Calculus Problem Of the Week

May 29 — June 04, 2006

Heights in a triangle. II

V. I. Arnold (`http://www.pdmi.ras.ru/~arnsem/Arnold/`) noted (in `http://pauli.uni-muenster.de/~munsteg/arnold.html`) that the Jacobi identity (2) of the cross product of vectors implies the theorem about heights in a triangle

Theorem. *The tree heights of a triangle intersect at one point.*

Let $\pi_{\mathbf{u}}$ denotes a plane through the origin O and perpendicular to the vector $\mathbf{u}$.

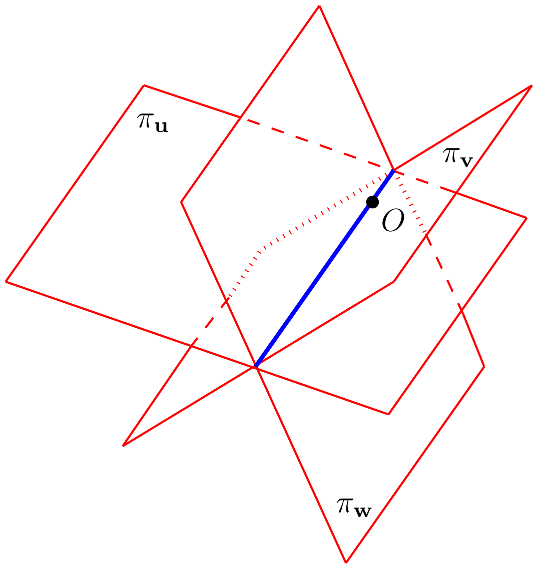
(1) *Suppose that three non collinear vectors* $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ *satisfy the equation*

$$\mathbf{u} + \mathbf{v} + \mathbf{w} = 0 \ .$$

Prove that the three planes $\pi_{\mathbf{u}}$, $\pi_{\mathbf{v}}$, *and* $\pi_{\mathbf{w}}$ *intersect on a line.*
The same is true for more general equation

$$\lambda\mathbf{u} + \mu\mathbf{v} + \nu\mathbf{w} = 0 \ ,$$

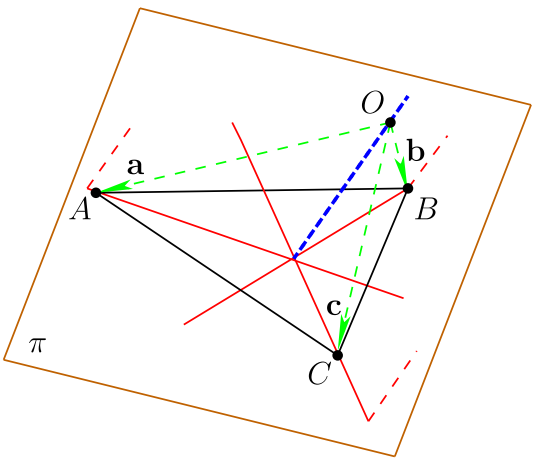
where λ, μ, ν are arbitrary not all zero numbers.



(2) ***Jacobi identity.*** *Prove that*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) \ + \ \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) \ + \ \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \ = \ 0 \ ,$$

for any three vectors $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$.



For a triangle ABC let π be the plane of the triangle. Pick a point O out of π and consider it as the origin. Let

$$\mathbf{a} = \overrightarrow{OA}, \quad \mathbf{b} = \overrightarrow{OB}, \quad \mathbf{c} = \overrightarrow{OC},$$

$$\mathbf{u} = \mathbf{a} \times (\mathbf{b} \times \mathbf{c}), \quad \mathbf{v} = \mathbf{b} \times (\mathbf{c} \times \mathbf{a}), \quad \mathbf{w} = \mathbf{c} \times (\mathbf{a} \times \mathbf{b}) \ .$$

(3) *Prove that*

(i) *the plane* $\pi_{\mathbf{u}}$ *contains* A *and perpendicular to* BC;

(ii) *the plane* $\pi_{\mathbf{v}}$ *contains* B *and perpendicular to* AC;

(iii) *the plane* $\pi_{\mathbf{w}}$ *contains* C *and perpendicular to* AB.