THE JONES POLYNOMIAL AND DESSINS D'ENFANT

OLIVER T. DASBACH, DAVID FUTER, EFSTRATIA KALFAGIANNI, XIAO-SONG LIN, AND NEAL W. STOLTZFUS

Abstract. The Jones polynomial of an alternating link is a certain specialization of the Tutte polynomial of the (planar) checkerboard graph associated to an alternating projection of the link. The Bollobás–Riordan–Tutte polynomial generalizes the Tutte plolynomial of planar graphs to graphs that are embedded in closed surfaces of higher genus (i.e. dessins d'enfant).

In this paper we show that the Jones polynomial of any link can be obtained from the Bollobás–Riordan–Tutte polynomial of a certain dessin associated to a link projection. We give some applications of this approach.

1. Introduction

Dessins d'enfant were introduced by Grothendieck and proved to be useful in many areas of mathematics (see e.g. [Sch94, LZ04] for introductions). Informally, dessins are graphs with a cyclic orientation around their vertices. Their genus is the minimal genus of an orientable surface in which the dessin embeds. Recently, Bollobás and Riordan extended the Tutte polynomial to dessins [BR01] as a three-variable polynomial, where the third variable is related to the genus of dessins.

The dessins of interest in this introductory paper arise naturally from link projections. We will show that the Jones polynomial, via the Kauffman bracket, is a specialization of the Bollobás-Riordan-Tutte polynomial of these dessins. Furthermore, this approach leads to a natural dessin-genus of a knot as the minimal genus of the dessin of a knot diagram over all knot projections. Knots of dessin-genus 0 are exactly the alternating knots.

Our approach is different from the one taken by Thistlethwaite [Thi87], where he gives a spanning tree expansion of the Jones polynomial via signed graphs. Using these ideas Kauffman defined a three-variable Tutte polynomial for signed graphs

Date: May 19, 2006.
The first author was supported in part by NSF grants DMS-0306774 and DMS-0456275 (FRG).
The second author was supported in part by NSF grant DMS-0353717 (RTG).
The third author was supported in part by NSF grants DMS-0306995 and DMS-0456155 (FRG).
The fourth author was supported in part by NSF grants DMS-0404511 and DMS-0456217 (FRG).
The fifth author was supported in part by NSF grant DMS-0456275 (FRG).

[Kau89]. For alternating knots both our approach and Thistlethwaite's approach coincide: the Jones polynomial of an alternating knot is a specialization of the Tutte polynomial of Tait's checkerboard graph of an alternating knot projection. The connection between the Tutte polynomial and the Jones polynomial for alternating knots was fruitfully used in [DL04]. The books [Bol98, Wel93] give a good introduction to the interplay between knots and graphs.

For non-alternating knots, however, we obtain a different type of spanning tree expansion for the Jones polynomial.

The paper is organized as follows: Section 2 recalls the definition of a dessin d'enfant. Section 3 describes the construction of a dessin from a diagram of a link. Section 4 gives a duality result for the constructed dessin. In Section 5, we define the Bollobás–Riordan–Tutte polynomial of this dessin, and show how the Kauffman bracket of the diagram can be obtained as a specialization of this polynomial. Passing from the Kauffman bracket to the Jones polynomial is then a matter of multiplying by a well-known diagrammatic factor.

As an application, we get a spanning tree and a spanning sub-dessin expansion for the Kauffman bracket in Section 6. Finally, Section 7 gives some implications for adequate links.

Acknowledgement: We would like to thank Sergei Chmutov for helpful discussions on the Bollobas-Riordan-Tutte polynomial.

2. Dessins d'enfant

Heuristically, a *dessin d'enfant* (oriented ribbon map) can be viewed as a multigraph (i.e. loops and multiple edges are allowed) equipped with a cyclic order on the edges at every vertex. Isomorphisms between dessins are graph isomorphisms that preserve the given cyclic order of the edges.

The definition of a dessin, however, highlights the permutation (and secondarily, the topological) structure.

Definition 2.1. A *dessin d'enfant* is a triple, $\mathbb{D} = (\sigma_0, \sigma_1, \sigma_2)$ of permutations of a finite set $\mathcal{B} = [2n] := \{1, 2, \ldots, 2n-1, 2n\}$. The triple must satisfy:

- σ_1 is a fixed point free involution, i.e. $\sigma_1(\sigma_1(b)) = b, \sigma_1(b) \neq b$ for all b.
- $\sigma_0(\sigma_1(\sigma_2(b))) = b$.

The elements of the set $\mathcal{B}$ will be called *half edges* (*les brins* en français). Note that it follows from the second condition that any two permutations determine the third. Often a third condition is imposed: the group generated by $\{\sigma_0, \sigma_1, \sigma_2\}$ acts transitively on $\mathcal{B}$. This is equivalent to the associated surface being connected, where by convention we also allow the dessin with a single vertex and an empty set of half-edges.

2.1. **Associated Surface and Genus of a Dessin.** Given a *dessin d'enfant* $\mathbb{D} = (\sigma_0, \sigma_1, \sigma_2)$, the orbits of σ_0 form the vertex set, of cardinality $v(\mathbb{D})$, the orbits of σ_1, the edge set, of cardinality $e(\mathbb{D})$ and σ_2, the face set, of cardinality $f(\mathbb{D})$, respectively. The associated graph of a dessin has an edge connecting the vertices in whose orbit its two *half edges* lie. In addition, this graph is embedded in an oriented surface with (cellular) faces corresponding to the orbits of σ_2 and oriented so that σ_0 cyclically rotates the *half edges* meeting at a vertex in the rotation direction determined by the orientation.

Definition 2.2. The genus $g(\mathbb{D})$ of a dessin $\mathbb{D}$ with k components is determined by its Euler characteristic: $v(\mathbb{D}) - e(\mathbb{D}) + f(\mathbb{D}) = 2k - 2g(\mathbb{D})$.

For the definitions and properties of dessins, we will follow the recent monograph: *Graphs in surfaces and their applications* [LZ04]. In particular, note that we will always assume the embedding surface is oriented (certain extensions to the non-orientable case are discussed in [BR02]). Historically, this concept has been rediscovered several times and explored in several distinct settings: combinatorics, topology, physics of fields, combinatorial group theory and algebraic number theory. Dessins d'enfant (Grothendieck's terminology) have also been called: cyclic graphs, fat graphs, rotation systems, (ribbon) maps and combinatorial maps, inter alia. In addition to the book of S.K. Lando and A. Zvonkin [LZ04], surveys of note are those by Robert Cori and Antonio Machi [CM92] and the book edited by Leila Schneps [Sch94], which contains a survey of G. Jones and D. Singmaster on group theoretical aspects. In the graph theory literature, the original source often cited is the 1891 article of L. Heffter [Hef91].

3. Constructing the Kauffman state dessin from a link diagram

We will associate a dessin with each Kauffman state of a (connected) link diagram. The dessin is constructed as follows: Given a link diagram $D(K)$ of a knot K we have, as in Figure 1, an A-splicing and a B-splicing at every crossing. For any state assignment of an A or B at each crossing we obtain a collection of non-intersecting circles in the plane, together with embedded arcs that record the crossing splice. Again, Figure 1 shows this situation locally. In particular, we will consider the state where all splicings are A-splicings.

To define the desired dessin associated to a link diagram, we need to define an orientation on each of the circles resulting from the A or B splicings, according to a given state assignment. (We note that these circles will become the vertices of our dessin.) For a somewhat similar situation, see Vogel's algorithm for transforming a link diagram into closed braid form [Vog90, BB05].

We will call the orientation of two circles in the plane *admissible* if their orientation is induced by an orientation of the plane. Thus, they are not admissible if and only

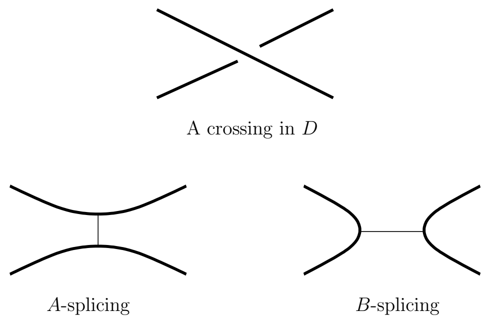

FIGURE 1. Splicings of a crossing, A-graph and B-graph.

if their orientation is induced by an orientation of the annulus that they bound. Two circles are *neighbors* of each other if they can be connected by an arc in the plane that does not intersect any other circle in the interior of the arc. We call a collection of oriented non-intersecting circles in the plane *admissible* if the orientation of every pair of neighbor circles is admissible.

We have the following:

Lemma 3.1. *Given a collection of pairwise disjoint circles in the plane, there is exactly one way to admissibly orient the given collection of circles so that the outermost circles have their orientation induced from a given orientation of the plane.*

Given a state assignment $s : E \to \{A, B\}$ on the crossings (the eventual edge set $E(\mathbb{D})$ of the dessin), the associated dessin is constructed by first resolving all the crossings according to the assigned states and then orienting the resulting circles according to a given orientation of the plane. The set of *half-edges* $\mathcal{B}$ will be the collection $\mathcal{B} = \{(C, \gamma)\}$, where C is a component circle of the resolution of the state s and γ is a directed edge from C corresponding to the chosen splice at a crossing. The permutation σ_0 permutes $\mathcal{B}$ according to the orientation order of the endpoints of the oriented arcs γ beginning on C. The permutation σ_1 matches the directed arc γ with the oppositely oriented arc beginning at the other end of the splice. We will denote the dessin associated to state s by $\mathbb{D}(s)$.

4. DUALITY

Given a link diagram and a state s, we can explicitly construct a surface $G(s)$ that realizes the genus of $\mathbb{D}(s)$. The construction uses both s and the *dual state* $\hat{s}$, in which every crossing is resolved in the opposite way from s.

Let $\Gamma \subset S^2$ be the planar, 4–valent graph of the link diagram. Thicken the projection plane to a slab $S^2 \times [-1, 1]$, so that Γ lies in $S^2 \times \{0\}$. Outside a neighborhood of the vertices (crossings), our surface will intersect this slab in $\Gamma \times [-1, 1]$. In the neighborhood of each vertex, we insert a saddle, positioned so that the boundary circles on

$S^2 \times \{1\}$ are the state circles of s, and the boundary circles on $S^2 \times \{\text{-}1\}$ are the state circles of $\hat{s}$. (See Figure 2.) Then, we cap off each state circle with a disk, obtaining an unknotted closed surface $G(s)$.

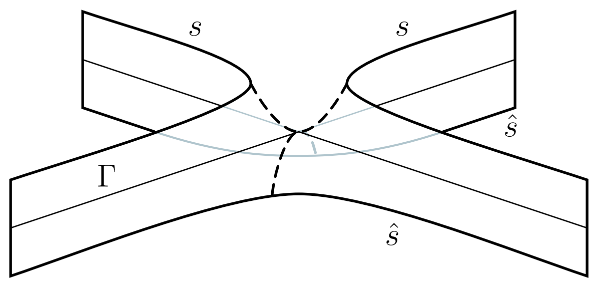

FIGURE 2. Near each crossing of the diagram, a saddle surface interpolates between state circles of s and state circles of $\hat{s}$. The edges of the dessins can be seen as gradient lines at the saddle.

Lemma 4.1. *The dessins $\mathbb{D}(s)$ and $\mathbb{D}(\hat{s})$ can both be embedded in $G(s)$. Furthermore, $\mathbb{D}(s)$ and $\mathbb{D}(\hat{s})$ are dual on $G(s)$: the vertices of one correspond to the faces of the other, and the edges of one correspond to the edges of the other.*

Proof. By construction, the surface $G(s)$ already has the structure of a cell complex, whose 1–skeleton is Γ. The disks of $G(s)$ can be two–colored, with the s–disks (above $S^2 \times \{0\}$) white and the $\hat{s}$–disks (below $S^2 \times \{0\}$) shaded.

We embed $\mathbb{D}(s)$ on $G(s)$ as follows. Pick a vertex in the interior of each white s–disk. Then, each time two s–disks touch each other on opposite sides of a crossing, connect the corresponding vertices by an edge. These edges correspond precisely to the splicing arcs in Figure 1. By construction, the orientation of $G(s)$ matches the orientation of the state circles of s, so the half-edges of $\mathbb{D}(s)$ are in fact embedded with the correct cyclic ordering.

We embed $\mathbb{D}(\hat{s})$ on $G(s)$ in a similar way, by picking a vertex in the interior of each $\hat{s}$–disk. Now, it is easy to observe that the two dessins are dual to each other. Every crossing of the diagram gives rise to two intersecting edges, one in each dessin. Every face of $\mathbb{D}(s)$ corresponds to a shaded disk in $G(s)$, which in turn corresponds to a vertex of $\mathbb{D}(\hat{s})$ – and vice versa. □

Corollary 4.2. *The genera of $G(s)$, $\mathbb{D}(s)$, and $\mathbb{D}(\hat{s})$ are all equal.*

Definition 4.3. Given a particular projection of a link L, we denote the dessin of the all-A state by $\mathbb{D}(A)$, and of the all-B state by $\mathbb{D}(B)$. Then, we define the *dessin-genus* of L to be the minimum value of $g(\mathbb{D}(A))$, taken over all projections of L. Note that by Corollary 4.2, this is also equal to the minimum value of $g(\mathbb{D}(B))$.

Lemma 4.4. *When s is the all-A state or the all-B state, the link has an alternating projection to $G(s)$.*

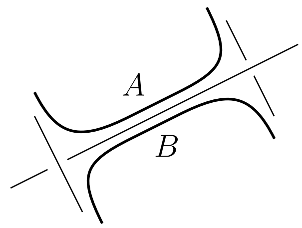

FIGURE 3. Because the A and B circles have disjoint projections, the link is alternating on $G(s)$.

Proof. When we project the link to $G(s)$, the image is the same 4–valent graph Γ. Furthermore, all the state circles of the A and B splicings have disjoint projections to $G(s)$, since we can draw each circle just inside the boundary of the corresponding disk. Now, if we follow an edge of Γ between consecutive crossings, the two state circles that share this edge must turn off in different directions, as in Figure 3. Given the definition of the A and B splicings in Figure 1, it follows that this edge of Γ connects an under–crossing to an over–crossing. □

Corollary 4.5. *A link diagram is alternating if and only if* $\mathbb{D}(A)$ *has genus* 0*. Thus a knot has dessin-genus* 0 *if and only if it is alternating.*

Proof. For an alternating diagram, the state circles of the A and B splicings correspond to the checkerboard coloring of the plane. In other words, the construction of $G(s)$ recovers the (compactified) projection plane, and $\mathbb{D}(A)$ and $\mathbb{D}(B)$ have genus 0. Conversely, if the genus of $\mathbb{D}(A)$ is 0, then by Lemma 4.4 the diagram is alternating on a sphere. □

Example 4.6. Figure 4 shows the non-alternating 8-crossing knot 8_{21}, as drawn by Knotscape [HTW98], and Figure 5 the all-A associated dessin.

With the numbering of the *half-edges* as given in the diagram, the fixed-point-free involution σ_1 is given in cycle notation by:

$$\sigma_1 = \{\{1,2\},\{3,4\},\{5,6\},\{7,8\},\{9,10\},\{11,12\},\{13,14\},\{15,16\}\}$$

With the admissible orientation on the circles induced from the counter-clockwise of the plane, the vertex permutation reads the half-edges around a circle of the state splicing. In cycle notation the permutation is:

$$\sigma_0 = \{\{2,6,12,10,14,16,8,4,15,13\},\{1,3,5\},\{7,9,11\}\}.$$

By the defining property relating all three permutations,

$$\sigma_2 = \{\{13,10,7,16,4,1\},\{5,2\},\{8,11,6,3\},\{12,9\},\{15,14\}\}$$

and we have five faces. The Euler characteristic of this dessin is

$$v(\mathbb{D}) - e(\mathbb{D}) + f(\mathbb{D}) = 3 - 8 + 5 = 0,$$

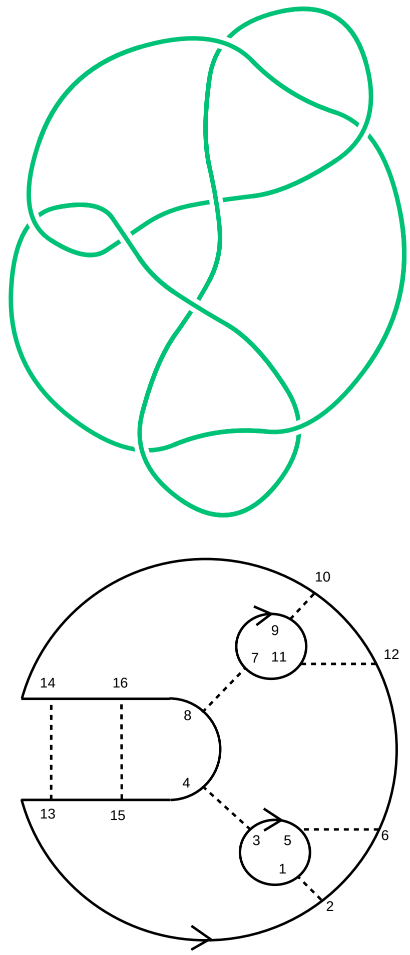

FIGURE 4. The eight-crossing knot 8_{21} with its all-A splicing projection diagram.

so we have that the genus is 1. Since the knot is non-alternating we know by Corollary 4.5 that the dessin-genus of the knot has to be 1.

5. THE BOLLOBÁS–RIORDAN–TUTTE POLYNOMIAL

In this section we will recall the definition of the Bollobás–Riordan–Tutte polynomial $C(\mathbb{D}) \in Z[X, Y, Z]$ of a dessin $\mathbb{D}$ from [BR01]. The definition requires several different combinatorial measurements of the dessin.

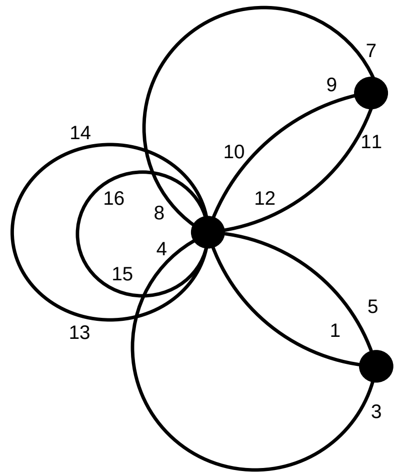

FIGURE 5. All-A splicing dessin for 8_{21}.

Definition 5.1. For a dessin $\mathbb{D}$, we define the following quantities:

$$\begin{aligned}
v(\mathbb{D}) &= \text{the number of vertices} = \text{the number of orbits of } \sigma_0,\\
e(\mathbb{D}) &= \text{the number of edges} = \text{the number of orbits of } \sigma_1,\\
f(\mathbb{D}) &= \text{the number of faces} = \text{the number of orbits of } \sigma_2,\\
k(\mathbb{D}) &= \text{the number of connected components of } \mathbb{D},\\
g(\mathbb{D}) &= \frac{2k(\mathbb{D}) - v(\mathbb{D}) + e(\mathbb{D}) - f(\mathbb{D})}{2}, \text{the } \textit{genus} \text{ of } \mathbb{D},\\
n(\mathbb{D}) &= e(\mathbb{D}) - v(\mathbb{D}) + k(\mathbb{D}), \text{the } \textit{nullity} \text{ of } \mathbb{D}.
\end{aligned}$$

The construction will be completed in two stages. For a dessin with an edge that is not a loop, i.e. does not connect a vertex with itself, the polynomial satisfies some contraction/deletion relation. This reduces the computation to the computation of the Bollobás–Riordan–Tutte polynomial for dessins with one vertex:

5.1. **The Bollobás–Riordan–Tutte polynomial for dessins with one vertex.**

For a dessin $\mathbb{D}$ with one vertex and $e(\mathbb{D})$ edges, we define the Bollobás–Riordan–Tutte polynomial $C(\mathbb{D})$ for one-vertex dessins as:

$$C(\mathbb{D}) := \sum_{\mathbb{H} \subset \mathbb{D}} Y^{n(\mathbb{H})} Z^{g(\mathbb{H})}.$$

Here, the summation is over all $2^{e(\mathbb{D})}$ sub-dessins of $\mathbb{D}$ obtained by deleting a subset of edges. These are the spanning sub-dessins on the same vertex set as $\mathbb{D}$.

5.2. **The Bollobás–Riordan–Tutte polynomial of a dessin with many vertices.** Given an edge e in a dessin, there is a naturally defined dessin $\mathbb{D}/e$ obtained by contracting the edge e to a vertex, with the cyclic order of the half-edges now meeting at that vertex given by amalgamating the two cyclic orders as shown below.

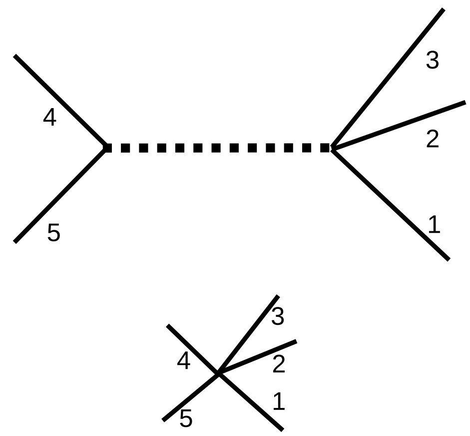

FIGURE 6. Edge contraction in a dessin.

Similarly, denote by $\mathbb{D} - e$ the dessin obtained by deleting an edge e and omitting both half-edges in the orbit e from the cyclic order at the corresponding vertices.

Recall that an edge of a graph is called a bridge when its deletion increases the number of components by 1.

Theorem 5.2 (Bollobás-Riordan [BR01]). *There is a well-defined invariant of a dessin $\mathbb{D}$, $C(\mathbb{D}; X, Y, Z) \in \mathbb{Z}[X, Y, Z]$, the Bollobás–Riordan–Tutte polynomial, satisfying:*

$$\begin{aligned} C(\mathbb{D}) &= C(\mathbb{D} - e) + C(\mathbb{D}/e) && \textit{for } e \textit{ neither a bridge nor a loop} \\ C(\mathbb{D}) &= X\, C(\mathbb{D}/e) && \textit{for } e \textit{ a bridge} \\ C(\mathbb{D}) &= \sum_{\mathbb{H} \subset \mathbb{D}} Y^{n(\mathbb{H})} Z^{g(\mathbb{H})} && \textit{if } \mathbb{D} \textit{ has one vertex} \end{aligned}$$

Remark 5.3. Note that our convention assigns the variable X to a bridge and $1 + Y$ to a loop, following that of Bollobás-Riordan [BR02]. The usual convention for the Tutte polynomial assigns X and Y, respectively.

5.3. **The Bollobás–Riordan–Tutte polynomial, the Kauffman bracket and the Jones polynomial.** Recall that the Jones polynomial of alternating knots and links can be considered as an evaluation of the Tutte polynomial of one of the checkerboard graphs, up to a unit in the coefficient ring of the Jones polynomial (see [Thi87]).

The following theorem is a generalization of this result to all links, not just alternating ones. It also extends the work of Chmutov and Pak [CP04]. It does not, however, require the machinery of signed graphs needed in [Thi87] for the general case. The Kauffman bracket is computed as an evaluation of the Bollobás–Riordan–Tutte polynomial for a single state!

Theorem 5.4. *Let $\langle P\rangle \in \mathbb{Z}[A, A^{-1}]$ be the Kauffman bracket of a connected link projection diagram P and $\mathbb{D}$ be the dessin of P associated to the all-A-splicing. Then the Bollobás–Riordan–Tutte polynomial $C(\mathbb{D}; X, Y, Z)$ and the Kauffman bracket are related by*

$$A^{-e(\mathbb{D})}\langle P\rangle = A^{2-2v(\mathbb{D})}C(\mathbb{D}; -A^4, A^{-2}\delta, \delta^{-2}),$$

where $\delta := (-A^2 - A^{-2})$, and $e(\mathbb{D})$ and $v(\mathbb{D})$ are as in Definition 5.1.

Proof. First we verify the desired equation for the special case where $\mathbb{D}$ has one vertex and all the edges are loops. For this case, using the Kauffman state sum formula, and abusing the notation to use $\langle\mathbb{D}\rangle$ for denoting the Kauffman bracket of a projection yielding $\mathbb{D}$ as the all-A dessin, we have

$$\begin{aligned}\langle\mathbb{D}\rangle &= \sum_{\mathbb{H}\subset\mathbb{D}} \delta^{f(\mathbb{H})-1}A^{e(\mathbb{D})-e(\mathbb{H})}A^{-e(\mathbb{H})}\\ &= A^{e(\mathbb{D})}\sum_{\mathbb{H}\subset\mathbb{D}} A^{-2e(\mathbb{H})}\delta^{f(\mathbb{H})-1}\\ &= A^{e(\mathbb{D})}\sum_{\mathbb{H}\subset\mathbb{D}} (A^{-2}\delta)^{2g(\mathbb{H})+f(\mathbb{H})-1}(\delta^{-2})^{g(\mathbb{H})}\\ &= A^{e(\mathbb{D})}\sum_{\mathbb{H}\subset\mathbb{D}} (A^{-2}\delta)^{n(\mathbb{H})}(\delta^{-2})^{g(\mathbb{H})}\end{aligned}$$

which is the desired specialization of the Bollobás–Riordan–Tutte polynomial.

To verify the equation in general, it is enough to check that the right hand side satisfies the axiomatic conditions of the Kauffman bracket. This is done using the axioms of the Bollobás–Riordan–Tutte polynomial in Theorem 5.2 and inducting on the number of crossings of P (equivalently, the number of edges of $\mathbb{D}$). To check the normalization (or boundary condition) for a trivial knot, we observe that the all-A dessin of the trivial knot projection (a single plane circle) is the dessin with a single vertex and no edges. Trivially $A^0\langle O\rangle = A^0 = 1$. Next, we check that the Kauffman skein relationship is satisfied at a crossing corresponding to an edge of the dessin that is not a bridge nor a loop: Suppose there is an edge e which is not a loop nor a bridge of the all-A dessin for the projection. Note that the all-A dessin of the result of an A-splicing at the chosen crossing will be the dessin $\mathbb{D} - e$; and the all-A dessin after performing the B-splicing will be the dessin with that crossing edge contracted, $\mathbb{D}/e$.

Since the projections obtained from P after each of these two splicings are connected, the induction hypothesis applies. Hence the following equalities hold:

$$\begin{aligned} A^{-e(\mathbb{D})}\langle \mathbb{D}\rangle &= A^{-1}(A^{-e(\mathbb{D})}\langle \mathbb{D}/e\rangle) + A(A^{-e(\mathbb{D})}\langle \mathbb{D}-e\rangle) \\ &= A^{-2}(A^{-(e(\mathbb{D})-1)}\langle \mathbb{D}/e\rangle) + (A^{-(e(\mathbb{D})-1)})\langle \mathbb{D}-e\rangle \\ &= A^{-2}(A^{2-2(v(\mathbb{D})-1)}C(\mathbb{D}/e;X,Y,Z)) + A^{2-2v(\mathbb{D})}C(\mathbb{D}-e;X,Y,Z) \\ &= A^{2-2v(\mathbb{D})}(C(\mathbb{D}/e;X,Y,Z) + C(\mathbb{D}-e;X,Y,Z)) \\ &= A^{2-2v(\mathbb{D})}C(\mathbb{D};X,Y,Z), \end{aligned}$$

where the last equation comes from the third axiom of the Bollobás–Riordan–Tutte polynomial. Therefore, the skein relation is verified.

Now suppose that $\mathbb{D}$ has no edge as above; that is all edges are either loops or bridges. If there are no bridges then, since $\mathbb{D}$ is connected, we are in the case of the one-vertex dessin discussed above. Suppose that $\mathbb{D}$ has an edge e that is a bridge. Then, the crossing of P corresponding to e is nugatory; i.e. there is a simple closed curve on the projection plane that intersects P exactly once at the double point of that crossing. The axioms of the Kauffman bracket imply that $\langle \mathbb{D}\rangle = (-A^3)\langle \mathbb{D}/e\rangle$. Using this and the inductive hypothesis we have

$$\begin{aligned} A^{-e(\mathbb{D})}\langle \mathbb{D}\rangle &= (-A^2)(A^{-(e(\mathbb{D})-1)}\langle \mathbb{D}/e\rangle) \\ &= (-A^2)(A^{2-2(v(\mathbb{D})-1)}C(\mathbb{D}/e;X,Y,Z)) \\ &= A^{2-2v(\mathbb{D})}((-A^4)C(\mathbb{D}/e;X,Y,Z)) \\ &= A^{2-2v(\mathbb{D})}C(\mathbb{D};X,Y,Z), \end{aligned}$$

where the last equation comes from the third axiom of the Bollobás–Riordan–Tutte polynomial.

To finish the proof, we check the Kauffman bracket axiom $\langle P \cup O\rangle = \delta\langle P\rangle$. For this we need to show that the right-hand formula behaves correctly under a type II move with a disjoint, trivial unknotted component. (This is necessary to make the diagram connected.) This all-A dessin changes by adding a linked pair of edges consisting of two adjacent interleaving loops with half-edges alternating in cyclic order. This adds a handle to the underlying surface, increasing the genus by one. After reduction to one vertex (using the same steps as applied for the computation of the bracket for the projection P, the one vertex dessin has an linked pair of edges $\{e, f\}$ consisting of two adjacent interleaving loops with half-edges alternating in cyclic order.

By the third axiom for the Bollobás–Riordan–Tutte polynomial computing the one vertex case, we separate the sum into four pieces which sum over the spanning sub-dessins of the original dessin. Each sum consists of subdiagrams containing a fixed subset of the linked edges $\{e, f\}$.

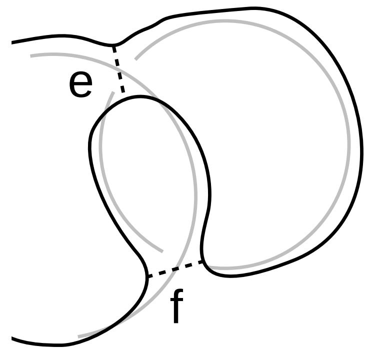

FIGURE 7. All A-splicing on a type II move with a trivial component.

$$\begin{aligned}
C(\mathbb{D}) &= \sum_{\mathbb{H}\subset\mathbb{D}} Y^{n(\mathbb{H})}Z^{g(\mathbb{H})} \\
&= \sum_{\mathbb{H}\subset\mathbb{D}\setminus\{e,f\}} Y^{n(\mathbb{H})}Z^{g(\mathbb{H})} + \sum_{\mathbb{H}\subset\mathbb{D}\setminus\{e,f\}} Y^{n(\mathbb{H}\cup\{e\})}Z^{g(\mathbb{H}\cup\{e\})} \\
&\quad + \sum_{\mathbb{H}\subset\mathbb{D}\setminus\{e,f\}} Y^{n(\mathbb{H}\cup\{f\})}Z^{g(\mathbb{H}\cup\{f\})} + \sum_{\mathbb{H}\subset\mathbb{D}\setminus\{e,f\}} Y^{n(\mathbb{H}\cup\{e,f\})}Z^{g(\mathbb{H}\cup\{e,f\})} \\
&= \sum_{\mathbb{H}\subset\mathbb{D}\setminus\{e,f\}} Y^{n(\mathbb{H})}Z^{g(\mathbb{H})} + Y \sum_{\mathbb{H}\subset\mathbb{D}\setminus\{e,f\}} Y^{n(\mathbb{H})}Z^{g(\mathbb{H})} \\
&\quad + Y \sum_{\mathbb{H}\subset\mathbb{D}\setminus\{e,f\}} Y^{n(\mathbb{H})}Z^{g(\mathbb{H})} + Y^2 Z \sum_{\mathbb{H}\subset\mathbb{D}\setminus\{e,f\}} Y^{n(\mathbb{H})}Z^{g(\mathbb{H})} \\
&= (1+2Y+Y^2Z) \sum_{\mathbb{H}\subset\mathbb{D}\setminus\{e,f\}} Y^{n(\mathbb{H})}Z^{g(\mathbb{H})} \ .
\end{aligned}$$

Next we note that the number of edges $e(\mathbb{D})$ has increased by two while the number of vertices $v(\mathbb{D})$ remained unchanged, hence the Kauffman bracket is multiplied by $A^2(1+2Y+Y^2Z)$, which equals δ under the given variable specializations. This finishes the proof of the theorem. □

Remark 5.5. It is not hard to see that the B-dessin of a link projection P is equal to the A-dessin of the mirror image of P (compare Figures 1 and 2). It follows that, in Theorem 5.4, replacement of $\mathbb{D}(A)$ by $\mathbb{D}(B)$ will compute the Kauffman bracket of the mirror image of the knot.

Remark 5.6. Let P be a connected projection of a link L, and let $w(P)$ denote the writhe of P. Recall that the Jones polynomial $J_L(t)$ is obtained from $(-A)^{-3w(P)}\langle P\rangle$ by substituting $A := t^{-1/4}$. Thus, by Theorem 5.4, the Jones polynomial of L is obtained as a specialization of the Bollobás–Riordan–Tutte polynomial of the A-dessin corresponding to P.

6. The Spanning Subgraph and Tree Expansions

The Bollobás–Riordan–Tutte polynomial has a spanning subgraph expansion and a spanning tree expansion, yielding the following corollaries.

For the spanning tree expansion we need an order $\prec$ of the edges of the dessin $\mathbb{D}$. A *spanning tree* T of $\mathbb{D}$ is a subdessin with the same vertex set, which is connected and has no cycles (equivalently zero nullity, or no homology). For an edge e of T the *cut* determined by e and T is the set of edges in $\mathbb{D}$ connecting one component of $T-e$ to the other. An edge E is called *internally active* if e is the smallest element of the cut in the prescribed order $\prec$. An edge e not in T is *externally active* if it is the smallest element in the unique cycle in $T\cup e$.

By [BR02], the spanning tree expansion of the Bollobás–Riordan–Tutte polynomial is given by:

$$\sum_T X^{i(T)} \sum_{S\subset\mathcal{E}(T)} Y^{n(T\cup S)} Z^{g(T\cup S)}$$

where $\mathcal{E}(T)$ is the set of externally active edges for a given tree T (and the order $\prec$). Therefore by Theorem 5.4, we have:

Corollary 6.1. *Let $\langle P\rangle \in \mathbb{Z}[A, A^{-1}]$ be the Kauffman bracket of a connected link projection diagram P and $\mathbb{D} := \mathbb{D}(A)$ be the dessin of P associated to the all-A-splicing. The Kauffman bracket can be computed using the fixed edge order $\prec$) by the following spanning tree expansion:*

$$A^{-e(\mathbb{D})}\langle P\rangle = A^{2-2v(\mathbb{D})} \sum_T X^{i(T)} \sum_{S\subset\mathcal{E}(T)} Y^{n(T\cup S)} Z^{g(T\cup S)}$$

under the following specialization: $\{X \to -A^4, Y \to A^{-2}\delta, Z \to \delta^{-2}\}$, where $\delta := (-A^2 - A^{-2})$ and $i(T)$ is the number of internally active edges in the spanning tree T.

The following spanning sub-dessin (sub-dessin with the same vertex set) expansion is obtained by specializing the expansion defined in [BR02] to the dessin case where all the edges are orientable. Again, by Theorem 5.4:

Corollary 6.2. *Let $\langle P\rangle \in \mathbb{Z}[A, A^{-1}]$ be the Kauffman bracket of a connected link projection diagram P and $\mathbb{D} := \mathbb{D}(A)$ be the dessin of P associated to the all-A-splicing. The Kauffman bracket can be computed by the following spanning sub-dessin $\mathbb{H}$ expansion:*

$$A^{-c(\mathbb{D})}\langle P\rangle = A^{2-2v(\mathbb{D})}(X-1)^{-k(\mathbb{D})}\sum_{\mathbb{H}\subset\mathbb{D}}(X-1)^{k(\mathbb{H})}Y^{n(\mathbb{H})}Z^{g(\mathbb{H})}$$

under the following specialization: $\{X \to -A^4, Y \to A^{-2}\delta, Z \to \delta^{-2}\}$ where $\delta := (-A^2 - A^{-2})$.

7. Span of the Polynomial of Adequate Knots

A connected link projection P is called A-adequate (resp. B-adequate) if and only if $\mathbb{D} := \mathbb{D}(A)$ (resp. $\mathbb{D}^* := \mathbb{D}(B)$) contains no loops (edges with both endpoints at the same vertex). Now P is called adequate if it is both A- and B-adequate ([Cro04]), and a link is called adequate if it admits an adequate projection. The class of adequate links contains that of alternating ones, but it is much more general. Let $e(\mathbb{D})$ denote the crossing number of P and let $v(\mathbb{D})$, $v'(\mathbb{D})$ denote the numbers of vertices of $\mathbb{D}(A)$, $\mathbb{D}(B)$, respectively. It is known that the span of the Kauffman bracket of an adequate projection P is given by $\mathrm{span}\langle P\rangle = 2e(\mathbb{D}) + 2v(\mathbb{D}) + 2v'(\mathbb{D}) - 4$. Next, we show how to derive this from the sub-dessin expansion of Proposition 6.2 for connected link projections.

Lemma 7.1. *For a connected link projection P, let $M(P)$ and $m(P)$ denote the maximum and minimum powers of A that occur in $\langle P\rangle$. We have:*

(a) $M(P) \le e(\mathbb{D}) + 2v(\mathbb{D}) - 2$, *with equality if P is A-adequate.*
(b) $m(P) \ge -e(\mathbb{D}) - 2v(\mathbb{D}) + 2$, *with equality if P is B-adequate.*

In particular, if P is adequate then

$$\mathrm{span}\langle P\rangle = M(P) - m(P) = 2e(\mathbb{D}) + 2v(\mathbb{D}) + 2v'(\mathbb{D}) - 4.$$

Proof. Let $\mathbb{D} := \mathbb{D}(A)$ denote the dessin corresponding to the all-A splicing of P. Given a spanning sub-dessin $\mathbb{H} \subset \mathbb{D}$ we have, by Definition 5.1,

$$2g(\mathbb{H}) = 2k(\mathbb{H}) - f(\mathbb{H}) - v(\mathbb{H}) + e(\mathbb{H}).$$

Now a straightforward computation, using $k(\mathbb{D}) = 1$, shows that after the substitutions for X, Y, Z given in Proposition 6.2 we obtain

$$A^{e(\mathbb{D})-2v(\mathbb{D})+2}(X-1)^{k(\mathbb{H})-k(\mathbb{D})}Y^{n(\mathbb{H})}Z^{g(\mathbb{H})} \;\to\; A^{e(\mathbb{D})-2e(\mathbb{H})}(-A^2 - A^{-2})^{f(\mathbb{H})-1}. \tag{7.1}$$

Then $M(\mathbb{H}) := e(\mathbb{D}) - 2e(\mathbb{H}) + 2f(\mathbb{H}) - 2$ is the highest power of A contributed by $\mathbb{H}$. Let $\mathbb{H}_0$ denote the spanning sub-dessin that contains only the vertices of $\mathbb{D}$ and no edges. Now every spanning sub-dessign $\mathbb{H} \subset \mathbb{D}$ is obtained from $\mathbb{H}_0$ by adding a number of edges. This can be done in stages so that there are sub-dessins $\mathbb{H}_0, \dots, \mathbb{H}_k$, with $\mathbb{H}_k = \mathbb{H}$ and such that, for $i = 1, \dots, k$, $\mathbb{H}_i$ is obtained from $\mathbb{H}_{i-1}$ by adding exactly one edge. Then, $e(\mathbb{H}_i) = e(\mathbb{H}_{i-1}) + 1$ and $f(\mathbb{H}_i) = f(\mathbb{H}_{i-1}) \pm 1$, and hence the difference $M(\mathbb{H}_{i-1}) - M(\mathbb{H}_i)$ is 0 or 4. Thus we obtain

$$M(\mathbb{H}) \leq M(\mathbb{H}_0) = e(\mathbb{D}) + 2v(\mathbb{D}) - 2, \tag{7.2}$$

for every spanning sub-dessin $\mathbb{H} \subset \mathbb{D}$. Now if P is A-adequate then, since an edge is added between two distinct vertices of $\mathbb{H}_0$, we have $f(\mathbb{H}_1) = f(\mathbb{H}_0) - 1$, and so $M(\mathbb{H}_i)$ decreases at the first step. By (7.2) it never increases, so we have

$$M(\mathbb{H}) < M(\mathbb{H}_0) = e(\mathbb{D}) + 2v(\mathbb{D}) - 2, \tag{7.3}$$

for every $\mathbb{H} \neq \mathbb{H}_0$. Thus the term with degree $M(\mathbb{H}_0)$ is never cancelled in the sub-dessin expansion of $\langle P \rangle$ and part (a) of the lemma follows. Part (b) follows by applying part (a) to the mirror image of P and using the observation in Remark 5.5. □

Corollary 7.2. *Let P be an adequate projection of a link L and let $\mathbb{D}(A)$ and $\mathbb{D}(B)$ be as above. Then the genus $g(\mathbb{D}(A)) = g(\mathbb{D}(B))$ is an invariant of the link L.*

Proof. It is known that $e(\mathbb{D})$ is actually the minimal crossing number of the link L and thus an invariant of L [Cro04, §9.5]. It is also known that the span of the Kauffman bracket of any link projection is an invariant of the link. Since P is adequate, by Lemma 7.1, we have $\mathrm{span}\langle P \rangle = 2e(\mathbb{D}) + 2v(\mathbb{D}) + 2v'(\mathbb{D}) - 4$. By Lemma 4.1, we have $v' = f(\mathbb{D}(A))$ and thus (Definition 5.1) $2 - 2g(\mathbb{D}(A)) = v(\mathbb{D}) + v'(\mathbb{D}) - e(\mathbb{D})$. Thus $4g(\mathbb{D}(A)) = 4e(\mathbb{D}) - \mathrm{span}\langle P \rangle$, and the conclusion follows from the previous observations. □

The span of the Kauffman bracket in the variable A of a knot projection is four times the span of the Jones polynomial in the variable t.

Corollary 7.3. *Let P be a c-crossing, connected projection of a link L, let s_L denote the span of the Jones polynomial of L and let g_L be the dessin-genus of the link. Then,*

$$g_L \leq c - s_L.$$

Proof. By Lemma 7.1 and its proof, we have $s_L \leq 4c - 4g(\mathbb{D}(A))$. Since the span of the Kauffman bracket in the variable A of a knot projection is four times the span of the Jones polynomial in the variable t, we have $g_L \leq g(\mathbb{D}(A)) \leq c - s_L$. □

Remark 7.4. The estimate in Corollary 7.3 is sharp for some families but not in general. For example, it is sharp for the family of non-alternating pretzel knots $P(a_1, \dots, a_r, b_1, \dots, b_s)$, where $a_i \geq 2$, $b_j \geq 2$, $r, s \geq 2$, as in Figure 8. On the one hand, these knots are non-alternating, so they have dessin-genus at least one. On the other hand, Lickorish and Thistlethwaite [LT88] prove that the span of the Jones polynomial is one less than the crossing number. Thus the estimate of Corollary 7.3 is sharp.

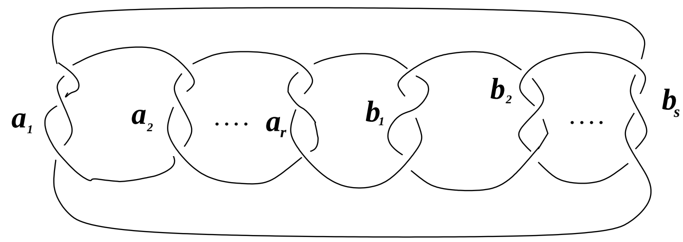

FIGURE 8. Non-alternating pretzel knots that realize the estimate of Corollary 7.3.

For a non-sharp example, consider the 8-crossing knot 8_{21} in Figure 4. The span of its Jones polynomial polynomial is 6 (Knotscape [HTW98]), and its dessin-genus is 1.

REFERENCES

[BB05] Joan S. Birman and Tara E. Brendle. Braids: a survey. In *Handbook of knot theory*, pages 19–103. Elsevier B. V., Amsterdam, 2005.

[Bol98] Béla Bollobás. *Modern graph theory*, volume 184 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[BR01] Béla Bollobás and Oliver Riordan. A polynomial invariant of graphs on orientable surfaces. *Proc. London Math. Soc. (3)*, 83(3):513–531, 2001.

[BR02] Béla Bollobás and Oliver Riordan. A polynomial of graphs on surfaces. *Math. Ann.*, 323(1):81–96, 2002.

[CM92] Robert Cori and Antonio Machì. Maps, hypermaps and their automorphisms: a survey. I, II, III. *Exposition. Math.*, 10(5):403–427, 429–447, 449–467, 1992.

[CP04] Sergei Chmutov and Igor Pak. The Kauffman bracket and the Bollobas-Riordan polynomial of ribbon graphs, 2004. math.GT/0404475.

[Cro04] Peter R. Cromwell. *Knots and links*. Cambridge University Press, Cambridge, 2004.

[DL04] Oliver T. Dasbach and Xiao-Song Lin. A volume-ish theorem for the Jones polynomial of alternating knots. math.GT/0403448, 2004.

[Hef91] Lothar Heffter. Über das Problem der Nachbargebiete. *Math. Ann.*, 38(4):477–508, 1891.

[HTW98] Jim Hoste, Morwen Thistlethwaite, and Jeff Weeks. The first 1,701,936 knots. *Math. Intelligencer*, 20(4):33–48, 1998.

[Kau89] Louis H. Kauffman. A Tutte polynomial for signed graphs. *Discrete Appl. Math.*, 25(1-2):105–127, 1989. Combinatorics and complexity (Chicago, IL, 1987).

[LT88] W. B. Raymond Lickorish and Morwen B. Thistlethwaite. Some links with nontrivial polynomials and their crossing-numbers. *Comment. Math. Helv.*, 63(4):527–539, 1988.

[LZ04] Sergei K. Lando and Alexander K. Zvonkin. *Graphs on surfaces and their applications*, volume 141 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2004. With an appendix by Don B. Zagier, Low-Dimensional Topology, II.

[Sch94] Leila Schneps. Dessins d'enfants on the Riemann sphere. In *The Grothendieck theory of dessins d'enfants (Luminy, 1993)*, volume 200 of *London Math. Soc. Lecture Note Ser.*, pages 47–77. Cambridge Univ. Press, Cambridge, 1994.

[Thi87] Morwen B. Thistlethwaite. A spanning tree expansion of the Jones polynomial. *Topology*, 26(3):297–309, 1987.

[Vog90] Pierre Vogel. Representation of links by braids: A new algorithm. *Comm. Math. Helv.*, 65:104–113, 1990.

[Wel93] Dominic J. A. Welsh. *Complexity: knots, colourings and counting*, volume 186 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1993.

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803

E-mail address: kasten@math.lsu.edu

Department of Mathematics, Michigan State University, East Lansing, MI 48824

E-mail address: dfuter@math.msu.edu

Department of Mathematics, Michigan State University, East Lansing, MI 48824

E-mail address: kalfagia@math.msu.edu

Department of Mathematics, University of California, Riverside, CA 92521

E-mail address: xl@math.ucr.edu

Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803

E-mail address: stoltz@math.lsu.edu