Math 6112 – Spring 2020
Problem Set 7
Due: 28 February 2020

28. Prove the **Five Lemma**: If we have a commutative diagram of R-modules so that each row is exact:

$$\begin{array}{ccccccccc} M_1 & \longrightarrow & M_2 & \longrightarrow & M_3 & \longrightarrow & M_4 & \longrightarrow & M_5 \\ {\scriptstyle d_1}\downarrow & & {\scriptstyle d_2}\downarrow & & {\scriptstyle d_3}\downarrow & & {\scriptstyle d_4}\downarrow & & {\scriptstyle d_5}\downarrow \\ N_1 & \longrightarrow & N_2 & \longrightarrow & N_3 & \longrightarrow & N_4 & \longrightarrow & N_5 \end{array}$$

Show that

(a) If d_1 is surjective and d_2, d_4 are injective, then d_3 is injective.

(b) If d_5 is injective and d_2, d_4 are surjective, then d_3 is surjective.

29. Let R be commutative. Let F be a flat R-module and suppose that

$$0 \longrightarrow N \longrightarrow M \longrightarrow F \longrightarrow 0$$

is en exact sequence of R-modules. Show that for any R-module E we have

$$0 \longrightarrow N \otimes E \longrightarrow M \otimes E \longrightarrow F \otimes E \longrightarrow 0$$

is exact.

[Hint. Represent E as the quotient of a flat module L (say a free module, which is flat)

$$0 \longrightarrow K \longrightarrow L \longrightarrow E \longrightarrow 0.$$

Then tensor the two sequences together to get a commutative square and then use the Snake Lemma to see that

$$0 \longrightarrow N \otimes E \longrightarrow M \otimes E$$

is exact.]

30. Use a similar technique to prove that if

$$0 \longrightarrow F' \longrightarrow F \longrightarrow F'' \longrightarrow 0$$

is exact and F'' is flat then F is flat iff F' is flat.

[Hint. Take an exact sequence

$$0 \longrightarrow E' \longrightarrow E$$

and tensor with the sequence of $F's$ to get a diagram to which you can apply the Snake. You will need to use the previous problem.]