq_1 \alpha + 1 + \eta + \tau$ for all τ. □

Definition 2.16 If $depth(\alpha) \neq 0$ define $length(\alpha)$ to be the unique $\lambda \leq \theta$ such that

$$[\alpha + 1, \alpha + depth(\alpha)] \cong \bigcup_{\xi \leq \lambda} I_\xi$$

Definition 2.17 *For $\alpha \leq \theta$, if $depth(\kappa_\alpha) \neq 0$ then define λ_α to be $length(\kappa_\alpha)$.*

If $X \subseteq \alpha$ we say that X is *cofinal* in α if for all $\xi < \alpha$ there is $\eta \in X$ with $\xi < \eta < \alpha$. Notice that X is never cofinal in α if α is a successor ordinal.

Define the function *logend* on the class of ordinals as follows. $logend(0) = 0$. If $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ then

$$logend(\omega^{\alpha_1} + \omega^{\alpha_2} + \cdots + \omega^{\alpha_n}) = \alpha_n$$

Note that if $\beta \neq 0$ then $logend(\alpha + \beta) = logend(\beta)$.

Another characterization of $logend(\alpha)$ when $\alpha \neq 0$ is as the least λ such that the collection of β with $logend(\beta) = \lambda$ is not cofinal in α.

Lemma 2.18 *If $\alpha < \theta$ then there is a finite $Z \subseteq [0, \kappa_\alpha + depth(\kappa_\alpha)]$ such that there is no isomorphic copy of Z contained in $\kappa_\alpha + depth(\kappa_\alpha)$.*

Proof. We argue by induction on α.

For $\alpha = 0$ the choice $Z = \{0\}$ works.

Suppose $\alpha \neq 0$. There are finite sets $X \subseteq \kappa_\alpha$ and $Y \subseteq [\kappa_\alpha, \kappa_\alpha + depth(\kappa_\alpha)]$ such that there is no $\tilde{Y} \subseteq \kappa_\alpha$ with $X < \tilde{Y}$ and $X \cup Y \cong X \cup \tilde{Y}$. Notice that $\xi \not\preceq_1 \eta$ for $\xi \in X$ and $\eta \in Y$. By lemma 2.7, there is $\beta < \alpha$ such that $X \subseteq [0, \kappa_\beta + depth(\kappa_\beta)]$. Notice that there can be no $\tilde{Y} \subseteq (\kappa_\beta + depth(\kappa_\beta), \kappa_\alpha)$ which is isomorphic to Y since in that case $X < \tilde{Y}$ and $X \cup Y \cong X \cup \tilde{Y}$. By the induction hypothesis, there is $Z' \subseteq [0, \kappa_\beta + depth(\kappa_\beta)]$ which has no isomorphic copy contained in $\kappa_\beta + depth(\kappa_\beta)$. Let $Z = Z' \cup Y$. In order to reach a contradiction, assume $\tilde{Z}$ is an isomorphic copy of Z with $\tilde{Z} \subseteq \kappa_\alpha + depth(\kappa_\alpha)$. Let $\tilde{Z}'$ and $\tilde{Y}$ correspond to Z' and Y respectively under the isomorphism of Z and $\tilde{Z}$. Since $\tilde{Z}' \not\subseteq \kappa_\beta + depth(\kappa_\beta)$ we must have $\tilde{Y} \subseteq (\kappa_\beta + depth(\kappa_\beta), \kappa_\alpha + depth(\kappa_\alpha))$. By using the fact that $\kappa_\alpha \preceq_1 \kappa_\alpha + depth(\kappa_\alpha)$ we may assume that $Y \subseteq (\kappa_\beta + depth(\kappa_\beta), \kappa_\alpha)$ – contradiction. □

The following lemma says, roughly, that I_α absorbs one new interval, I_λ, beyond those which have been absorbed by cofinally many I_β with $\beta < \alpha$.

Lemma 2.19 *If $\alpha \leq \theta$ is a limit ordinal then $depth(\kappa_\alpha) \neq 0$ and $1 + \lambda_\alpha = logend(\alpha)$.*

Proof. We argue by induction on α.

Suppose $\alpha \leq \theta$ is a limit ordinal. Note that $logend(\alpha) \neq 0$ so there is an ordinal λ such that $1 + \lambda = logend(\alpha)$. Choose α' such that $\alpha = \alpha' + \omega^{1+\lambda}$. To see that $\lambda = \lambda_\alpha$ we must show that $depth(\kappa_\alpha) = 1 + \kappa_\lambda + depth(\kappa_\lambda)$.

We first establish that $\kappa_\alpha \preceq_1 \kappa_\alpha + 1 + \delta$ by induction on δ for $\delta \leq \kappa_\lambda + depth(\kappa_\lambda)$. Assume that $X \subseteq \kappa_\alpha$ and $Y \subseteq [\kappa_\alpha, \kappa_\alpha + 1 + \delta)$. Note that $\xi \not\preceq_1 \eta$ whenever $\xi \in X$ and $\eta \in Y$. Choose $\beta < \alpha$ such that $X \subseteq \kappa_\beta$. We will find an isomorphic copy $\tilde{Y}$ of Y contained in I_γ for some γ with $\beta \leq \gamma < \alpha$. Since $\xi \not\preceq_1 \eta$ for $\xi < \kappa_\beta$ and $\eta \in I_\gamma$, $X \cup Y \cong X \cup \tilde{Y}$.

We may assume without loss of generality that $\kappa_\alpha \in Y$. Let $Y' = Y \setminus \{\kappa_\alpha\}$. By lemma 2.11 and the induction hypothesis, $[\kappa_\alpha + 1, \kappa_\alpha + 1 + \delta) \cong \delta$. This implies that δ contains an isomorphic copy of Y'. We can conlude that κ_λ contains an isomorphic copy of Y'. This is clear if $\delta \leq \kappa_\lambda$. If $\kappa_\lambda < \delta$ then we can use the fact that $\kappa_\lambda \preceq_1 \delta$ to obtain an isomorphic copy of Y' in κ_λ from one in δ.

We now consider two cases.

First, suppose that $\lambda = 0$. In this case, $\kappa_\lambda + depth(\kappa_\lambda) = 0$. So we must have $\delta = 0$ implying that $Y = \{\kappa_\alpha\}$. $\tilde{Y} = \{\kappa_\beta\}$ satisfies our requirements.

Now suppose that $\lambda \neq 0$. If we fix an isomorphic copy of Y' contained in κ_λ we see that copy must be contained in $[0, \kappa_{\lambda'} + depth(\kappa_{\lambda'})]$ for some $\lambda' < \lambda$. Note that $\beta + \omega^{1+\lambda'} < \alpha$ since $end(\alpha) = \omega^{1+\lambda}$. By the induction hypothesis, $depth(\kappa_{\beta+\omega^{1+\lambda'}}) = 1 + \kappa_{\lambda'} + depth(\kappa_{\lambda'})$ and $[0, \kappa_{\lambda'} + depth(\kappa_{\lambda'})] \cong [\kappa_{\beta+\omega^{1+\lambda'}} + 1, \kappa_{\beta+\omega^{1+\lambda'}} + 1 + \kappa_{\lambda'} + depth(\kappa_{\lambda'})]$. Therefore, $[\kappa_{\beta+\omega^{1+\lambda'}} + 1, \kappa_{\beta+\omega^{1+\lambda'}} + 1 + \kappa_{\lambda'} + depth(\kappa_{\lambda'})]$ contains some $\tilde{Y}'$ which is isomorphic to Y'. $\tilde{Y} = \{\kappa_{\beta+\omega^{1+\lambda'}}\} \cup \tilde{Y}'$ is isomorphic to Y and is a subset of $I_{\beta+\omega^{1+\lambda'}}$.

This concludes the induction and establishes that $\kappa_\alpha \preceq_1 \kappa_\alpha + 1 + \kappa_\lambda + depth(\kappa_\lambda)$.

We now show that $\kappa_\alpha \not\preceq_1 \kappa_\alpha + 1 + \kappa_\lambda + depth(\kappa_\lambda) + 1$. By lemma 2.18, there is a finite subset Z of $[0, \kappa_\lambda + depth(\kappa_\lambda)]$ with no isomorphic copy in $[0, \kappa_\lambda + depth(\kappa_\lambda))$. Note that Z cannot be empty. Lemma 2.11 implies that there is a subset Y' of $[\kappa_\alpha + 1, \kappa_\alpha + 1 + \kappa_\lambda + depth(\kappa_\lambda)]$ which is isomorphic to Z. Let $Y = Y' \cup \{\kappa_\alpha\}$. Recall that $\alpha' < \alpha$ and $\alpha = \alpha' + \omega^{1+\lambda}$. Let $X = \{\kappa_{\alpha'+1}\}$.

We claim there is no isomorphic copy $\tilde{Y}$ of Y contained in κ_α with $X < \tilde{Y}$. Argue by contradiction and assume that $\tilde{Y}$ is a counterexample. Let $\tilde{Y}'$ correspond to Y' under the isomorphism of Y and $\tilde{Y}$. Since $\kappa_\alpha \preceq_1 \xi$ for all $\xi \in Y$ we see by part 3 of lemma 2.7 that $\tilde{Y}$ must be contained in I_β for some $\beta < \alpha$. The fact that $X < \tilde{Y}$ implies that $\alpha' < \beta$. So we have $\alpha' < \beta < \alpha' + \omega^{1+\lambda}$. This implies that $logend(\beta) < 1 + \lambda$. Since $\tilde{Y} \subseteq [\kappa_\beta, \kappa_\beta + depth(\kappa_\beta)]$, we must have $\tilde{Y}' \subseteq (\kappa_\beta, \kappa_\beta + depth(\kappa_\beta)]$. $\tilde{Y}'$, being isomorphic to Z, is nonempty. So we must have $depth(\kappa_\beta) \neq 0$ implying β is a limit ordinal. There is λ' such that $logend(\beta) = 1 + \lambda'$. By the induction hypothesis and lemma 2.11, $(\kappa_\beta, \kappa_\beta + depth(\kappa_\beta)] = [\kappa_\beta + 1, \kappa_\beta + 1 + \kappa_{\lambda'} + depth(\kappa_{\lambda'})] \cong [0, \kappa_{\lambda'} + depth(\kappa_{\lambda'})]$. Since $(\kappa_\beta, \kappa_\beta + depth(\kappa_\beta)]$ contains $\tilde{Y}'$ which is isomorphic to Z, $[0, \kappa_{\lambda'} + depth(\kappa_{\lambda'})]$ contains an isomorphic copy of Z – contradiction. □

Lemma 2.20 *$\kappa_{\varepsilon_0} \cdot \eta \preceq_1 \infty$ for all $\eta > 0$. Moreover, $\theta = \varepsilon_0$.*

Proof. We first show that $\varepsilon_0 \leq \theta$. In order to reach a contradiction, assume $\theta < \varepsilon_0$. λ_θ is defined since θ is a limit ordinal, and $1 + \lambda_\theta = logend(\theta)$ by the previous lemma. Since $\theta < \varepsilon_0$, $logend(\theta) < \theta$ implying $\lambda_\theta < \theta$ and $depth(\kappa_\theta) = 1 + \kappa_{\lambda_\theta} + depth(\kappa_{\lambda_\theta}) < \infty$ – contradiction.

We will now show that

$$(*) \quad \kappa_{\varepsilon_0} \cdot \eta \preceq_1 \kappa_{\varepsilon_0} \cdot (\eta + 1)$$

for $\eta > 0$. This implies that $\kappa_{\varepsilon_0} \cdot \eta \preceq_1 \infty$ for all $\eta > 0$. Note that $(*)$ is equivalent to $depth(\kappa_{\varepsilon_0} \cdot \eta) \geq \kappa_{\varepsilon_0}$.

We establish $(*)$ by induction on η. By the previous lemma, $\lambda_{\varepsilon_0} = \varepsilon_0$. Therefore, $depth(\kappa_{\varepsilon_0}) = 1 + \kappa_{\varepsilon_0} + depth(\kappa_{\varepsilon_0}) = \kappa_{\varepsilon_0} + depth(\kappa_{\varepsilon_0})$. This immediately implies that $(*)$ holds with $\eta = 1$. So we may assume that $1 < \eta$.

We first consider the case where $\eta = \eta' + 1$ for some $\eta' \neq 0$. By the induction hypothesis, $\kappa_{\varepsilon_0} \cdot \eta' \preceq_1 \kappa_{\varepsilon_0} \cdot \eta' + \kappa_{\varepsilon_0}$. Applying lemma 2.15 we see $depth(\kappa_{\varepsilon_0} \cdot \eta' + \kappa_{\varepsilon_0}) = depth(\kappa_{\varepsilon_0} \cdot \eta' + 1 + \kappa_{\varepsilon_0}) = depth(\kappa_{\varepsilon_0}) \geq \kappa_{\varepsilon_0}$.

Now consider the case where η is a limit ordinal. We show by induction on $\xi \leq \kappa_{\varepsilon_0}$ that $\kappa_{\varepsilon_0} \cdot \eta \preceq_1 \kappa_{\varepsilon_0} \cdot \eta + \xi$. For $\xi = 0$ this is clear. So we may assume $\xi \neq 0$. Suppose $X \subseteq \kappa_{\varepsilon_0} \cdot \eta$ and $Y \subseteq [\kappa_{\varepsilon_0} \cdot \eta, \kappa_{\varepsilon_0} \cdot \eta + \xi)$ are finite sets. We need to show there is $\tilde{Y} \subset \kappa(\varepsilon_0) \cdot \eta$ with $X < \tilde{Y}$ and $X \cup Y \cong X \cup \tilde{Y}$. If ξ is a limit ordinal then the induction hypothesis allows us to find $\tilde{Y}$. So we may assume $\xi = \tau + 1$ for some τ. Choose $\nu < \eta$ such that $X \subseteq \kappa_{\varepsilon_0} \cdot \nu$. By lemma 2.12, $[\kappa_{\varepsilon_0} \cdot (\nu + 1), \kappa_{\varepsilon_0} \cdot (\nu + 1) + \tau] \cong [\kappa_{\varepsilon_0} \cdot \eta, \kappa_{\varepsilon_0} \cdot \eta + \tau]$. Therefore, there is a subset $\tilde{Y}$ of $[\kappa_{\varepsilon_0} \cdot (\nu + 1), \kappa_{\varepsilon_0} \cdot (\nu + 1) + \tau]$ which is isomorphic to Y. Since the induction hypothesis implies that $\kappa_{\varepsilon_0} \cdot \nu \preceq_1 \kappa_{\varepsilon_0} \cdot \eta + \tau$, we can use lemma 2.9 to conclude that $X \cup Y \cong X \cup \tilde{Y}$. □

Lemma 2.21 *If $\alpha, \beta < \varepsilon_0$ and $\beta \neq 0$ then*

1. *$depth(\kappa_{\alpha+\beta}) = depth(\kappa_\beta)$.*

2. $\kappa_{\alpha+\beta} = \kappa_\alpha + depth(\kappa_\alpha) + \kappa_\beta$.

3. $depth(\kappa_\alpha) = \kappa_{logend(\alpha)} + depth(\kappa_{logend(\alpha)})$.

Proof. Notice that the first part of the lemma is trivial if β is a successor ordinal since than both κ_β and $\kappa_{\alpha+\beta}$ are successor ordinals and both have the value 0 under *depth*. So we may assume β is a limit ordinal. Using lemma 2.19, $1 + \lambda_{\alpha+\beta} = logend(\alpha + \beta) = logend(\beta) = 1 + \lambda_\beta$. Therefore, $\lambda_{\alpha+\beta} = \lambda_\beta$ which implies the desired equality.

We prove the second part by induction on $\beta > 0$. Notice that $\kappa_1 = \kappa_0 + depth(\kappa_0) + 1 = 1$. For the case $\beta = 1$ we have $\kappa_{\alpha+1} = \kappa_\alpha + depth(\kappa_\alpha) + 1 = \kappa_\alpha + depth(\kappa_\alpha) + \kappa_1$. If β is a limit ordinal then the desired equality follows from the induction hypothesis and the continuity of the enumeration κ_ξ $(\xi \in \varepsilon_0)$. So we may assume that $\beta = \beta' + 1$ for some $\beta' \neq 0$. Using the induction hypothesis we see $\kappa_{\alpha+\beta'+1} = \kappa_{\alpha+\beta'} + depth(\kappa_{\alpha+\beta'}) + 1 = \kappa_{\alpha+\beta'} + depth(\kappa_{\beta'}) + 1 = \kappa_\alpha + depth(\kappa_\alpha) + \kappa_{\beta'} + depth(\kappa_{\beta'}) + 1 = \kappa_\alpha + depth(\kappa_\alpha) + \kappa_{\beta'+1}$. The second equality uses the first part of this lemma.

Towards proving the third part, notice that if $\alpha = 0$ or α is a successor ordinal then $depth(\kappa_\alpha) = 0 = \kappa_0 + depth(\kappa_0) = \kappa_{logend(\alpha)} + depth(\kappa_{logend(\alpha)})$. So we may assume that α is a limit ordinal. We see that $depth(\kappa_\alpha) = 1 + \kappa_{\lambda_\alpha} + depth(\kappa_{\lambda_\alpha}) = \kappa_1 + depth(\kappa_1) + \kappa_{\lambda_\alpha} + depth(\kappa_{\lambda_\alpha}) = \kappa_{1+\lambda_\alpha} + depth(\kappa_{1+\lambda_\alpha}) = \kappa_{logend(\alpha)} + depth(\kappa_{logend(\alpha)})$. In the third equality, the case $\lambda_\alpha = 0$ is easy while we can use the previous two parts of this lemma for the case $\lambda_\alpha \neq 0$. □

Lemma 2.22 *If $\alpha < \varepsilon_0$ then $depth(\kappa_\alpha) < \kappa_\alpha$.*

Proof. Suppose $\alpha < \varepsilon_0$. Since $\alpha < \varepsilon_0$, $logend(\alpha) < \alpha$. By the third part of the previous lemma, $depth(\kappa_\alpha) = \kappa_{logend(\alpha)} + depth(\kappa_{logend(\alpha)}) < \kappa_\alpha$. □

Proof of theorem 2.5: The verification that $depth = d$ is closely related to the calculation of κ_α.

Suppose $\alpha < \varepsilon_0$ and $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ where $\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_n$ are indecomposable. By the second part of the previous lemma

$$\kappa_\alpha = \kappa_{\alpha_1} + depth(\kappa_{\alpha_1}) + \cdots + \kappa_{\alpha_{n-1}} + depth(\kappa_{\alpha_{n-1}}) + \kappa_{\alpha_n}$$

Combined with the following claim, κ_α can be calculated in terms of *depth*.

Claim: If $\alpha \leq \varepsilon_0$ then $\kappa_{\omega^\alpha} = \omega^\alpha$.

We prove the claim by induction on α.

For the case $\alpha = 0$,

$$\kappa_{\omega^0} = \kappa_1 = 1 = \omega^0$$

When α is a limit ordinal, the claim follows from the induction hypothesis by using the continuity of the enumeration κ_ξ $(\xi \leq \varepsilon_0)$.

Suppose $\alpha = \alpha' + 1$. Using the continuity of κ_ξ $(\xi \leq \varepsilon_0)$

$$(*) \quad \kappa_{\omega^\alpha} = sup\{\kappa_{\omega^{\alpha'}\cdot n} \,|\, n \in \omega\}$$

By the remark preceding the statement of the claim,

$$\kappa_{\omega^{\alpha'}\cdot(n+1)} = (\kappa_{\omega^{\alpha'}} + depth(\kappa_{\omega^{\alpha'}})) \cdot n + \kappa_{\omega^{\alpha'}}$$

By the previous lemma and the induction hypothesis, we obtain the following bounds for $\kappa_{\omega^{\alpha'}\cdot(n+1)}$.

$$\omega^{\alpha'} \cdot (n+1) \leq \kappa_{\omega^{\alpha'}\cdot(n+1)} \leq \omega^{\alpha'} \cdot (2n+1)$$

By $(*)$, $\kappa_{\omega^\alpha} = \omega^\alpha$.

This concludes the proof of the claim.

We now prove that $depth(\alpha) = d(\alpha)$ by induction on α.

If $\alpha = 0$ or α is a successor ordinal then $depth(\alpha) = 0 = d(\alpha)$. If α is divisible by ε_0 then $depth(\alpha) = \infty = d(\alpha)$ by lemma 2.20 and the definition of d.

Assume α is a limit ordinal which is not divisible by ε_0.

Case 1: $\alpha \geq \varepsilon_0$.

There are $\nu \neq 0$ and $\mu < \varepsilon_0$ such that $\alpha = \varepsilon_0 \cdot \nu + \mu$.

Since α is not divisible by ε_0, $\mu \neq 0$. Let τ be given by $\mu = 1 + \tau$.

$$\begin{aligned} depth(\alpha) &= depth(\tau) && \text{(by lemma 2.15)} \\ &= d(\tau) && \text{(by the induction hypothesis)} \\ &= d(\alpha) && \text{(by lemma 2.4)} \end{aligned}$$

Case 2: $\alpha < \varepsilon_0$.

There is $\beta < \varepsilon_0$ such that $\alpha \in I_\beta$ i.e. $\kappa_\beta \leq \alpha \leq \kappa_\beta + depth(\kappa_\beta)$.

Suppose $\alpha \neq \kappa_\beta$. We have $\alpha = \kappa_\beta + 1 + \tau$ for some τ. Lemma 2.22 implies that $\tau < \kappa_\beta < \alpha$. A calculation as in case 1 shows $depth(\alpha) = d(\alpha)$. So we may assume $\alpha = \kappa_\beta$.

Since α is a limit ordinal, so is β. There are indecomposable $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_n$ such that $\beta = \beta_1 + \beta_2 + \cdots + \beta_n$. Let $\rho = logend(\beta)$ i.e. $\beta_n = \omega^\rho$. Since β is a limit ordinal, $\rho \neq 0$. So there are indecomposable $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_k$ such that $\rho = \rho_1 + \rho_2 + \cdots + \rho_k$. Notice that for each i, $\rho_i < \beta \leq \kappa_\beta$ so, by the induction hypothesis, $depth(\rho_i) = d(\rho_i)$.

$$\begin{aligned} depth(\kappa_\beta) &= \kappa_{\rho_1+\cdots+\rho_k} + depth(\kappa_{\rho_1+\cdots+\rho_k}) \\ &\quad \text{(by part 3 of lemma 2.21)} \\ &= \kappa_{\rho_1+\cdots+\rho_k} + depth(\kappa_{\rho_k}) \\ &\quad \text{(by part 2 of lemma 2.21)} \\ &= \kappa_{\rho_1} + depth(\kappa_{\rho_1}) + \cdots + \kappa_{\rho_k} + depth(\kappa_{\rho_k}) \\ &\quad \text{(by the remark before the claim above)} \\ &= \rho_1 + depth(\rho_1) + \cdots + \rho_k + depth(\rho_k) \\ &\quad \text{(by the claim above)} \\ &= \rho_1 + d(\rho_1) + \cdots + \rho_k + d(\rho_k) \end{aligned}$$

(by the induction hypothesis)
$= d(\beta_n)$
(by definition of d)
$= d(\kappa_{\beta_n})$
(by the claim above)
$= d(\kappa_{\beta_1} + depth(\kappa_{\beta_1}) + \cdots + \kappa_{\beta_{n-1}} + depth(\kappa_{\beta_{n-1}}) + \kappa_{\beta_n})$
(by lemma 2.4)
$= d(\kappa_\beta)$
(by the remark before the claim above)

□

3 Concluding Remarks

The characterization of $\preceq_1$ in the previous section implies that $(\varepsilon_0 \cdot \omega, <, \preceq_1)$ is isomorphic to a recursive structure (this will be needed in [1]). One might ask whether this fact can be proved in weak theories. To answer this question, we replace the ordinals below $\varepsilon_0 \cdot \omega$ with notations of the form $\varepsilon_0 \cdot n + t$ where n and t are notations from the usual system of notations for ε_0 and n is a notation for a natural number. From now on we will use lower case Greek letters to range over notations for ordinals below $\varepsilon_0 \cdot \omega$ and we now view d as being defined on elements of the notation system. The proof given above uses inductive constructions over the ordinals. We will now sketch an argument that $\preceq_1$ is primitive recursive in PRA, the theory of pimitive recursive arithmetic. The disadvantage of this approach is that it cannot be motivated as well as the approach taken in this paper and it ignores much of the structure uncovered here. In particular, this approach gives little idea where the definition of d comes from.

We must first give a definition of $\preceq_1$ within PRA. Notice that d is primitive recursive. The definition of $d(\alpha)$ is given by induction on the buildup of the notation α. We use theorem 2.5 to define $\preceq_1$ within PRA:

$$\alpha \preceq_1 \beta \quad \text{iff} \quad \alpha \leq \beta \leq \alpha + d(\alpha)$$

We now argue, within PRA, that $\alpha \preceq_1 \beta$ iff $(\alpha, <, \preceq_1) \preceq_{\Sigma_1} (\beta, <, \preceq_1)$ for all α and β (where $<$ denotes the ordering of the notations in type $\varepsilon_0 \cdot \omega$).

Using lemma 2.4, we see that for all λ,β, and $\alpha \neq 0$

$$\alpha \preceq_1 \beta \text{ iff } \lambda + \alpha \preceq_1 \lambda + \beta$$

A somewhat tedious but straightforward exercise in ordinal arithmetic shows that $\preceq_1$ is transitive. This argument will use induction on the buildup of notations. One can define a function which from α, β, and finite sets X and Y satisfying $\alpha \preceq_1 \beta$, $X \subseteq \alpha$, and $Y \subseteq [\alpha, \beta)$ will produce $\tilde{Y} \subseteq \alpha$ with $X < \tilde{Y}$ such that $X \cup Y \cong X \cup \tilde{Y}$. This shows that $\alpha \preceq_{\Sigma_1} \beta$ whenever $\alpha \preceq_1 \beta$. To establish the converse one defines a function which given $\alpha < \varepsilon_0$ will produce a finite subset of $[\omega^\alpha, \omega^\alpha + d(\omega^\alpha)]$ with no isomorphic copy below $\omega^\alpha + d(\omega^\alpha)$. This can be used to find examples showing that $\alpha \not\leq_{\Sigma_1} \alpha + d(\alpha) + 1$ for all α. The functions described here are defined by induction on the buildup of notations and, under the usual coding of finite sets, are seen to be primitive recursive.

References

1. T. Carlson, *Can a machine know that it is a machine?*, preprint.