Math 6112 – Spring 2020
Problem Set 1
Due: Friday 17 January 2020

1. Let G be a group. Use G to define a category $\underline{G}$ with a single object as we did with monoids. Show that in this (small) category all morphisms are isomorphisms. [We then define a *groupoid* to be a small category in which all morphisms are isomorphisms.]

2. An object A of a category $\mathcal{C}$ is called an *initial* object if for every $X \in Ob(\mathcal{C})$ the set $Hom(A, X)$ consists of a single element. An object A of a category $\mathcal{C}$ is called a *terminal* object if for every object $X \in Ob(\mathcal{C})$ the set $Hom(X, A)$ has a single element. An object that is both an initial and terminal object is called a *zero* object of $\mathcal{C}$.

   (a) If A and A' are both initial objects in a category $\mathcal{C}$ then show that there is a unique isomorphism in $Hom(A, A')$, and similarly for terminal objects. Hence these objects are "unique up to isomorphism".

   (b) Show that there exist zero objects in $R - \underline{mod}$ and in $\underline{Grp}$.

3. Let $\mathcal{C}$ and $\mathcal{D}$ be categories, and $F : \mathcal{C} \to \mathcal{D}$ be a (covariant) functor that is faithful and full. For any $f \in Hom_{\mathcal{C}}(A, B)$ show that if $F(f)$ is monic (resp. epic) then so is f.

4. Let M and N be moniods, and associate to each a category $\underline{M}$ and $\underline{N}$ with a single object. Show that with this identification, a (covariant) functor $F : \underline{M} \to \underline{N}$ is simply a homomorphism $F : M \to N$. Show that a natural transformation $\eta : F \to G$ for two functors F and G corresponds to an element $b \in N$ such that $b \cdot F(x) = G(x) \cdot b$ for all $x \in M$.

5. Use the previous exercise to construct a functor F and a monomorphism f such that $F(f)$ is not a monomorphism and an epimorphism g such that $F(g)$ is not an epimorphism.