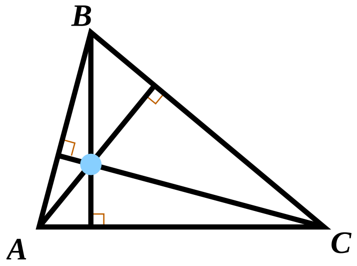
Calculus Problem Of the Week

May 08 — 14, 2006

Heights in a triangle. I

Using the dot product of vectors we can prove a celebrated theorem of elementary geometry.



Theorem. *The tree heights of a triangle intersect at one point.* It is called the ***orthocenter*** of the triangle.

For points P and Q on the the plane let $\overrightarrow{PQ}$ denote the vector with the initial point P and the terminal point Q:

(1) *For the vertices A, B, C of a triangle show that* $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = 0$.

(2) *For an arbitrary point M on the plane prove that* $\overrightarrow{AB} \cdot \overrightarrow{CM} + \overrightarrow{BC} \cdot \overrightarrow{AM} + \overrightarrow{CA} \cdot \overrightarrow{BM} = 0$.

(3) Let H be the point of intersection of the two heights from vertices A and B. In particular, it means that $\overrightarrow{BC} \cdot \overrightarrow{AH} = 0$ and $\overrightarrow{CA} \cdot \overrightarrow{BH} = 0$. Using (2) *show that* $\overrightarrow{AB} \cdot \overrightarrow{CH} = 0$.
In other words, the line from C to the point H is perpendicular to AB, i.e. it is the third height of the triangle.