The spread and extreme terms of Jones polynomials

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Abstract

We adapt Thistlethwaite's alternating tangle decomposition of a knot diagram to identify the potential extreme terms in its bracket polynomial, and give a simple combinatorial calculation for their coefficients, based on the intersection graph of certain chord diagrams.

Introduction

One of the most striking combinatorial applications of the Jones polynomial has been the result of Murasugi and Thistlethwaite which characterises alternating knots by relating the spread of the Jones polynomial to the number of crossings in the knot diagram. The result follows from the identification of the two potential extreme terms in the bracket polynomial arising from the A-splitting and the B-splitting of the diagram and the calculation that each term occurs with coefficient ± 1.

Lickorish and Thistlethwaite [4] were able to widen the class of knots for which a similar exact bound for the spread of the polynomial could be given. In this wider class of 'adequate' diagrams they were again able to specify extreme terms and show that their coefficients were ± 1. Later Thistlethwaite [6] made use of a natural decomposition of a given link diagram into maximal alternating pieces, and formulated a bound for the spread purely in terms of combinatorial data from the non-alternating part of the diagram.

Our aim here is to determine the coefficients of the two potential extreme terms in the bracket polynomial of a general diagram. These are the terms of extreme degree which arise from the A-splitting or the B-splitting of the diagram and possibly also from other choices of splitting. In the case of alternating or adequate diagrams they only occur once, from the A-splitting or the B-splitting alone.

Where a diagram is largely alternating the contributions to the extreme terms are determined by a relatively small part of the diagram around the non-alternating edges. Our motivation is to show how this readily identified part of the diagram, along with a short combinatorial calculation, leads to the extreme coefficients. We adopt the term *non-alternating skeleton* in section 3 to describe the key part of the diagram for our analysis. Section 2 includes a reformulation of the calculation of the extreme coefficients in terms of a simple combinatorial function on a related graph. Qualitatively the alternating parts of the diagram can only contribute certain 'edge effects' to the extreme terms, as shown in section 4, and it then becomes easy to find the coefficients of these terms using a considerably reduced diagram derived from a neighbourhood of the non-alternating skeleton. In consequence it is often possible to find the coefficients of the extreme terms for a large diagram by hand, when it has plenty of alternating edges, without involving a complete computer calculation of the Jones polynomial. This avoids the need for either a mouse drawn input, where the calculation can grow significantly with the crossing number, or a braid presentation.

Even before calculating the extreme coefficients, knowledge of the non-alternating skeleton can improve Thistlethwaite's bound on the spread of the polynomial with little extra work, as shown in section 5.

Among our examples in section 6 we have included calculations of the extreme coefficients for the 10-crossing knots, to illustrate the extent of the information arising in this way without a full Jones polynomial calculation. From this we get exact knowledge of the spread of the polynomial in 20 of the 55 non-alternating diagrams, while only 3 of them are fully adequate.

1 The extreme states bound

We recall the states sum description of the Kauffman bracket polynomial $\langle D \rangle$ for an unoriented link diagram D.

Label the four quadrants at each crossing A or B, according to the rule that the overcrossing strand sweeps out the A quadrants when turned anticlockwise. The two possible local splittings are termed the A-split and the B-split.

A *state* s of D is a labelling of each crossing by either A or B. Making the corresponding split for each crossing gives a number of disjoint embedded closed curves, called the *state circles* for s. We write $|s|$ for the number of state circles, and $a(s)$, $b(s)$ respectively for the number of crossings labelled A, B in the state s. Following Kauffman [3] we retain information about the original crossings, in the form of chords on the split diagram, which we call

A-chords or B-chords according to the splitting, as in figure 1.

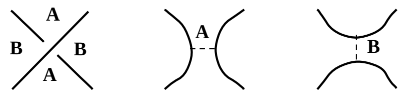


Figure 1.

The Kauffman bracket polynomial $\langle D \rangle \in \mathbf{Z}[A^{\pm 1}]$ is defined by

$$\langle D \rangle = \sum_{\text{states } s} \varphi_s,$$

where $\varphi_s = A^{a(s)-b(s)}(-A^2 - A^{-2})^{|s|-1}$.

We write $\max(s)$ for the maximum degree of φ_s, so that

$$\max(s) = a(s) - b(s) + 2|s| - 2,$$

and similarly $\min(s) = a(s) - b(s) - 2|s| + 2$.

Suppose that a state s' is given by changing k of the A-splittings of a state s to B-splittings. It can be readily shown that

$$\max(s') \leq \max(s),$$

and that equality occurs if and only if $|s'| = |s| + k$. Similarly, $\min(s') \leq \min(s)$, with equality if and only if $|s'| = |s| - k$.

Write s_A and s_B for the *extreme states* in which all crossings are labelled A or all are labelled B. It follows at once [3] that for any diagram D with $c(D)$ crossings

$$\begin{aligned} \max\deg\langle D \rangle &\leq \max(s_A) = c(D) + 2|s_A| - 2 \\ \text{and } \min\deg\langle D \rangle &\geq \min(s_B) = -(c(D) + 2|s_B| - 2). \end{aligned}$$

We will call these bounds, $\max(s_A)$ and $\min(s_B)$, the *extreme degrees* for D. They may of course be different from the actual minimum and maximum degrees. Indeed both depend on the diagram chosen to represent a given link, and they can be changed substantially by Reidemeister moves, whereas $< D >$ is unaltered by moves II and III and is simply multiplied by a power of $-A^3$ under move I.

The *spread* (or Laurent degree)

$$\beta(L) = \max\deg\langle D \rangle - \min\deg\langle D \rangle,$$

which depends only on the Jones polynomial of the link L represented by D, then satisfies the *extreme states bound*

$$\beta(L) \leq 2c(D) + 2(|s_A| + |s_B| - 2).$$

We now give an algorithm for calculating the coefficients of the terms of extreme degree, $\max(s_A)$ and $\min(s_B)$, for a chosen diagram in terms of combinatorial features of the state curves and splitting chords for each extreme state. Thistlethwaite [6] showed that for an *adequate* diagram both extreme coefficients are ± 1. Adequate diagrams include reduced alternating diagrams, and so the bound shown gives an exact count of the spread for any link with such a diagram. Thistlethwaite also found a bound for the spread expressed in terms of simple features of the decomposition of a given diagram into maximal alternating tangles. We show also how the non-alternating skeleton of the diagram can be used to calculate the extreme states bound, which is in general stronger than Thistlethwaite's bound.

2 The extreme coefficients

In this section we give a combinatorial formula for the coefficient a_{s_A} of the extreme term of degree $\max(s_A)$.

As noted in section 1, the only contributions to a_{s_A} come from states s with k B-splittings such that $|s| = |s_A| + k$, or equivalently $|s| = |s_A| + b(s)$. Each such state s then contributes $(-1)^{|s_A|+k-1}$ to the extreme coefficient a_{s_A}.

Perform the A-splitting at each crossing of D to get the $|s_A|$ state circles for the extreme state s_A, marking all the A-chords, as in figure 1.

Each state s corresponds to a selection of a subset of $b(s)$ of the A-chords at which the B-splitting is to be done in place of the A-splitting. When a change of splitting takes place the number of circles either increases or decreases by 1, depending on whether the ends of the selected A-chord lie on the same component or not. In order to finish with an extra $b(s)$ circles after performing $b(s)$ changes of splitting we must increase the number of circles at each change.

Theorem 1 *Necessary and sufficient conditions for changing a splitting using a set of k A-chords of the state circles of s_A to yield k extra curves are*

- *(i) the ends of each chord lie on the same state circle of s_A,*
- *(ii) the ends of each pair of chords which lie on the same circle do not alternate in order round the circle.*

Proof : These conditions are clearly necessary, to ensure that the number of circles increases after each change. They are also sufficient, by induction on the number of chords, since after one change the conditions are maintained. □

Definition. A subset C of the A-chords of the diagram D is *independent* if it satisfies conditions (i) and (ii) above. We include the case $C = \phi$.

Theorem 2 *The coefficient of the term of degree* $\max(s_A)$ *is given by*

$$a_{s_A} = (-1)^{|s_A|-1} \sum (-1)^{|C|},$$

where the sum is taken over all independent sets of A-chords C in the state circles for s_A.

Proof : Each set of A-chords corresponds to a state of D, and the set C makes a contribution of $(-1)^{|s_A|+|C|-1}$ to the term of degree $\max(s_A)$ if and only if C is independent, as observed above. □

A diagram is + *adequate* in Thistlethwaite's sense when $\max(s_A) > \max(s)$ for all states $s \neq s_A$. This is the case when there are no non-empty independent sets of A-chords.

An exactly similar analysis of the term of degree $\min(s_B)$ can be made in terms of the B-chords joining the state circles of the state s_B. Here the coefficient is $(-1)^{|s_B|-1} \sum (-1)^{|C|}$, with the sum taken over independent sets of B-chords. A diagram is − adequate when $\min(s_B) < \min(s)$ for all $s \neq s_B$, or equivalently when there are no non-empty independent sets of B-chords.

In these calculations the only chords that need to be considered are those whose ends lie on the same state circle of s_A (or s_B). Indeed we can reformulate theorem 2 in a graph theory context as follows.

Definition. Let K be a graph and let C be a subset of the vertices of K. Say that C is *independent* if no two vertices of C are joined by an edge of K. Define an integer-valued function f on graphs by $f(K) = \sum (-1)^{|C|}$, where the sum is taken over all independent subsets C of vertices of K, including the empty set.

Theorem 2 can then be restated in terms of the function f for a suitable graph.

Theorem 3 (graphical version) *Let K be the 'intersection graph', in the sense of Lando [1], of the A-chords with endpoints on the same state circle for s_A, namely the graph whose vertices are these chords, with an edge joining each pair of chords whose ends occur alternately on the same circle. Then the required coefficient a_{s_A} is just $(-1)^{|s_A|-1} f(K)$.*

Calculation of f for a general graph K is simplified by the use of some readily established properties.

Property 1 (Recursion) *Let $K - v$ be the subgraph of K given by deleting a vertex v and its incident edges, and let $K - N(v)$ be given by deleting the immediate neighbours of v along with their incident edges. Then*

$$f(K) = f(K - v) - f(K - N(v)).$$

Proof : This follows by grouping the subsets C into those which do and those which do not contain the vertex v. $\square$

Property 2 (Multiplication under disjoint union) *Let $K = K_1 \cup K_2$ with $K_1 \cap K_2 = \phi$ then $f(K) = f(K_1)f(K_2)$.*

Proof : This follows by induction on the number of vertices in K_2, and recursion. $\square$

Property 3 (Duplication) *If an extra vertex v is inserted which is* not *joined to one vertex w but which is joined to* all *the neighbours of w (and possibly to other vertices as well) then f is unchanged.*

Proof : This follows from the first two properties, noting that here $K - N(v)$ has an isolated vertex w and that $f = 0$ on a graph with a single vertex. $\square$

Remark. The intersection graphs which arise here are naturally bipartite graphs, since the planar diagram of chords has one set of non-intersecting chords inside each circle, and another set of non-intersecting chords outside the circle. Intersections in the graph can only take place between the two different types of chord, and can be realised by redrawing so that both sets of chords lie inside the circle, when the condition that the endpoints occur alternately will correspond to an intersection of a pair of chords. Such graphs are sometimes known as *circle graphs.*

While the calculation of $f(K)$ clearly depends only on the intersection matrix of the graph K, we do not have a simple formula for f in terms of this matrix. Clearly if there is just a single chord then $f = 0$, and equally if there is no intersection among any of the chords then again $f = 0$ by the multiplicative property. On the other hand there are plenty of examples where the graph is non-empty and the value of f is non-zero, giving us exact bounds on the spread of the bracket polynomial beyond the cases of $\pm$ adequate diagrams.

3 Alternating tangle decompositions

In using theorem 2 to calculate a_{s_A} it is enough to consider the state circles individually, because of the multiplicative property 2 of f.

Where a diagram has a substantial number of alternating edges there will in general be many of the extreme state circles with no cross-chords. These can therefore be ignored completely in the calculation.

Consider the projection of the diagram D as a 4-valent planar graph, which we call the *projection graph* of D. Each edge is either *alternating* or *non-alternating* according to the crossings at its ends. The non-alternating edges are of two types, *over* and *under*, indicated by $+$ and $-$ respectively.

The state circles of s_A and the A-chords are constructed by separating the vertices of the projection graph slightly and inserting the appropriate chord. They can be generated dynamically as circuits in the projection graph by turning right at each undercrossing of D, and left at each overcrossing. We assume throughout that D is not a split diagram, and is *reduced* in the sense that it has no cut-vertex. The closure of each complementary region of its graph in S^2 is then a disc. Any state circle for s_A consisting entirely of alternating edges forms the boundary of one of these discs, as the remainder of the graph lies entirely on the same side of the state circle. This consequently has no cross-chords. In using theorem 2 we need then only consider chords with ends on those state circles which include some non-alternating edges.

To find these systematically we draw the graph G which is dual to the non-alternating edges in the projection graph of D; there is one vertex of G in each complementary disc of the projection graph whose boundary contains non-alternating edges.

When we superimpose G on the original knot diagram D we find that the complementary regions of G are discs if G is connected, or more generally discs with holes, which meet D in alternating tangles. The intersections of D with these complementary regions are the maximal alternating pieces as defined by Thistlethwaite [6]; when G is not connected some of them will be tangles in a disc with holes, rather than a classical tangle in a disc. In our setting, Thistlethwaite's 'channels' are the planar neighbourhoods of the components of G.

In any event, the major part of each tangle can be ignored in making our calculations, and we concentrate on the graph G, which we call the *non-alternating skeleton* of D.

Each state circle for s_A made up of alternating edges bounds a disc lying entirely in one of the alternating tangles. Only the state circles with non-alternating edges intersect the skeleton G; we show how to recover them up to isotopy by making a standard splitting of the graph G. We then add the 'boundary information' about the A-chords with ends on these circles to complete the data needed in the calculations of theorem 2.

4 The non-alternating extreme state circles

The non-alternating edges of the diagram D and hence the edges of its non-alternating skeleton G come in two types, *over*, labelled $+$ and *under*, labelled $-$. As we trace out a state circle of s_A which contains some non-alternating edges we will come to a non-alternating over edge, labelled $+$, where the circle will cross G. It will then continue past some crossings, turning left each time, until it reaches the next non-alternating edge, necessarily an under edge, with sign $-$, where it again crosses G. These two edges of G have a common vertex, lying in some complementary region of the projection graph, and so the local picture of the projection graph and G will look like figure 2.

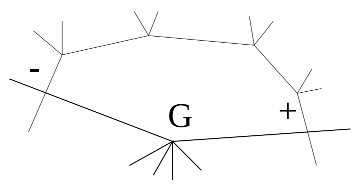


Figure 2

The segment of the s_A state circle between these adjacent crossings with G is then isotopic, relative to its endpoints, to the union of the two half-edges of G, through the complementary domain of $G \cup D$.

The $+$ and $-$ edges must alternate around each vertex of G, and all pairs of half-edges will correspond in this way to pieces of states circles. So when we break the graph G apart at each vertex by pairing adjacent $+$ and $-$ edges, matching each $+$ edge with the next $-$ edge in the anticlockwise sense, as in figure 3,

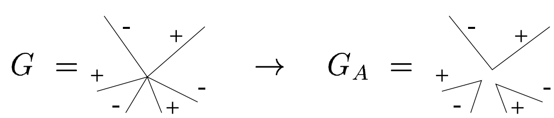


Figure 3

the resulting curves, which we denote by G_A, are isotopic to those s_A circles with some non-alternating edges. These are the only s_A circles which can appear in our extreme term calculation.

Separating all the vertices of G in the opposite sense will similarly yield curves G_B isotopic to the non-alternating s_B circles.

4.1 Boundary information

Having found all the s_A circles needed for theorem 2 we can identify those A-chords which may be involved in the formula. Recalling that only A-chords with both ends on the same circle will contribute, we may treat the components of G separately, and combine the results by use of the multiplicative property of f.

From our picture of the construction of the non-alternating s_A circles we see that the possible A-chords occur in a complementary region of G where there is an arc across the region which passes through just one crossing of the projection graph, as in figure 4.

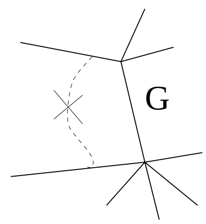


Figure 4

To give an A-chord, the crossing must be approached through the B-quadrants of the original diagram D. The A-chord which arises from the crossing is then isotopic to this arc when the s_A circle is isotoped to G_A. The union of such arcs drawn across the complementary regions of G makes up the 'boundary information' which we need for the A-chords, with a similar union of arcs drawn for the B-chords.

4.2 Algorithm for finding the extreme coefficients

Given a diagram D we can assemble the data needed to apply theorem 2 as follows.

Step 1. Construct the non-alternating skeleton G.

Step 2. Add the boundary information, as follows.

For each component of G consider separately each tangle defined by D in the complementary regions of this component. Draw any arcs across each tangle which meet D in just one crossing approached through the B-quadrants. In general there will be a relatively small number of these, as they can only involve the complementary regions of D which are adjacent to the boundary of the tangle, hence the term 'boundary information'. The result is to decorate G with a number of non-intersecting chords drawn across the complementary regions.

Step 3. Separate G into the circles G_A by splitting apart at the vertices, as in figure 3, while retaining the decorating chords. These circles are isotopic to the non-alternating s_A circles.
Step 4. Ignore any chords between different circles. For each circle separately calculate the value of f on the intersection graph given by its chords. Multiply the values, to give the extreme coefficient a_{s_A}, up to the sign $(-1)^{|s_A|-1}$.

Repeat steps 2-4 with the B-chords across the tangles, and the splitting of G into s_B circles, to find the lowest degree extreme coefficient similarly.

5 States surfaces

One of the neatest techniques in the proof of the original results about alternating diagrams is the use of 'states surfaces'. Each state s of a diagram D has a dual state $\hat{s}$ defined by changing all the markers of s. In particular the extreme states s_A and s_B are dual to each other. The states surface for s is a closed orientable surface with Euler characteristic $|s| + |\hat{s}| - c(D)$. The *extreme states surface* F for the states s_A or s_B then has Euler characteristic $\chi(F) = |s_A| + |s_B| - c(D)$ and the extreme states bound for the spread of $\langle D \rangle$, which is $2(c(D) + |s_A| + |s_B| - 2)$, can be written as $4c(D) - 4g(F)$ in terms of the genus $g(F)$ of F.

In [7] Turaev gives a construction for the extreme states surface F in which discs round each crossing of D are connected by an untwisted band for each alternating edge, and a half-twisted band for each non-alternating edge. This gives a surface with $|s_A| + |s_B|$ boundary components, and yields F when they are capped off by discs.

Make this construction with the non-alternating skeleton G in place, inserting first only the bands for the alternating edges. The boundary of the resulting planar surface includes all state circles for s_A and s_B made of alternating edges only. These all lie in complementary regions of G, along with circles parallel to the boundary of each complementary region. Capping off the alternating state circles then gives the complement of a neighbourhood of G.

The surface F is completed by adding a twisted band across each edge of G and capping off the boundary of the resulting surface. The boundary curves of this surface can be readily identified with the non-alternating state circles given by separating the vertices of G to yield the curves G_A and G_B. Then

$$\chi(F) \quad = \quad 2 - \chi(G) - e(G) + |G_A| + |G_B|$$

$$= \quad 2 - v(G) + |G_A| + |G_B|,$$

where G has $v(G)$ vertices and $e(G)$ edges. The extreme states bound is then

$$\begin{aligned} 4c(D) - 4g(F) &= 4c(D) + 2\chi(F) - 4 \\ &= 4c(D) + 2(|G_A| + |G_B|) - 2v(G). \end{aligned}$$

The extreme states bound for the spread can thus be found readily in terms of the non-alternating skeleton G, as an embedded graph (so as to find G_A and G_B).

Theorem 4 *The extreme states bound is lower than Thistlethwaite's bound in general.*

Proof : Thistlethwaite's bound for the spread $\max\deg\langle D\rangle - \min\deg\langle D\rangle$ is given in terms of the number of alternating tangles n, (the number of complementary regions of G), and the number of non-alternating edges ν $(=e(G))$. Explicitly, his bound is $4c(D) + 4(n-1) - 2\nu$.

Suppose that G has r components, so that $n = r + 1 - v(G) + e(G)$. Now construct a surface from G by putting a disc at each vertex, and joining them by a twisted band for each edge. The boundary can again be regarded as the curves G_A and G_B. Cap these off to give a closed surface F^* with r components, so that $\chi(F^*) \leq 2r$. Then

$$0 \leq 4r - 2\chi(F^*) = 4r - 2(v(G) - e(G) + |G_A| + |G_B|),$$

and so the extreme states bound of $4c(D) - 4g(F)$ satisfies

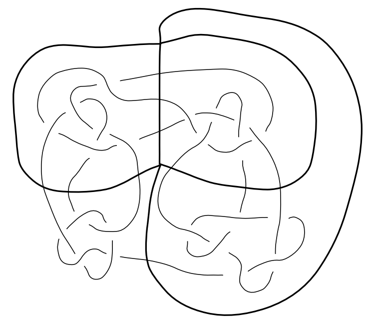
$$\begin{aligned} 4c(D) - 4g(F) &= 4c(D) + 2(|G_A| + |G_B|) - 2v(G) \\ &\leq 4c(D) + 4r - 4v(G) + 4e(G) - 2e(G) \\ &= 4c(D) + 4(n-1) - 2\nu. \end{aligned}$$

The example in the next section shows that the inequality can be strict.
□

6 Some examples

In figure 5 we show a diagram with its non-alternating skeleton G and the two sets of curves G_A and G_B resulting from splitting G. In this case $e(G) = 10$ and $\chi(G) = -2$, while $|G_A| = |G_B| = 1$. The extreme states bound for the spread of the bracket polynomial is then $4c+2\times 2-2(\chi(G)+e(G)) = 4c-12$, giving a bound of $c-3=20$ for the spread of its Jones polynomial.

By comparison, Thistlethwaite's bound with 4 alternating tangles and 10 non-alternating edges gives the weaker bound $4c+12-20=4c-8$ for the spread of the bracket polynomial.



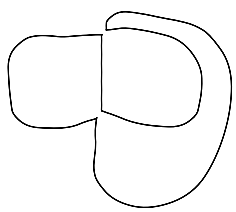
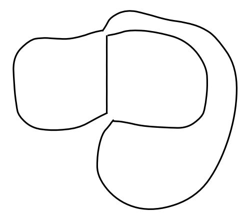
$G_A =$  , $G_B =$  .

Figure 5

Figure 6 shows the reducing A-chords on G, and the resulting chords on G_A after splitting.

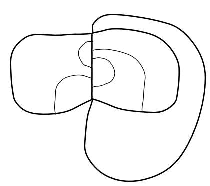
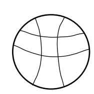
A-chords,  , $G_A =$  .

Figure 6

In view of the four chords on G_A the diagram is not + adequate. Apply the function f to the circle graph of G_A to calculate $\hat{a}_{s_A} = (-1)^{|s_A|-1} a_{s_A} = -1$.

A similar picture is needed in general to give the B-chords on G_B, but in this case there are no B-chords, so the diagram is $-$ adequate. Both extreme coefficients are then non-zero and we deduce that the exact spread of the Jones polynomial is $c-3$.

The diagram in figure 7 has the same non-alternating skeleton G and only differs from figure 5 in that the alternating tangle in one of the complementary regions of G has been rotated.

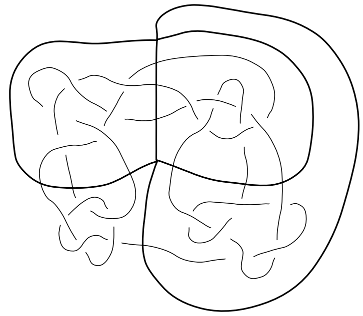


Figure 7

Thus G_A and G_B, and the bound of $c - 3$ for the spread of the Jones polynomial are unaltered. However the two reducing chords in the rotated tangle now lie in a different way relative to G, as shown in figure 8 together with the boundary information for G_A.

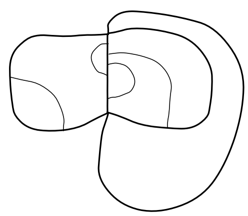
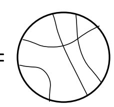
A-chords,  , $G_A =$  .

Figure 8

Here $f = 0$ for the circle graph of G_A and so $a_{s_A} = 0$. The spread of the Jones polynomial is then at most $c - 4 = 19$.

6.1 Calculations on Rolfsen's tables

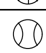
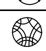
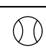
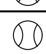
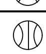
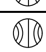
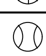
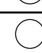
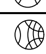
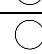
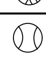
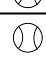
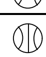
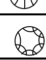
Table 1 shows the reducing chords, and the corresponding coefficients, for the diagrams of knots up to 10 crossings. The source of the table is Rolfsen's knot diagram table of 10 crossings or less. Since the maximal and minimal terms of the alternating knots are already known, we list data for non-alternating knots only. The non-alternating skeleton for all these diagrams in Rolfsen's table consists of a single circle, so G_A and G_B are a single curve in each case.

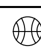
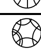
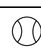
In each row of the table we show the diagrams of the A-reducing and B-reducing chords. The extreme coefficients are given by $a_{s_A} = (-1)^{|s_A|-1}\hat{a}_{s_A}$ and $a_{s_B} = (-1)^{|s_B|-1}\hat{a}_{s_B}$, where $\hat{a}_{s_A}$ and $\hat{a}_{s_B}$ are calculated directly from the intersection graph using the function f. The number $\hat{\beta}$ is the extreme states bound for the spread of the Jones polynomial coming from Rolfsen's diagram. This is equal to the actual spread β when $a_{s_A} \neq 0$ and $a_{s_B} \neq 0$. The value of β in other cases, calculated directly from the Jones polynomial, is noted for comparison in the final column of the table.

Where the diagram of G_A or G_B has no chords the knot diagram is $\pm$ adequate, so we can see that all knots of 10 crossings or less are $+$ adequate or $-$ adequate. In particular, $10_{152}, 10_{153}, 10_{154}$ are the only adequate knots of 10 crossings or less which are non-alternating, as noted by Thistlethwaite. Our table shows that the spread of a further 17 knots is given by the extreme states bound.

Rolfsen's diagram for 10_{144} was incorrect. The entry in the table here refers to a correct diagram of the knot. The Perko duplicate $10_{162} = 10_{161}$ is omitted.

Table 1. Reducing chords and extreme coefficients up to 10 crossings.

Knot	$\hat{\beta}$	$\hat{a}_{s_B}$	chords on G_B	$\hat{a}_{s_A}$	chords on G_A	$\beta(\langle L\rangle)$
8_{19}	6	0		1		5
8_{20}	6	1		1		
8_{21}	6	2		1		
9_{42}	8	0		1		6
9_{43}	8	0		1		7
9_{44}	7	1		1		
9_{45}	8	0		1		7
9_{46}	8	0		1		6
9_{47}	8	0		1		7
9_{48}	7	2		1		
9_{49}	8	0		1		7
10_{124}	9	0		1		6
10_{125}	9	0		1		8
10_{126}	9	0		1		8
10_{127}	9	0		1		8
10_{128}	8	0		1		6
10_{129}	9	0		1		8
10_{130}	9	0		1		8
10_{131}	9	0		1		8
10_{132}	7	0		1		5

Knot	$\hat{\beta}$	$\hat{a}_{s_B}$	chords on G_B	$\hat{a}_{s_A}$	chords on G_A	$\beta(\langle L \rangle)$
10_{133}	8	1		1		
10_{134}	8	1		1		
10_{135}	9	0		1		8
10_{136}	9	0		1		7
10_{137}	9	0		1		8
10_{138}	9	0		1		8
10_{139}	8	-1		1		
10_{140}	9	0		1		7
10_{141}	8	1		1		
10_{142}	9	0		1		6
10_{143}	9	0		1		8
10_{144}	8	2		1		
10_{145}	8	-1		1		
10_{146}	9	0		1		8
10_{147}	8	1		1		
10_{148}	9	0		1		8
10_{149}	9	0		1		8
10_{150}	8	1		1		
10_{151}	9	0		1		8
10_{152}	9	1		1		
10_{153}	9	1		1		
10_{154}	9	1		1		
10_{155}	9	1		0		8
10_{156}	8	1		1		
10_{157}	8	1		2		
10_{158}	9	0		1		8
10_{159}	9	0		1		8
10_{160}	8	0		1		7
10_{161}	8	-1		1		
10_{163}	8	2		1		
10_{164}	9	0		1		8
10_{165}	8	2		1		
10_{166}	9	0		1		8

6.2 Realisation of extreme coefficients

We finish with some results about the range of possible values of the extreme coefficients. It is certainly possible to find a graph K with $f(K) = n$ for any chosen integer n. Indeed it is easy to see that $f(K_{n+1}) = -n$, where K_{n+1} is the complete graph on $n+1$ vertices. On the other hand the circle graphs which determine the extreme coefficients form a proper subset of all bipartite graphs, and $f(K_{m,n}) = -1$ for the complete bipartite graph $K_{m,n}$.

We initially wondered whether any values of f besides 0 and $\pm 2^k$ were possible for the extreme coefficients. We then managed to find a circle graph with $f = 3$, illustrated in figure 9 along with its realisation by chords, and used it to produce a link with extreme coefficient 3.

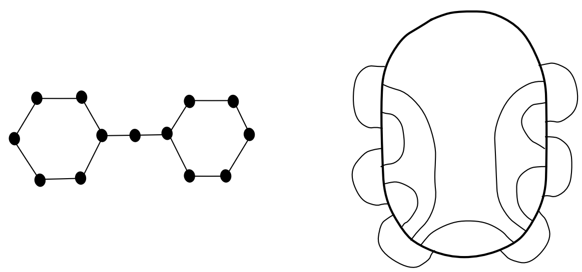


Figure 9.

This can be done by replacing each chord, , in the set $E \cup F$ of chords in the realisation, by a single crossing . The circle graph of figure 9 produces in this way the link shown in figure 10, whose bracket polynomial is $3A^{13} - 2A^9 + 4A^5 - A + 4A^{-3} - A^{-7} + A^{-11}$.

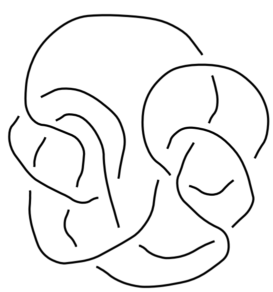


Figure 10.

More generally, given any set E of non-intersecting chords inside a circle, and another set F of non-intersecting chords outside the same circle the same procedure will construct a knot having a single curve G_A with E and F as its A-reducing chords, hence with one extreme coefficient given by the intersection graph of these chords. As in the example above, the other extreme coefficient may be zero. A more elaborate construction, using for example  in place of some or all of the chords, can be made to ensure that the tangles used in the construction are B-reduced, while retaining the same or parallel families of A-reducing chords. Starting from a circle graph with value f this will lead to a knot or link with a $-$adequate diagram, whose extreme coefficients are then f and 1, up to sign. It is equally easy to extend this so that both extreme coefficients are any chosen values of f for a circle graph.

The coefficients of the maximal and minimal degree terms in the Jones polynomial may not in general be values of f, since the knot may not have a diagram for which they appear as the extreme coefficients. Since the original version of this paper was written Manchon [5] has given a nice construction for circle graphs (or equivalently the families of chords E and F) which realise every integer value of f.

Acknowledgments

The presentation here is a substantial revision of an earlier version in which we used the dual graph of the non-alternating skeleton. We are grateful to the referee for pointing out that our treatment did not deal with the case where the non-alternating skeleton had more than one component, and for further suggested improvements.

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