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A polynomial of graphs on surfaces

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Abstract. We consider *ribbon graphs*, i.e., graphs realized as disks (vertices) joined together by strips (edges) glued to their boundaries, corresponding to neighbourhoods of graphs embedded into surfaces. We construct a four-variable polynomial invariant of these objects, the *ribbon graph polynomial*, which has all the main properties of the Tutte polynomial. Although the ribbon graph polynomial extends the Tutte polynomial, its definition is very different, and it depends on the topological structure in an essential way.

Mathematics Subject Classification (1991): 05C10

1. Introduction

The objects we consider in this paper, namely *ribbon graphs*, arise naturally as neighbourhoods of graphs embedded into surfaces. One can think of a ribbon graph as consisting of disks (vertices) attached to each other by thin strips (edges) glued to their boundaries; precise definitions are given in the next section. Unfortunately there is some variation in the terminology used in the literature; the term ribbon graph is sometimes used for related but different objects (in [9], for example), as is the alternative *fatgraph*. Our aim in this paper is to introduce a polynomial invariant of ribbon graphs we call the *ribbon graph polynomial* and denote by R. This is a polynomial in four variables which generalizes the Tutte polynomial in an essential way. The analogous task for *cyclic graphs*, i.e., orientable ribbon graphs, was accomplished in [2], although the approach taken there is very different from that used here.

The Tutte polynomial is an important graph invariant which can be defined in three different ways. The first is from a state-space expansion, or sum over all subgraphs. The second is from contraction-deletion relations, or *skein* relations as they are often called; both these descriptions are in [11]. The third definition

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is from a spanning tree expansion [12]; the relationships between the different definitions are discussed in [1]. Generalizations of the Tutte polynomial to graphs with additional structure, such as signs or weights on the edges, are important in many contexts. Examples include the signed Tutte polynomial of Kauffman [5], with its close relationship to the Jones polynomial, and the weighted Tutte polynomials of [4] and [10] connected to statistical physics. In each case one or more of the definitions of the Tutte polynomial is generalized, always including the contraction-deletion relations. In [1] graphs with arbitrary labels on the edges are considered, and the question of how far each approach to the Tutte polynomial may be generalized for such objects is answered. In particular, a universal Tutte polynomial for these objects is obtained. This includes the signed and weighted Tutte polynomials mentioned above as special cases.

A rather different generalization is that given by Noble and Welsh [8] to graphs with weights on the vertices; this is related to Vassiliev invariants. A more topological example is the Bott polynomial for CW-complexes given in [13]. Here we shall generalize the Tutte polynomial to ribbon graphs, taking account of the topological structure, simplifying and extending the results of [2]. The resulting ribbon graph polynomial turns out to have properties corresponding to all three definitions of the Tutte polynomial mentioned above.

The rest of this paper is organized as follows. In Sect. 2 we give precise definitions of ribbon graphs, and of the effect of deleting or contracting an edge e of a ribbon graph $\mathbf{G}$ to obtain $\mathbf{G}-e$ and $\mathbf{G}/e$ respectively. (Contracting an edge means using it to merge the two disks it joins to form a single vertex.) In Sect. 3 we define the ribbon graph polynomial R as a sum over all subgraphs, counting subgraphs by their graph rank and nullity, as well as by the topology of the closed surface naturally associated to each subgraph. We also state our main result, that the polynomial so defined satisfies the relations $R(\mathbf{G}) = R(\mathbf{G}/e) + R(\mathbf{G} - e)$ for an edge e which is neither a bridge nor a loop (definitions in Sect. 2), and $R(\mathbf{G}) = XR(\mathbf{G}/e)$ for a bridge e.

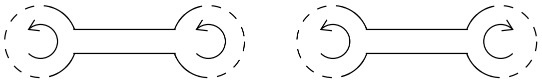
In Sect. 4 we discuss the value of R on ribbon graphs with only one vertex, thought of as signed chord diagrams. In Sect. 5 we use this notation to show that R is the universal ribbon graph invariant satisfying the contraction-deletion results given above: the spirit of this result is similar to that of Tutte's in [11], but the result is considerably more complicated and the proof very different. In Sect. 6 we give the spanning tree expansion of R, again very different from that of the Tutte polynomial. Finally, in Sect. 7 we consider the notion of a *dual* ribbon graph, showing that a certain specialization of the ribbon graph polynomial takes the same value on a ribbon graph as on its dual.

2. Ribbon graphs

Ribbon graphs can be considered either as geometric or as abstract combinatorial objects, and we shall give both definitions. The geometric definition we start with merely formalizes the idea of 'disks attached to each other by strips' described in the introduction.

A *geometric ribbon graph* **G** is a surface S with boundary, together with two finite sets of closed disks in S, a set $V(S)$ of vertices, and a set $E(S)$ of edges, subject to the following restrictions. The surface S is covered by the disks of $V(S) \cup E(S)$, and these disks intersect only in certain disjoint line segments in S. Each such line segment lies in the boundary of one vertex and one edge, and meets no other vertex or edge. Also, every edge contains exactly two such line segments. Two geometric ribbon graphs are isomorphic if there is a homeomorphism from one to the other mapping vertices to vertices and edges to edges. Note that geometric ribbon graphs need not be connected, and that edges joining a vertex to itself are allowed.

In general, geometric ribbon graphs need not be orientable; however, an oriented geometric ribbon graph **G** can be coded combinatorially in a very simple way, namely by an abstract graph with a cyclic order on the edges meeting at each vertex (a loop appearing twice in this order). These objects are known by several different names, for example 'cyclic graphs' in [2], 'graphs with rotation systems' in [7] and 'fatgraphs' in [6]. To represent a general geometric ribbon graph **G**, we orient each vertex in an arbitrary way and label each edge e with $+$ if the orientations of the two vertices incident with e are consistent across e, and $-$ otherwise. This is illustrated below.



A positive edge A negative edge

Note that changing the orientation of a vertex v reverses the cyclic order of the edges at v and also changes the sign of all edges from v to other vertices. (The sign of an edge from v to itself is unchanged.) We call such an operation a *vertex flip*. We may thus define an *abstract ribbon graph* as a cyclic graph G together with an assignment of a sign $+$ or $-$ to each edge of G. Two abstract ribbon graphs are isomorphic if one can be obtained from the other by an isomorphism of cyclic graphs composed with a series of vertex flips.

As the isomorphism classes of geometric ribbon graphs correspond naturally to those of abstract ribbon graphs, one can work with whichever definition is most convenient at any given point. For the rest of the paper we shall only distinguish geometric ribbon graphs from abstract ribbon graphs when there is some danger of confusion, and by a *ribbon graph* shall mean either. Also, as we are always interested in properties preserved by isomorphism, we shall not distinguish a ribbon graph of either type from its isomorphism class.

For a ribbon graph $\mathbf{G}$ we shall write $V(\mathbf{G})$ for the set of vertices of $\mathbf{G}$ and $E(\mathbf{G})$ for the set of edges. We shall use the following standard graph parameters: $v(\mathbf{G})$, the number of vertices, $e(\mathbf{G})$, the number of edges, $k(\mathbf{G})$, the number of components, and the combinations $r(\mathbf{G}) = v(\mathbf{G}) - k(\mathbf{G})$, the *rank* of $\mathbf{G}$, and $n(\mathbf{G}) = e(\mathbf{G}) - r(\mathbf{G})$, the *nullity* of $\mathbf{G}$. The nullity of $\mathbf{G}$ is the dimension of the cycle space of $\mathbf{G}$ as a graph, i.e., the dimension of $H_1(\mathbf{G})$ when $\mathbf{G}$ is thought of as a surface.

In addition to these graph paramaters, we consider two topological parameters of ribbon graphs. The first, $bc(\mathbf{G})$, is the number of components of the boundary of $\mathbf{G}$ considered as a geometric ribbon graph, and hence as a surface with boundary. Thus if $\mathbf{G}$ is the neighbourhood of a graph embedded into a surface, $bc(\mathbf{G})$ is the number of faces of the embedding. The second parameter is the orientability of $\mathbf{G}$ (as a surface); it will turn out to be more convenient to consider the non-orientability, so we set $t(\mathbf{G}) = 0$ if $\mathbf{G}$ is orientable and $t(\mathbf{G}) = 1$ otherwise. (The letter t can be taken to stand for twistedness.) Although $bc(\mathbf{G})$ and $t(\mathbf{G})$ are most naturally defined for geometric ribbon graphs, they can be easily read off from an abstract ribbon graph. For example, $t(\mathbf{G}) = 1$ if and only if the product of the signs of the edges of some cycle in $\mathbf{G}$ is negative.

In passing from graphs embedded in closed surfaces to ribbon graphs, the only topological information we lose is the topology of each face of the embedding. There is thus a natural way to assign a unique closed surface $\sigma(\mathbf{G})$ to each ribbon graph, namely to take the faces to be disks. Thinking of $\mathbf{G}$ as a geometric ribbon graph, we form $\sigma(\mathbf{G})$ from $\mathbf{G}$ by glueing one disk to each component of the boundary of $\mathbf{G}$. It is easy to see that $\sigma(\mathbf{G})$ is orientable if and only if $\mathbf{G}$ is. Also, if $\mathbf{G}$ is connected then the Euler characteristic of $\sigma(\mathbf{G})$ is $bc(\mathbf{G}) - n(\mathbf{G}) + 1$. Thus, for connected $\mathbf{G}$, $\sigma(\mathbf{G})$ is determined by $bc(\mathbf{G}) - n(\mathbf{G})$ and $t(\mathbf{G})$.

In order to describe the properties of the ribbon graph polynomial we consider four natural operations on ribbon graphs: contracting and deleting edges, and taking disjoint unions and connected sums. Let e be an edge of a ribbon graph $\mathbf{G}$. Then e is a *loop* in $\mathbf{G}$ if the two ends of e are adjacent to the same vertex v of $\mathbf{G}$. The edge e is a *bridge* in $\mathbf{G}$ if its removal disconnects a component of $\mathbf{G}$, and e is an *ordinary edge* of $\mathbf{G}$ if it is neither a bridge nor a loop. For any edge e of $\mathbf{G}$, we write $\mathbf{G} - e$ for the ribbon graph obtained from $\mathbf{G}$ by deleting e. The 'dual' operation of contraction is slightly more complicated: if e is not a loop, then it is incident to two distinct vertices v_1 and v_2 of $\mathbf{G}$. Considering $\mathbf{G}$ as a geometric ribbon graph, $e \cup v_1 \cup v_2$ is a disk, and the ribbon graph obtained from $\mathbf{G}$ by *contracting* e is defined by deleting e, v_1 and v_2 from $\mathbf{G}$ and replacing them by a new vertex $e \cup v_1 \cup v_2$. Thus $\mathbf{G}$ and $\mathbf{G}/e$ have the same underlying surface with boundary, but different structures as graphs. We can also define contraction directly on abstract ribbon graphs: as e is not a loop we may assume that the sign of e is positive (flipping v_1, say, if it is not). Then $\mathbf{G}/e$ is formed from $\mathbf{G}$ by deleting e, and uniting v_1 and v_2 to form a single vertex. The cyclic order is obtained from those at v_1 and v_2 by uniting them using the position of e in each order. In either case there is also a natural definition of contraction for a loop e, but as this is rather different we postpone it until Sect. 7.

For later use we note how the basic ribbon graph parameters change when an edge e of $\mathbf{G}$ which is not a loop is contracted: $k(\mathbf{G})$, $n(\mathbf{G})$, $bc(\mathbf{G})$ and $t(\mathbf{G})$ are all unchanged, while $v(\mathbf{G})$ and $r(\mathbf{G})$ decrease by one.

There are two natural ways of combining two ribbon graphs $\mathbf{G_1}$ and $\mathbf{G_2}$. The first, the *disjoint union* $\mathbf{G_1} \dot{\cup} \mathbf{G_2}$, needs no comment. The second, the *one-point join* or *connected sum* $\mathbf{G_1} \cdot \mathbf{G_2}$, is formed by taking disjoint ribbon graphs (isomorphic to) $\mathbf{G_1}$ and $\mathbf{G_2}$, choosing any vertices $v_1 \in V(\mathbf{G_1})$ and $v_2 \in V(\mathbf{G_2})$, and identifying a segment of the boundary of v_1 with a segment of the boundary of v_2, these segments being disjoint from the edges of the ribbon graphs. Combinatorially, we unite the vertices v_1 and v_2, combining their cyclic orders so that all the edges of v_1 are consecutive in the new order. As with other notions of connected sum (for example for chord diagrams or singular knots) this definition is ambiguous—there are in general many non-isomorphic one-point joins of $\mathbf{G_1}$ and $\mathbf{G_2}$, depending on which vertices are chosen and how they are united. Note that all the parameters $v(\mathbf{G})$, $e(\mathbf{G})$, $k(\mathbf{G})$, $r(\mathbf{G})$, $n(\mathbf{G})$ and $bc(\mathbf{G})$ are additive with respect to taking disjoint unions. Also, $e(\mathbf{G})$, $r(\mathbf{G})$ and $n(\mathbf{G})$ are additive with respect to taking one-point joins, while for $v(\mathbf{G})$, $k(\mathbf{G})$ and $bc(\mathbf{G})$ we 'lose one'—thus $bc(\mathbf{G_1} \cdot \mathbf{G_2}) = bc(\mathbf{G_1}) + bc(\mathbf{G_2}) - 1$. Also we have $t(\mathbf{G_1} \dot{\cup} \mathbf{G_2}) = t(\mathbf{G_1} \cdot \mathbf{G_2}) = \max\{t(\mathbf{G_1}), t(\mathbf{G_2})\}$.

The final notion we shall need is that of a *spanning subgraph* of a ribbon graph $\mathbf{G}$. This is a ribbon graph $\mathbf{H}$ formed from $\mathbf{G}$ by deleting some set of the edges, keeping all vertices. It is when we consider deleting edges or taking subgraphs that the graph structure of a ribbon graph becomes important; $\mathbf{G}$ and $\mathbf{G}/e$ correspond to the same surface but have very different collections of subgraphs.

3. The ribbon graph polynomial

As mentioned in the introduction, the ribbon graph polynomial can be defined in several ways. We are now ready to give the most direct definition, in terms of a state-space expansion, or sum over all subgraphs.

For a ribbon graph $\mathbf{G}$ we define the *ribbon graph polynomial* of $\mathbf{G}$ to be

$$\begin{aligned} R(\mathbf{G}) &= R(\mathbf{G}; X, Y, Z, W) \\ &= \sum_{\mathbf{H} \subset \mathbf{G}} (X-1)^{r(\mathbf{G})-r(\mathbf{H})} Y^{n(\mathbf{H})} Z^{k(\mathbf{H})-bc(\mathbf{H})+n(\mathbf{H})} W^{t(\mathbf{H})}, \end{aligned} \tag{1}$$

considered as an element of the quotient of $\mathbb{Z}[X, Y, Z, W]$ by the ideal generated by $W^2 - W$. The sum is over all $2^{e(\mathbf{G})}$ spanning subgraphs of $\mathbf{G}$. Before turning to the properties of the ribbon graph polynomial, we discuss the normalization, which at first sight looks rather arbitrary.

The exponents above are non-negative, and for the subgraph $\mathbf{H}$ with no edges each of them is zero, so $\mathbf{H}$ contributes 1 to the sum giving $R(\mathbf{G})$. Also, when $\mathbf{H}$ is connected the exponent of Z is $2-\chi(\sigma(\mathbf{H}))$, where χ is the Euler characteristic. When $\mathbf{H}$ is orientable this is twice the genus of $\sigma(\mathbf{H})$, otherwise it is exactly the genus of $\sigma(\mathbf{H})$. This quantity is additive with respect to taking the connected sum of two surfaces, so $k(\mathbf{H}) - bc(\mathbf{H}) + n(\mathbf{H})$ is additive with respect to one-point joins, which will be useful later. We can think of the part $Z^{k(\mathbf{H})-bc(\mathbf{H})+n(\mathbf{H})}W^{t(\mathbf{H})}$ in the definition of $R(\mathbf{G})$ as coding the topology of $\sigma(\mathbf{H})$ when this surface is connected, and the topology of the connected sum of its components otherwise.

The part $(X-1)^{r(\mathbf{G})-r(\mathbf{H})}Y^{n(\mathbf{H})}$ in the definition of $R(\mathbf{G})$ is similar to the rank generating function expansion of the Tutte polynomial. We have written $X-1$ rather than X because this is more natural when we come to the contraction-deletion relations, and also because the coefficients obtained are non-negative. The coefficients would also be non-negative with $X-1$ replaced by X, but this is a much weaker statement. Similarly, we write Y rather than $Y-1$ (which would be closer to the Tutte polynomial, and would be more natural in Sect. 7) because with $Y-1$ the coefficients would no longer be non-negative.

We now turn to the main property of the ribbon graph polynomial, namely the contraction-deletion relations, or skein relations as they are often called. First we consider the behaviour of R with respect to disjoint unions and one-point joins.

When $\mathbf{H}$ is a spanning subgraph of $\mathbf{G}$, $k(\mathbf{H}) - k(\mathbf{G}) = r(\mathbf{G}) - r(\mathbf{H})$, so

$$R(\mathbf{G}) = (X-1)^{-k(\mathbf{G})} \sum_{\mathbf{H}\subset\mathbf{G}} M(\mathbf{H}),$$

where

$$M(\mathbf{H}) = (X-1)^{k(\mathbf{H})}Y^{n(\mathbf{H})}Z^{k(\mathbf{H})-bc(\mathbf{H})+n(\mathbf{H})}W^{t(\mathbf{H})}.$$

This way of writing $R(\mathbf{G})$ will be more convenient as $M(\mathbf{H})$ depends only on $\mathbf{H}$ and not on $\mathbf{G}$. Now from the behaviour of the basic graph parameters with respect to disjoint unions and one-point joins we have $M(\mathbf{H_1} \dot\cup \mathbf{H_2}) = M(\mathbf{H_1})M(\mathbf{H_2}) = (X-1)M(\mathbf{H_1}\cdot\mathbf{H_2})$ (bearing in mind that $W^2 = W$ in the ring we are working in). If $\mathbf{G_1}$ and $\mathbf{G_2}$ are ribbon graphs, then as $\mathbf{H_1}$ and $\mathbf{H_2}$ run over all spanning subgraphs of $\mathbf{G_1}$ and $\mathbf{G_2}$ respectively, $\mathbf{H_1}\dot\cup\mathbf{H_2}$ and $\mathbf{H_1}\cdot\mathbf{H_2}$ run respectively over all spanning subgraphs of $\mathbf{G_1}\dot\cup\mathbf{G_2}$ and of $\mathbf{G_1}\cdot\mathbf{G_2}$. Thus

$$R(\mathbf{G_1}\dot\cup\mathbf{G_2}) = R(\mathbf{G_1})R(\mathbf{G_2}), \tag{2}$$

and

$$R(\mathbf{G_1}\cdot\mathbf{G_2}) = R(\mathbf{G_1})R(\mathbf{G_2}). \tag{3}$$

We now turn to the contraction-deletion relations themselves.

Theorem 1. *Let* $\mathbf{G}$ *be any ribbon graph. Then*

$$R(\mathbf{G}) = R(\mathbf{G}/e) + R(\mathbf{G} - e) \tag{4}$$

for every ordinary edge e *of* $\mathbf{G}$*, and*

$$R(\mathbf{G}) = XR(\mathbf{G}/e) \tag{5}$$

for every bridge e *of* $\mathbf{G}$*.*

Proof. The subgraphs of $\mathbf{G}$ not containing e are precisely the subgraphs of $\mathbf{G}-e$. Also, if e is not a loop then the map $\mathbf{H} \mapsto \mathbf{H}/e$ gives a bijection from the subgraphs of $\mathbf{G}$ containing e to the subgraphs of $\mathbf{G}/e$. From the properties of the ribbon graph parameters described in Sect. 2 we have $M(\mathbf{H}) = M(\mathbf{H}/e)$ whenever e is an edge of $\mathbf{H}$ which is not a loop. Thus if e is an ordinary edge of $\mathbf{G}$, so that $k(\mathbf{G} - e) = k(\mathbf{G}/e) = k(\mathbf{G})$, then

$$\begin{aligned} R(\mathbf{G}) &= (X-1)^{-k(\mathbf{G})} \sum_{\mathbf{H}\subset\mathbf{G}, e\notin E(\mathbf{H})} M(\mathbf{H}) + (X-1)^{-k(\mathbf{G})} \sum_{\mathbf{H}\subset\mathbf{G}, e\in E(\mathbf{H})} M(\mathbf{H}) \\ &= (X-1)^{-k(\mathbf{G}-e)} \sum_{\mathbf{H}\subset\mathbf{G}-e} M(\mathbf{H}) + (X-1)^{-k(\mathbf{G}/e)} \sum_{\mathbf{H}'\subset\mathbf{G}/e} M(\mathbf{H}') \\ &= R(\mathbf{G}-e) + R(\mathbf{G}/e), \end{aligned}$$

proving (4). If e is a bridge in $\mathbf{G}$ then $k(\mathbf{G} - e) = k(\mathbf{G}) + 1$, so

$$\begin{aligned} R(\mathbf{G}) &= (X-1)^{-k(\mathbf{G})} \sum_{\mathbf{H}\subset\mathbf{G}-e} M(\mathbf{H}) + (X-1)^{-k(\mathbf{G})} \sum_{\mathbf{H}'\subset\mathbf{G}/e} M(\mathbf{H}') \\ &= (X-1)R(\mathbf{G}-e) + R(\mathbf{G}/e). \end{aligned}$$

However, $\mathbf{G} - e$ can be written as the disjoint union of two ribbon graphs $\mathbf{G_1}$ and $\mathbf{G_2}$ such that $\mathbf{G}/e = \mathbf{G_1} \cdot \mathbf{G_2}$. From (2) and (3) we thus have $R(\mathbf{G} - e) = R(\mathbf{G_1})R(\mathbf{G_2}) = R(\mathbf{G}/e)$, so (5) follows. □

As for the Tutte polynomial, or many link invariants, the contraction-deletion relations can be taken as a definition of the ribbon graph polynomial, if we also impose a suitable ‘boundary condition’, discussed in the next section. Note that we cannot impose the relation $R(\mathbf{G}) = YR(\mathbf{G} - e)$ satisfied by the Tutte polynomial—if we did, R would become exactly the Tutte polynomial. However, as a consequence of our definition (and thus, as we shall see later, of the relations in Theorem 1) we do obtain a simple relation for certain special loops.

We shall say that a loop e at a vertex v of a ribbon graph $\mathbf{G}$ is *trivial* if there is no cycle in $\mathbf{G}$ which can be contracted to form a loop f interlaced with e, i.e, a loop f at v such that the ends of e and f alternate in the cyclic order at v. We say that a loop e at v is *twisted* if $v \cup e$ forms a Möbius band as opposed to an annulus. As an easy consequence of (1) we have for an untwisted trivial loop that

$$R(\mathbf{G}) = (1 + Y)R(\mathbf{G} - e),$$

and for a twisted trivial loop that

$$R(\mathbf{G}) = (1 + YZW)R(\mathbf{G} - e),$$

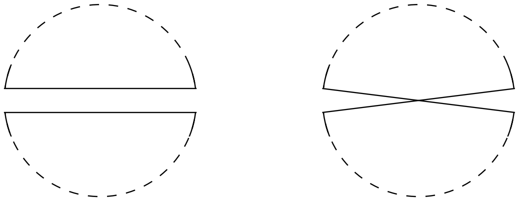
since adding a trivial loop e to any subgraph $\mathbf{H}$ of $\mathbf{G}$ increases $n(\mathbf{H})$ by one, and increases $bc(\mathbf{H})$ by one if and only if e is untwisted.

4. One-vertex ribbon graphs

Together with (2), the relations given in Theorem 1 allow one to calculate the ribbon graph polynomial of any ribbon graph from the values of R on ribbon graphs with only one vertex. Here we shall give an alternative description of R on one-vertex ribbon graphs, using the notion of a signed chord diagram. Although this will be equivalent to our earlier definition, chord diagrams are easier to draw and to visualize, which will be useful in Sect. 5, for example.

A *chord diagram* consists of $2n$ distinct points on the unit circle paired off by n chords. Two chord diagrams are usually thought of as isomorphic if there is an orientation preserving homeomorphism from the unit circle to itself mapping one to the other. These objects arise in many contexts, for example in the study of Vassiliev invariants, where they correspond to singular knots. A *signed chord diagram* is a chord diagram together with an assignment of a sign $+$ or $-$ to each chord. We consider two signed chord diagrams to be isomorphic if there is *any* homeomorphism from the circle to itself mapping one to the other, i.e., we consider mirror image signed chord diagrams to be isomorphic. With this definition the isomorphism classes of signed chord diagrams correspond naturally to the isomorphism classes of one-vertex ribbon graphs: each chord corresponds to an edge, the order of the endpoints of the chords round the circle corresponds to the cyclic order at the vertex, and the sign of a chord is just the sign of the corresponding edge. As every edge is a loop, when the single vertex is flipped the cyclic order is reversed but the signs of the edges do not change.

If $\mathbf{G}$ is a one-vertex ribbon graph and D the corresponding signed chord diagram then the number of chords of D is equal to $e(\mathbf{G})$. As $\mathbf{G}$ has only one vertex, $n(\mathbf{G}) = e(\mathbf{G})$. Thus we shall write $n(D)$ for the number of chords of D, which is also the nullity of the corresponding ribbon graph. We can also read off $bc(\mathbf{G})$ from D using the following doubling operation: replace each chord of D by two edges joining the parts of the circle on each side of each end of the chord D as shown below (these edges do not meet even when they are drawn crossing each other).



A doubled positive chord   A doubled negative chord

We shall write $bc(D)$ for the number of components of the resulting figure, as this is equal to $bc(\mathbf{G})$. This function on unsigned chord diagrams is considered for example in [3]. We also write $t(D)$ for $t(\mathbf{G})$, so $t(D) = 0$ if all chords of D have a positive sign, and $t(D) = 1$ otherwise.

A *subdiagram* of a signed chord diagram D is a signed chord diagram D' obtained from D by deleting some subset of the chords of D. Thus D has exactly $2^{n(D)}$ subdiagrams, some of which may be isomorphic. With this notation we can re-write the definition (1) of $R(\mathbf{G})$ for a one-vertex ribbon graph $\mathbf{G}$ as

$$R(\mathbf{G}) = \sum_{D' \subset D} Y^{n(D')} Z^{1-bc(D')+n(D')} W^{t(D')} \tag{6}$$

where D is the signed chord diagram corresponding to $\mathbf{G}$ and the sum is over all subdiagrams of D. As mentioned in the introduction, this boundary condition and the relations (2), (4) and (5) can be taken as the definition of the ribbon graph polynomial. A priori it is not clear that there is a function with all these properties; this can be proved directly using the methods of [2], but as we have an explicit formula for such a function R, no such proof is needed.

If $\mathbf{G}$ is an orientable ribbon graph, then so are $\mathbf{G}/e$ and $\mathbf{G} - e$ whenever they are defined (we have not yet defined $\mathbf{G}/e$ for e a loop). Thus the contraction-deletion relations for R can be used to calculate the ribbon graph polynomial of $\mathbf{G}$ from the values of R on orientable one-vertex ribbon graphs, corresponding to signed chord diagrams in which every edge is positive. Comparing the boundary condition (6) for the ribbon graph polynomial with that for the cyclic graph polynomial C given in [2], we see that $R(\mathbf{G}; X, Y, Z^{1/2}, W)$ and $C(\mathbf{G}; X, Y, Z)$ coincide exactly on orientable one-vertex ribbon graphs, and hence on all orientable ribbon graphs. Thus all the properties of the cyclic graph polynomial are inherited by the ribbon graph polynomial. In particular, the ribbon graph polynomial specializes to the Tutte polynomial of the underlying graph, but depends very strongly on the extra structure of the ribbon graph.

5. Universality

In this section we shall show that the ribbon graph polynomial R is the universal invariant of connected ribbon graphs satisfying (4) and (5), in that any other such invariant can be calculated from R. To prove this we use the definition of R from the contraction-deletion relations and the boundary condition (6). As the method is very similar to that used to prove the corresponding result for the cyclic graph polynomial in [2] we shall only sketch the details. As in the previous section, we consider only connected ribbon graphs; the extension to all ribbon graphs using (2) requires very little extra work.

Let $\mathcal{G}$ be the set of isomorphism classes of connected ribbon graphs. We shall write R_{ijk} for the coefficient of $Y^i Z^j W^k$ in R, so R_{ijk} is a map from $\mathcal{G}$ to $\mathbb{Z}[X]$. Given a ring $\mathbf{R}$ and an element x of $\mathbf{R}$, by $R_{ijk}(x)$ or $R_{ijk}(\mathbf{G}; x)$ we shall mean the map from $\mathcal{G}$ to $\mathbf{R}$ given by composing R_{ijk} with the natural ring homomorphism from $\mathbb{Z}[X]$ to $\mathbf{R}$ mapping X to x. Note that infinite sums of such functions make sense, as only finitely many are non-zero on any given ribbon graph.

Theorem 2. *Let $\mathbf{R}$ be a commutative ring, x an element of $\mathbf{R}$, and ϕ a map from $\mathcal{G}$ to $\mathbf{R}$ satisfying*

$$\phi(\mathbf{G}) = \begin{cases} \phi(\mathbf{G}/e) + \phi(\mathbf{G} - e) & \text{if } e \text{ is ordinary,} \\ x\phi(\mathbf{G}/e) & \text{if } e \text{ is a bridge.} \end{cases} \tag{7}$$

Then there are elements λ_{ijk}, $i \geq 0$, $0 \leq j \leq i$, $0 \leq k \leq 1$, such that

$$\phi = \sum_{i,j,k} \lambda_{ijk} R_{ijk}(x). \tag{8}$$

Proof. We use the signed chord diagram notation of Sect. 4, sometimes omitting the sign of positive chords for clarity in the figures. We shall use Greek letters to denote parts of the diagrams which are the same in all diagrams in which they appear. Also, we shall write $\overline{\alpha}$ for part of a diagram obtained by *reflecting* α, i.e., by reversing the order of the endpoints of the chords, and changing the sign of any chord from α to the rest of the diagram. (Think of reflecting part of the doubled chord diagram.) This is illustrated in Fig. 1.

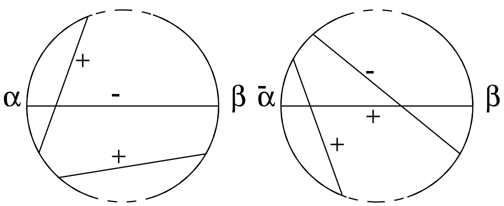
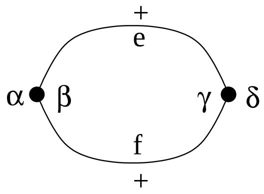


Fig. 1. Corresponding realizations of chord diagrams $\alpha\beta$ and $\overline{\alpha}\beta$

Let $\mathbf{G}$ be a two-vertex ribbon graph, and let e and f be edges of $\mathbf{G}$ which are not loops. Suppose for the moment that e and f are positive edges. Let us write

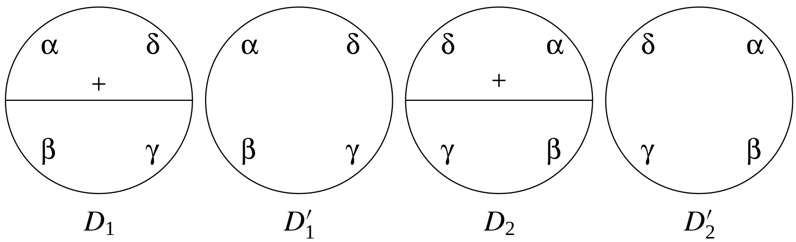
α, β, γ and δ for the sections into which e and f divide the cyclic orders at the vertices of **G**, so **G** is as shown below.



Applying relation (7) gives two different expressions for $\phi(\mathbf{G})$, depending on whether we first apply the relation to the edge e and then to f (if it is not a loop), or the other way round. Equating these expressions shows that

$$\phi(D_1) - a\phi(D_1') = \phi(D_2) - a\phi(D_2') \tag{9}$$

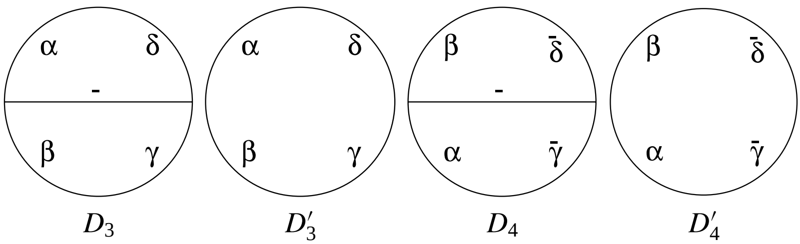
whenever D_1, D_1', D_2, D_2' are signed chord diagrams related as shown below, with $a = 1$ if there is some chord from $\alpha \cup \beta$ to $\gamma \cup \delta$ and $a = x$ otherwise.



Similarly, considering the case when f is negative shows that

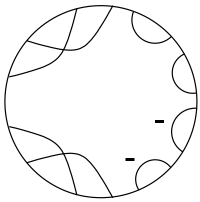
$$\phi(D_3) - a\phi(D_3') = \phi(D_4) - a\phi(D_4') \tag{10}$$

whenever D_3, D_3', D_4, D_4' are signed chord diagrams related as shown below, with the definition of a as above.



We shall say that two signed chord diagrams (or *diagrams* for short) are related by a *rotation about the chord e* if they are related as D_1 and D_2 above, and that they are related by a *twist* about e if they are related as D_3 and D_4.

Note that we only rotate about positive chords, and only twist about negative ones. We shall say that two diagrams are *related* if they are related by a sequence of rotations and twists. As in [2], the key fact is that any diagram is related to a *canonical* diagram; for signed chord diagrams we take as canonical the diagrams D_{ijk}, $0 \le k \le 2$, consisting of $i - 2j - k$ positive chords intersecting no other chords, j pairs of intersecting positive chords, and k negative chords intersecting no other chords, arranged as shown below.



$D_{8,2,2}$

As in [2], we can prove that any diagram is related to a canonical one by induction on the number of chords—it suffices to consider adding a single chord to a canonical diagram. As the new chord intersects at most two components of the diagram, this reduces to a finite check, in fact to one easy to complete by hand.

Suppose now that $n \ge 0$ and that we have chosen elements $(\lambda_{ijk})_{i<n}$ of $\mathbf{R}$ in such a way that the quantity $\phi' = \phi - \sum_{i<n} \lambda_{ijk} R_{ijk}(x)$ vanishes on chord diagrams with fewer than n chords. Since ϕ and the R_{ijk} satisfy (7), so does ϕ'. Thus the R_{ijk} and ϕ' satisfy (9) and (10). As ϕ' vanishes on diagrams with $n - 1$ chords, ϕ' thus takes the same value on any two related diagrams with n chords. Now the canonical diagrams with n chords are distinguished by the quantities $bc(D_{njk})$ and $t(D_{njk})$. Thus for each j and k there is an $R_{nj'k'}$ such that $R_{nj'k'}(D_{nj''k''})$ is one if $j'' = j$ and $k'' = k$, and zero otherwise. This implies that we can choose the λ_{njk} so that (8) holds on the D_{njk}, and hence on all chord diagrams with n chords.

Proceeding by induction we obtain λ_{ijk} such that (8) holds on all one-vertex ribbon graphs. Using (7), the result follows. □

As with other generalizations of the Tutte polynomial, one can ask for an equivalent of Theorem 2 with condition (7) replaced by

$$\phi(\mathbf{G}) = \begin{cases} \sigma\phi(\mathbf{G}/e) + \tau\phi(\mathbf{G} - e) & \text{if } e \text{ is ordinary,} \\ x\phi(\mathbf{G}/e) & \text{if } e \text{ is a bridge,} \end{cases} \tag{11}$$

for fixed elements x, σ and τ of $\mathbf{R}$. When σ and τ are invertible, we obtain nothing new. In this case if $\phi(\mathbf{G})$ satisfies (11) then $\phi'(\mathbf{G}) = \sigma^{-r(\mathbf{G})}\tau^{-n(\mathbf{G})}\phi(\mathbf{G})$ satisfies (7) with x replaced by $x\sigma^{-1}$, so we may apply Theorem 2 to this function. At the other extreme, if σ and τ are zero then ϕ can take arbitrary values on one-vertex ribbon graphs. For general σ and τ the situation seems more complicated and less interesting.

6. Spanning tree expansion

As can be seen from the alternative definition given in Sect. 4, the coefficients of the ribbon graph polynomial are always non-negative. It is thus natural to ask whether there is a direct interpretation for these coefficients as counting something. As for the Tutte polynomial, the answer is yes, although the interpretation is more complicated. The argument given below is a modification of the one concerning cyclic graphs given in [2].

We shall restrict our attention to connected ribbon graphs just for notational convenience; everything that follows can be easily translated to the general case. Suppose **G** is a connected ribbon graph. A *spanning tree* T of **G** is a spanning subgraph of **G** which is connected and has zero nullity. Suppose an order $\prec$ on the edges of **G** is given. For an edge e of T the *cut* determined by e and T (and **G**) is the set of edges of **G** between the two components of $T - e$. The edge e is said to be *internally active* if e is the first edge of this cut, in the order $\prec$. An edge $e \in E(G) \setminus E(T)$ is *externally active* if e is the first edge of the unique cycle in $T \cup e$. These definitions were introduced by Tutte in [12] in order to define the Tutte polynomial.

We define the *weight* of a spanning tree T (depending on **G** and $\prec$) to be

$$w(T, \mathbf{G}, \prec) = X^{i(T)} \sum_{S} Y^{n(T\cup S)} Z^{1-bc(T\cup S)+n(T\cup S)} W^{t(T\cup S)},$$

where $i(T)$ is the number of edges of T which are internally active, the sum is over all subsets of the set of externally active edges, and $T \cup S$ is the spanning subgraph of **G** formed by the edges of T together with those of S. Let

$$w(\mathbf{G}, \prec) = \sum_{T} w(T, \mathbf{G}, \prec),$$

where the sum is over all spanning trees of **G**. It turns out that $w(\mathbf{G}, \prec)$ is independent of the order $\prec$, and coincides with $R(\mathbf{G})$.

Theorem 3. *For a ribbon graph* **G** *and any order* $\prec$ *on* $E(\mathbf{G})$ *we have* $w(\mathbf{G}, \prec) = R(\mathbf{G})$.

Proof. We use induction on the number of edges of **G** which are not loops: if there are no such edges, **G** is a one-vertex ribbon graph. Thus there is only one spanning tree (with no edges), every edge is externally active, and the definition of $w(\mathbf{G}, \prec)$ coincides with (6).

Otherwise, let e be the last edge of **G** in the order $\prec$ which is not a loop. If e is a bridge then every spanning tree of **G** contains e, e is always internally active, and as the activities of other edges are not affected by contracting e, $w(\mathbf{G}, \prec) = Xw(\mathbf{G}/e, \prec)$. If e is an ordinary edge then e is never active either internally or externally. The spanning trees of **G** containing e are in bijection with those of $\mathbf{G}/e$, while those not containing e are exactly the spanning trees of $\mathbf{G}-e$. Again neither deleting nor contracting e affects the activity of other edges, so $w(\mathbf{G}, \prec) = w(\mathbf{G}/e, \prec) + w(\mathbf{G}-e, \prec)$. Combining this with the contraction-deletion relations (4) and (5) for R, we see by induction that $w(\mathbf{G}, \prec) = R(\mathbf{G})$, completing the proof. □

7. Dual ribbon graphs

If G is an abstract graph which can be embedded into the plane, then one can use such an embedding to define a dual graph G^*, which turns out to be essentially unique. The Tutte polynomial has the fundamental property that for such a dual graph $T(G; X, Y) = T(G^*; Y, X)$. For non-planar graphs there is no corresponding notion, as one does not know which surface to embed the graph into. For a ribbon graph $\mathbf{G}$, on the other hand, there is a very natural notion of the dual ribbon graph $\mathbf{G}^*$. (This and everything else in this section is true without modification if we consider cyclic graphs instead of ribbon graphs.)

Let $\mathbf{G}$ be a geometric ribbon graph. In forming the closed surface $\sigma(\mathbf{G})$ from $\mathbf{G}$ we glue $bc(\mathbf{G})$ disks to $\mathbf{G}$ along its boundary. We take these as the vertices of $\mathbf{G}^*$, and the edges of $\mathbf{G}$ as the edges of $\mathbf{G}^*$. Then it is easy to see that $\mathbf{G}^*$ is a ribbon graph, realised as a subspace of the surface $\sigma(\mathbf{G})$. Also, we have $\sigma(\mathbf{G}^*) = \sigma(\mathbf{G})$ and $(\mathbf{G}^*)^* = \mathbf{G}$.

In the planar case, an edge e is a bridge in $\mathbf{G}$ if and only if it is a loop in $\mathbf{G}^*$, and vice versa. Thus e is ordinary in $\mathbf{G}$ if and only if it is ordinary in $\mathbf{G}^*$. Also, as long as we avoid deleting bridges or contracting loops, contracting e in $\mathbf{G}$ corresponds to deleting e in $\mathbf{G}^*$ and vice versa. This, together with a check on graphs with no edges, gives the duality property of the Tutte polynomial.

For ribbon graphs the situation is more complicated: a bridge in $\mathbf{G}$ corresponds to a special kind of loop in $\mathbf{G}^*$, namely a trivial loop (defined at the end of Sect. 3). Thus an edge e may be ordinary in $\mathbf{G}$ but a non-trivial loop in $\mathbf{G}^*$. This suggests that there will be no simple transformation taking $R(\mathbf{G})$ into $R(\mathbf{G}^*)$—the former obeys a contraction-deletion relation for every ordinary edge of $\mathbf{G}$, and hence for every non-trivial loop of $\mathbf{G}^*$. The ribbon graph polynomial does not in general obey such a relation. On the other hand, it turns out that there is a specialization of the ribbon graph polynomial which does.

Let e be a loop at a vertex v of a ribbon graph $\mathbf{G}$. We define the operation of contracting e as follows: consider the union of e and v in $\mathbf{G}$ (thought of as a surface). This is either a Möbius band or an annulus, so its boundary consists of one or two circles. Form $\mathbf{G}/e$ from $\mathbf{G}$ by deleting e and v, and adding one or two new vertices, these being disks the union of whose boundaries is the boundary of $e \cup v$. That this operation is the natural definition can be seen in two ways. Firstly, the corresponding combinatorial definition is almost exactly that for contracting a non-loop e: taking e to be positive for simplicity, we contract

e by uniting the cyclic orders at the vertices adjacent to v, setting the successor of the predeccessor of one end of e to be the successor of the other end of e. Performing the same operation for a loop at v splits the cyclic order at v into two cycles, which we take to represent two vertices into which v is split. Secondly, considering $\mathbf{G}$ and $\mathbf{G}^*$ as geometric ribbon graphs realised as subspaces of the same surface $\sigma(\mathbf{G})$, we see that for any edge e of $\mathbf{G}$ we now have $(\mathbf{G}-e)^* = \mathbf{G}^*/e$ and $(\mathbf{G}/e)^* = \mathbf{G}^* - e$.

Consider the polynomial $p(\mathbf{G})$ defined by

$$p(\mathbf{G}) = \sum_{\mathbf{H}\subset\mathbf{G}} t^{bc(\mathbf{H})-1}, \tag{12}$$

where the sum is over all spanning subgraphs of a ribbon graph $\mathbf{G}$. Following the first part of the proof of Theorem 1 we have that

$$p(\mathbf{G}) = p(\mathbf{G}/e) + p(\mathbf{G} - e) \tag{13}$$

for any edge e of $\mathbf{G}$. This together with $p(\mathbf{E_n}) = t^{n-1}$ for the ribbon graph $\mathbf{E_n}$ with n vertices an no edges can be taken as a definiton of p. Writing $q(\mathbf{G}) = p(\mathbf{G}^*)$, from (13) we obtain $q(\mathbf{G}) = q(\mathbf{G}/e) + q(\mathbf{G}-e)$ for any edge e of a ribbon graph $\mathbf{G}$. Since $\mathbf{E_n}$ is its own dual, $q(\mathbf{E_n}) = p(\mathbf{E_n}) = t^{n-1}$, so q and p agree on all ribbon graphs, i.e., $p(\mathbf{G})$ takes the same value on any ribbon graph $\mathbf{G}$ and on its dual. For a one-vertex ribbon graph $\mathbf{G}$ we have $bc(\mathbf{G})-1 = n(\mathbf{G})-(k(\mathbf{G})-bc(\mathbf{G})+n(\mathbf{G}))$, so comparing the definition (12) of p and with equation (6) for R, we see that

$$p(\mathbf{G}) = R(\mathbf{G}; t+1, t, t^{-1}, 1) \tag{14}$$

on one-vertex ribbon graphs. Considering the behaviour of p with respect to taking disjoint unions and one-point joins we see that if e is a bridge of $\mathbf{G}$, then $p(\mathbf{G} - e) = tp(\mathbf{G}/e)$, so $p(\mathbf{G}) = (t+1)p(\mathbf{G}/e)$. Comparing this relation and (13) with the contraction-deletion relations for R shows that (14) holds for all ribbon graphs. Thus we have

$$R(\mathbf{G}; t+1, t, t^{-1}, 1) = R(\mathbf{G}^*; t+1, t, t^{-1}, 1).$$

It is in such contexts that it would be more natural to take X rather than $X-1$ in the definition of R, or $Y-1$ rather than Y. Then the roles of X and Y would more nearly interchange when we pass to dual ribbon graphs.

Although the only specialization of the ribbon graph polynomial behaving well with respect to taking duals is that given above, it may well be that there is a less restrictive such specialization, perhaps having two variables. It might be possible to obtain such a polynomial by considering different contraction-deletion relations for a loop e according to whether e is trivial or not. In fact, it is possible that there is a generalization of the ribbon graph polynomial behaving well with respect to duals. This might be obtained by considering the relation (4) only for edges e which are ordinary both in $\mathbf{G}$ and in $\mathbf{G}^*$.

References

1. B. Bollobás, O. Riordan: A Tutte polynomial for coloured graphs, in Recent trends in combinatorics (Mátraháza, 1995). Combin. Probab. Comput. **8**, 45–93 (1999)
2. B. Bollobás, O. M. Riordan: A polynomial invariant of graphs on orientable surfaces. Proc. London Math. Soc. **83**, 513–531 (2001)
3. S. Chmutov: A proof of the Melvin-Morton conjecture and Feynman diagrams. J. Knot Th. Ramifications. **7**, 23–40 (1998)
4. C. M. Fortuin, P. W. Kasteleyn: On the random-cluster model. I. Physica. **57**, 536–564 (1972)
5. L. H. Kauffman: A Tutte polynomial for signed graphs. Discrete Applied Mathematics. **25**, 105–127 (1989)
6. R.J. Milgram, R. C. Penner: Riemann's moduli space and the symmetric groups, in Mapping class groups and moduli spaces of Riemann surfaces. Contemp. Math. **150**, 247–290 (1993)
7. I. Nikolaev: Graphs and flows on surfaces. Ergodic Th. and Dynamical Systems. **18**, 207–220 (1998)
8. S. D. Noble, D. J. A. Welsh: A weighted graph polynomial from chromatic invariants of knots. Annales de l'Institute Fourier (Grenoble). **49**, 1057–1087 (1999)
9. V. Yu. Reshetikhin, V. G. Turaev: Ribbon graphs and their invariants derived from quantum groups. Comm. Math. Phys. **127**, 1–26 (1990)
10. L. Traldi: A dichromatic polynomial for weighted graphs and link polynomials. Proceedings of the American Mathematical Society. **106**, 279–286 (1989)
11. W. T. Tutte: A ring in graph theory. Proceedings of the Cambridge Philosophical Society. **43**, 26–40 (1947)
12. W. T. Tutte: A contribution to the theory of chromatic polynomials. Canadian Journal of Mathematics. **6**, 80–91 (1954)
13. Z. Wang: On Bott polynomials. J. Knot Theory Ramifications. **3**, 537–546 (1994)