Math 6112 – Spring 2020
Problem Set 2
Due: Friday 24 January 2020

6. Define the *center* of a category $\mathcal{C}$ to be the class of natural transformations of the identity functor $1_{\mathcal{C}}$ to itself. Let $\mathcal{C} = R - \underline{mod}$ and let c be an element of the center of R. For any $M \in Ob(R - \underline{mod})$ let $\eta(c)_M : M \to M$ denote the map $x \mapsto cx$ for $x \in M$. Show that the map $\eta(c) : M \mapsto \eta(c)_M$ is a natural transformation in the center of $R - \underline{mod}$ and every element of the center of $R - \underline{mod}$ is of this form. Show that $c \mapsto \eta(c)$ is a bijection of the center of R with the center of $R - \underline{mod}$ and hence the center of $R - \underline{mod}$ is a set.

7. Let $\mathcal{C}$ and $\mathcal{D}$ be categories and let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ define an equivalence of categories. Let $f \in Hom_{\mathcal{C}}(A, B)$. Show that any of the following properties of f implies the same property for $F(f)$:

   (a) f is monic;

   (b) f is epic;

   (c) f has a right (or left) inverse;

   (d) f is an isomorphism.

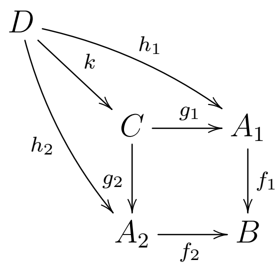
Let $\mathcal{C}$ be a category and $f_i : A_i \to B$ be two morphisms in $\mathcal{C}$. A *pullback* of $\{f_1, f_2\}$ or a *fibered product* of A_1 and A_2 is a triple (C, g_1, g_2) consisting of an object C of $\mathcal{C}$ and morphisms $g_i : C \to A_i$ such that the following diagram commutes

$$\begin{CD} C @>{g_1}>> A_1 \\ @V{g_2}VV @VV{f_1}V \\ A_2 @>>{f_2}> B \end{CD}$$

and such that if

$$\begin{CD} D @>{h_1}>> A_1 \\ @V{h_2}VV @VV{f_1}V \\ A_2 @>>{f_2}> B \end{CD}$$

is any commutative diagram containing f_1 and f_2 then there is a unique $k : D \to C$ such that



is commutative. We usually denote C by $A_1 \times_B A_2$. As in the above exercise if a pull back exists it is unique up to isomorphism. [The dual notion to a pull back is a *push out* and can be defined by reversing all the arrows above.]

8. Now take $\mathcal{C} = \underline{Grp}$ and $f_i : G_i \to H$ in $\underline{Grp}$. Let M be the subgroup of the product $G_1 \times G_2$ defined by

$$M = \{(g_1, g_2) \in G_1 \times G_2 \mid f_1(g_1) = f_2(g_2)\}.$$

   Let $m_i = p_i|_M$ where the p_i are the projections of $G_1 \times G_2$ onto G_i. Show that (M, m_1, m_2) is a pull back of $\{f_1, f_2\}$, i.e., $M = G_1 \times_H G_2$.

9. Let p be a prime number and let $I = \{1, 2, 3, \dots\}$ be the directed set of positive integers with their usual order. For each $n \in I$ let $\mathbb{Z}/p^n\mathbb{Z}$ be the ring of integers mod p^n. If $m \leq n$ we have the projections $\varphi_{nm} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$. The *ring of p-adic integers* is the inverse limit ring $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.

   (i) Show that $\mathbb{Z}_p$ can be constructed as sequences of residue classes $a = (a_1 \pmod{p}, a_2 \pmod{p^2}, a_3 \pmod{p^3}, \dots)$ where the a_i are integers and for $\ell \geq k$ we have $a_k \equiv a_\ell \pmod{p^k}$ with componentwise addition and multiplication. The maps $\eta_n : \mathbb{Z}_p \to \mathbb{Z}/p^n\mathbb{Z}$ are given by $a \mapsto a_n \pmod{p^n}$.

   (ii) Show that every element of $\mathbb{Z}_p$ can be represented by a representative of the form $(r_0, r_0 + r_1 p, r_0 + r_1 p + r_2 p^2, \dots)$ with $0 \leq r_i < p$.

   (iii) Show that if we associate to $(r_0, r_0 + r_1 p, r_0 + r_1 p + r_2 p^2, \dots)$ the formal sum $r_0 + r_1 p + r_2 p^2 + \cdots = \sum r_i p^i$, called a p-adic number, then the addition and multiplication in $\mathbb{Z}_p$ correspond to the usual sum and product of series with the usual rules of "carrying".