Evaluating the Mahler measure of linear forms via the Kronecker limit formula on complex projective space

James Cogdell Jay Jorgenson * Lejla Smajlović

March 10, 2021

Abstract

In Cogdell et al., *LMS Lecture Notes Series* **459,** 393–427 (2020), the authors proved an analogue of Kronecker's limit formula associated to any divisor $\mathcal{D}$ which is smooth in codimension one on any smooth Kähler manifold X. In the present article, we apply the aforementioned Kronecker limit formula in the case when X is complex projective space $\mathbb{CP}^n$ for $n \geq 2$ and $\mathcal{D}$ is a hyperplane, meaning the divisor of a linear form $P_D(z)$ for $z = (\mathcal{Z}_j) \in \mathbb{CP}^n$. Our main result is an explicit evaluation of the Mahler measure of P_D as a convergent series whose each term is given in terms of rational numbers, multinomial coefficients, and the L^2-norm of the vector of coefficients of P_D.

1 Introduction

1.1 Mahler measure

Let $P(x_1, \cdots, x_n) \in \mathbb{C}[x_1, \cdots, x_n]$ be a polynomial in n variables with complex coefficients; we assume that P is not identically equal to zero. The Mahler measure $M(P)$ of P is defined by the expression

$$M(P) = \exp\left(\frac{1}{(2\pi)^n}\int_0^{2\pi}\int_0^{2\pi}\cdots\int_0^{2\pi}\log\left(\left|P(e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})\right|\right)d\theta_1\, d\theta_2 \cdots d\theta_n\right). \tag{1}$$

If $n = 1$, we can write $P(x) = a_d x^d + \cdots + a_1 x + a_0 = a_d \prod_{k=1}^{d}(x - \alpha_k)$, in which case Jensen's formula implies that

$$M(P) = |a_d| \prod_{|\alpha_j|>1} |\alpha_j|. \tag{2}$$

As usual, one sets $m(P) = \log M(P)$ to denote the logarithmic Mahler measure of P.

Amongst the numerous articles involving Mahler measures, we shall highlight a few which we find particularly motivating. In [Sm08] the author presents an excellent survey of the many ways in which the Mahler measure of polynomials in one variable is related to various questions in mathematics, including problems in algebraic number theory, ergodic theory, knot theory, transcendental number theory and diophantine approximation, just to name a few. In [De97]

*The second named author acknowledges grant support from several PSC-CUNY Awards, which are jointly funded by the Professional Staff Congress and The City University of New York.

the author established a fascinating connection between Mahler measures and Deligne periods associated to mixed motives; see [De09] and [De12] for subsequent development of the insight from [De97]. In [Bo98], the author undertakes a study of numerical methods by which one can estimate Mahler measures and, as a result, is able to investigate some of the ideas from [De97]. Since then, many authors have extended the study of Mahler measures both in the numerical direction as in [Bo98] and in the theoretical framework as in [De97].

On page 22 of [BG06] the authors stated the definition of Mahler measure (1) in the context of heights of polynomials, though subsequent discussion only considers the setting of one variable polynomials. In [Ma00] the author computed the arithmetic height, in the sense of Arakelov theory, of divisors in projective space. Specifically, it was shown that a hypersurface defined over $\mathbb{Z}$ has canonical height which was expressed by the Mahler measure of a defining polynomial; see page 107 of [Ma00].

The following observation summarizes a considerable part of the aforementioned work: In many instances, Mahler measures can be expressed as special number, such as norms of algebraic numbers, arithmetic heights or special values of L-functions. As such, the study of Mahler measures is intrinsically interesting.

1.2 Mahler measure of a linear polynomial

For this article, we will consider the specific setting of linear polynomials, which itself has been the focus of attention; see, for example, [R-VTV04] or [Sm81]. Let

$$P_D(\mathcal{Z}_0, \ldots, \mathcal{Z}_n) = \mathcal{W}_0\mathcal{Z}_0 + \mathcal{W}_1\mathcal{Z}_1 + ... + \mathcal{W}_n\mathcal{Z}_n \tag{3}$$

denote the linear polynomial in $n+1$ complex projective coordinates variables, and assume for now that $n \geq 2$. We will parameterize the polynomial P_D through the $(n+1)$-tuple

$$D = (\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_n)$$

of its coefficients. Of course, we assume that some $\mathcal{W}_j$ is not zero, and we set

$$\|D\|^2 = |\mathcal{W}_0|^2 + \cdots + |\mathcal{W}_n|^2.$$

Assuming that $\mathcal{W}_0 \neq 0$, one has that, after dehomogenization, the (logarithmic) Mahler measure $m(P_D)$ can be evaluated as

$$m(P_D) = \frac{1}{(2\pi)^n} \int_0^{2\pi}\int_0^{2\pi} \cdots \int_0^{2\pi} \log\Big(\big|P_D(1, e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n})\big|\Big)\, d\theta_1\, d\theta_2 \cdots d\theta_n. \tag{4}$$

In [R-VTV04], the authors derived the bounds

$$\log||D|| - \frac{1}{2}\gamma - 2 \leq m(P_D) \leq \log\|D\|, \tag{5}$$

where γ denotes Euler's constant. The upper bound in (5) is trivial; however, the lower bound follows from reasonably extensive computations stemming from an infinite series expansion of the Mahler measure in terms of certain weighted integrals of J-Bessel functions. Indeed, one of the points made in [R-VTV04] is that their results are amenable to numerical estimation of $m(P_D)$ for any linear polynomial P_D; see, in particular, Corollary 1.4 on page 476 of [R-VTV04].

In [Sm81] it is shown that for certain classes of linear polynomials in $n+1$ variables one can evaluate the corresponding Mahler measure. Some of the main results of [Sm81] follow from clever applications of Jensen's formula, thus the resulting formulas are similar to (2).

To specialize further, let us now assume that for each j we have that $\mathcal{W}_j = 1$. In [BSWZ12] it is shown that $m(P_D) = \frac{d}{ds} W_{n+1}(s)\Big|_{s=0}$ where

$$W_{n+1}(s) = \int_0^1 \cdots \int_0^1 \left| \sum_{k=1}^{n+1} e^{2\pi i t_k} \right|^s dt_1 \cdots dt_{n+1}.$$

When studying the Mahler measure of P_D the authors of [BSWZ12] employed arguments from probability theory to analyze $W_{n+1}(s)$, which is viewed as the s-th moment of an $(n+1)$-step random walk. In doing so, it is asserted on page 982 of [BSWZ12] that

$$m(P_D) = \log(n+1) - \sum_{j=1}^{\infty} \frac{1}{2j} \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k}{(n+1)^{2k}} W_{n+1}(2k) \quad \text{when } D = (1, 1, \cdots, 1). \tag{6}$$

As it turns out, equation (6) is a special case of our Theorem 1 as stated below.

Finally, let us note that in certain special instances the values of the Mahler measure of a linear polynomial P_D, have been computed explicitly. When $n = 2$ and $D = (1, 1, 1)$, it is proved in [Sm81] that

$$m(1 + z_1 + z_2) = \frac{3\sqrt{3}}{4\pi} L(2, \chi_3)$$

where $L(s, \chi_3)$ is the Dirichlet $L-$function associated to the non-principal odd character χ_3 modulo 3. If $n = 3$, then it also is proved in [Sm81] that

$$m(1 + z_1 + z_2 + z_3) = \frac{7}{2\pi^2} \zeta(3)$$

where $\zeta(s)$ denotes the Riemann zeta function. There are many other examples of explicit evaluations of Mahler measures, far too many to provide an exhaustive listing. However, it should be noted that each new evaluation is in and of itself interesting and aids in the understanding the importance of Mahler measures.

1.3 Our main results

The purpose of this article is to develop a different means to evaluate Mahler measures of linear polynomials in $n+1$ complex variables. Our approach is based on the following observation. A holomorphic section of a power of the canonical bundle on n-dimensional complex projective space $\mathbb{CP}^n$ can be realized as a homogeneous polynomial in $n+1$ projective coordinates. Therefore, the log-norm of such a polynomial, which appears in the definition of the Mahler measure, can be expressed in terms of the log-norm of a holomorphic form on $\mathbb{CP}^n$ which, from the results of [CJS20], are related to an integral over its divisor of a "truncated" Green's function, or resolvent kernel, on $\mathbb{CP}^n$ by way of its Kronecker limit formula. The spectral expansion of Green's function on $\mathbb{CP}^n$ yields a representation of the log-norm of the polynomial in terms of a certain infinite series which we are able to explicitly evaluate.

Our first main result is the following theorem.

Theorem 1 *With the notation as above, let* $c(D)^2 = (n+1)||D||^2$ *and set*

$$a(n,k,D) = \sum_{\ell_0+\ldots+\ell_n=k, \ell_m \geq 0} \binom{k}{\ell_0, \ell_1, \ldots, \ell_n}^2 |\mathcal{W}_0|^{2\ell_0} \cdots |\mathcal{W}_n|^{2\ell_n}$$

where

$$\binom{k}{\ell_0, \ell_1, \ldots, \ell_n} = \frac{k!}{\ell_0! \ell_1! \cdots \ell_n!}.$$

is the multinomial coefficient. Then for $n \geq 3$ *the logarithmic Mahler measure* $m(P_D)$ *of the linear polynomial* P_D *is given by*

$$m(P_D) = \log c(D) - \frac{1}{2} \sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k a(n,k,D)}{c(D)^{2k}}. \tag{7}$$

The sum over j on the right-hand side of (7) is absolutely convergent. However, it is not possible to view the series as a double series in j and k and then interchange the order of summation. In particular, the series diverges when viewed as a sum over j and for a fixed k.

We also obtain the following expressions for $m(P_D)$.

Theorem 2 *For any integer* $\ell \geq 1$, *let* $H_\ell = 1 + \frac{1}{2} \cdots + \frac{1}{\ell}$ *and set*

$$S_D(\ell) = \sum_{j=1}^{\infty} \frac{2j+\ell}{j(j+\ell)} \sum_{k=0}^{j} \binom{j+\ell+k-1}{k} \binom{j}{k} \frac{(-1)^k a(n,k,D)}{c(D)^{2k}},$$

which is defined for $\ell \geq 0$. *Then for any* $n \geq 3$ *and any* D, *we have that*

$$m(P_D) = \log c(D) - \frac{1}{2} H_1 - \frac{1}{2} S_D(1). \tag{8}$$

Further, for any $n \geq 3$ *and* $\ell \geq 2$ *we have that*

$$m(P_D) = \log c(D) - \frac{1}{2} H_\ell - \frac{1}{2} S_D(\ell) \tag{9}$$

provided $D \neq r(1,1,\cdots,1)$ *for some* $r \neq 0$.

From (8), we will prove that for all $n \geq 3$ and all D one has that

$$m(P_D) = \log c(D) - \frac{1}{2} S_D(0). \tag{10}$$

In summary, we will prove that (9) holds in the following cases:

(i) All $n \geq 3$ and all D when $\ell = 0$;

(ii) All $n \geq 3$ and all D when $\ell = 1$;

(iii) All $n \geq 3$ and all $\ell \geq 2$ provided $D \neq r(1,1,\cdots,1)$ for some $r \neq 0$.

In our concluding comments to this paper, we will discuss the exceptional instance in case (iii) as well as the general setting when $n = 2$.

As stated, [Sm81] obtains some explicit evaluations of Mahler measures for certain linear polynomials. Thus, by combining our main theorem with the formulas from [Sm81], we obtain many intriguing identities. Along this line, let us point out the following "amusing" corollary which comes from comparing our results to those from [R-VTV04].

Corollary 1 *For any non-zero vector $D = (\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_n) \in \mathbb{C}^{n+1}$ one has that*

$$\sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k a(n,k,D)}{||D||^{2k}} \left(\frac{1}{(n+1)^k} - \frac{1}{k!} \right) = \log(n+1) + \gamma, \tag{11}$$

where γ denotes the Euler constant.

It is interesting that the right-hand side of (11) is independent of D. When $n = 0$, equation (11) still makes sense and will follow from equation (2.4) of [R-VTV04] after one would show that (11) converges. As such, it is possible that (11) could be proved directly, at least for some "small" values of n.

A further discussion of additional identities is given in the concluding section of this article, see for example equation (60).

In our proof of Theorem 1 and Theorem 2 we obtain precise bounds for the rates of convergence of the infinite series involved. Specifically, we obtain the following estimates.

Theorem 3 *With notation as above, assume $n \geq 3$ and choose any $D \neq 0$. Then there is an explicitly computable constant $G(n,D)$, which depends solely on n and D, such that for any $N \geq 1$ we have the bounds*

$$|m(P_D) - E_1(N;n,D)| \leq \frac{\Gamma(3/4)}{3} \frac{G(n,D)}{N^{3/4}} \quad \text{and} \quad |m(P_D) - E_2(N;n,D)| \leq 2\sqrt[4]{2}\frac{G(n,D)}{\sqrt{N}}, \tag{12}$$

where

$$E_1(N;n,D) = \log c(D) - \frac{1}{2} \sum_{j=1}^{N} \frac{1}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k a(n,k,D)}{c(D)^{2k}} \tag{13}$$

and

$$E_2(N;n,D) = \log c(D) - \frac{1}{2} - \frac{1}{2} \sum_{j=1}^{N} \frac{2j+1}{j(j+1)} \sum_{k=0}^{j} \binom{j+k}{k} \binom{j}{k} \frac{(-1)^k a(n,k,D)}{c(D)^{2k}}. \tag{14}$$

The bound we derive for $G(n,D)$ will be given in terms of the J-Bessel function, see formula (47). As such, an elementary and explicit bound for $G(n,D)$ can be derived. In particular, Theorem 3 leads to an explicit computational means by which one can estimate $m(P_D)$ as accurately as one may wish.

We will derive explicit bounds for the tail of the series in (9) for all $\ell \geq 2$ and for all $D \neq r(1,1,\cdots,1)$. We refer the reader to Section 6 for the statements. In the course of the proof it becomes clear why this sole D is singled out; it is the only instance where an application of the Cauchy-Schwarz inequality is an equality rather than a strict inequality.

As is evident from equations (12) through (14) and the statements of our main theorems, the approximating sum $E_1(N;n,D)$ is somewhat simpler and with a better rate of convergence to $m(P_D)$ than the sum $E_2(N;n,D)$. However, we find the estimate of $m(P_D)$ by $E_2(N;n,D)$ to be theoretically interesting as well. Indeed, when combining the various expressions for $m(P_D)$ derived above one has a potential source of combinatorial identities amongst weighted series of sums of binomial and multinomial coefficients.

Finally, we would like to emphasize that our main theorem is the first result of which we are aware which gives an explicit expression of the (logarithmic) Mahler measure of a linear form in terms of an absolutely convergent series which involves elementary quantities, such as binomial and multinomial coefficients. The explicit and effective upper bound for the approximation of $m(P_D)$ by a partial sum of this series provides a tool which can be used in estimating the size of $m(P_D)$, hence the canonical height of the divisor $\mathcal{D}$ of P_D; see [Ma00] and the discussion in Section 7.5.

1.4 Outline of the proofs

In general terms, the analysis of the present paper involves a detailed investigation of the Kronecker-type limit formula which was proved in [CJS20]. In [CJS20], we considered an arbitrary smooth Kähler manifold X with divisor $\mathcal{D}$ which was assumed to be smooth up to codimension two. For this article, we take X to be n-dimensional complex projective space $\mathbb{CP}^n$ for $n \geq 2$ and $\mathcal{D}$ to be a hyperplane, meaning the divisor of a degree one polynomial $P_D(z)$. We equip $\mathbb{CP}^n$ with its natural Fubini-Study metric.

With this setup, we employ a representation of the associated Green's function in terms of the heat kernel; see [HI02]. The spectral expansion of the Green's function also can be computed explicitly; see [Lu98]. In order to evaluate the Kronecker-type limit function as in [CJS20], we need to integrate the Green's function on $\mathbb{CP}^n$ along a hyperplane. In doing so, the evaluation of such integrals amounts to the Radon transform on projective space for which we use results from [Gr83]. Ultimately, we are able to express the log-norm of the polynomial P_D as an absolutely convergent series of Jacobi polynomials. The evaluation of the Mahler measure then reduces to the problem of evaluating certain integrals involving Jacobi polynomials, which yields the results stated above. The different expressions for the Mahler measure $m(P_D)$ yield various identities involving Jacobi polynomials.

We wish to emphasize here that our main result holds for all linear polynomials provided $n \geq 3$; all our results, except possibly (9) with $\ell \geq 3$ and $D = r(1, 1, \ldots, 1)$ for some $r \neq 0$, hold when the the divisor of $P_D(z)$ intersects the domain of integration in (1). Previous authors such as [Sm81] used techniques of complex analysis to obtain their results. Their computations are important and interesting, but are limited because of the logarithmic-type singularities which naturally occur. From out point of view, such singularities are L^2, hence can be addressed when using real analytic methods.

1.5 Organization of the paper

In Section 2 we state additional notation and relevant results from the literature. In Section 3 we recall the general Kronecker-type limit formula from [CJS20], which holds for a reasonably general Kähler manifold X, and make the result explicit in the case $X = \mathbb{CP}^n$. In Section 4 we study the results from Section 3 and obtain various expressions for $\log \|P_D\|_\mu$ where the norm is with respect to the Fubini-Study metric. In Section 5 we derive a change of variables formula which is an important ingredient in the proof of our main result, as carried out in Section 6. We conclude with Section 7 where we present several comments regarding the analysis of this article.

2 Preliminaries

In this section we will introduce some necessary notation and prove certain intermediate results related to the representation of the resolvent kernel on $\mathbb{CP}^n$ and its associated Kronecker limit formula as proved in [CJS20].

2.1 Some special functions

For any non-negative integers α, β and j, we let $P_j^{(\alpha,\beta)}$ denote the Jacobi polynomial, which is defined for $x \in (-1,1)$ by

$$P_j^{(\alpha,\beta)}(x) := \frac{(-1)^j}{2^j j!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^j}{dx^j}\left[(1-x)^{\alpha+j}(1+x)^{\beta+j}\right]. \tag{15}$$

If $\alpha = \beta = 0$, then the Jacobi polynomials specialize to the Legrendre polynomials, which can given by

$$P_j(x) = \frac{1}{2^j j!}\frac{d^j}{dx^j}(x^2-1)^j.$$

Many fascinating properties of Jacobi and Legrendre polynomials are developed in the classical text [Sz74]. In the present article, we will use the following bound which we quote from [HS14]. For any $j \geq 1$ and for all $x \in [-1,1]$, we have that

$$(1-x^2)^{\frac{1}{4}}|P_j(x)| \leq \sqrt{4/\pi}(2j+1)^{-\frac{1}{2}}. \tag{16}$$

This bound is referred to as the the sharp form of the Bernstein's inequality for Legendre polynomials $P_j = P_j^{(0,0)}$; see Theorem 3.3 of [Sz74], [Lo82/83], or the discussion on page 228 of [HS14]. More generally, we will use the main theorem of [HS14] which gives a uniform upper bound, namely that

$$(1-x^2)^{\frac{1}{4}}\left(\frac{1-x}{2}\right)^{(m-1)/2}|P_j^{(m-1,0)}(x)| \leq 12\cdot(2j+m)^{-\frac{1}{2}}, \tag{17}$$

which holds for all positive integers j and m and for $x \in [-1,1]$; see Theorem 1.1 and subsequent discussion on page 228 of [HS14].

For complex numbers ν and z with $|\arg z| < \pi$, the Bessel function of the first kind $J_\nu(z)$ is defined by absolutely convergent power series

$$J_\nu(z) := \frac{z^\nu}{2^\nu}\sum_{k=0}^{\infty}\frac{(-1)^k}{2^{2k}k!\Gamma(\nu+k+1)}z^{2k}.$$

For non-integral complex ν and any complex z with $|\arg z| < \pi$ one defines the Bessel function of the second kind $Y_\nu(z)$ by $Y_\nu(z) := (\sin\pi\nu)^{-1}(\cos(\pi\nu)J_\nu(z) - J_{-\nu}(z))$; when $\nu = n$ is a non-negative integer, then

$$\begin{aligned}\pi Y_n(z) :&= 2J_n(z)\log\left(\frac{z}{2}\right) - \sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\left(\frac{z}{2}\right)^{2k-n}\\ &\quad - \sum_{k=0}^{\infty}\frac{(-1)^k(z/2)^{2k+n}}{k!(k+n)!}\left(\frac{\Gamma'}{\Gamma}(k+1)+\frac{\Gamma'}{\Gamma}(k+n+1)\right),\end{aligned}$$

with the convention that the empty sum when $n = 0$ is zero. A thorough analysis of Bessel functions and functions associated with them can be found in the seminal book [Wa66]. The article [Kr06] contains very explicit pointwise bounds for Bessel functions. For our purposes, we will use the inequality from 7.31.2 [Sz74] which states that

$$|J_0(2x)| \leq (\max\{1, |\pi x|\})^{-\frac{1}{2}}. \tag{18}$$

2.2 Complex projective space

Let $\mathbb{CP}^n$ denote the n-dimensional complex projective space with the usual projective coordinates $(\mathcal{Z}_0, \cdots, \mathcal{Z}_n)$. If U is any open set in $\mathbb{CP}^n$ and $z : U \to \mathbb{C}^{n+1} \setminus \{0\}$ a holomorphic lifting of U for a holomorphically varying choice of homogeneous coordinates z, then the local Kähler potential is given by $\rho(z) = \log \|z\|^2 = \log(|\mathcal{Z}_0|^2 + \cdots + |\mathcal{Z}_n|^2)$. The Kähler $(1,1)$ form $\frac{i}{2}\partial_z \partial_{\bar{z}} \rho$ will be denoted by ω, and we equip $\mathbb{CP}^n$ with the Fubini-Study metric $\mu = \mu_{FS}$ associated to ω. The Fubini-Study distance between two points $z, w \in \mathbb{CP}^n$ will be denoted by $d_{\mathrm{FS}}(z, w)$. Explicitly, the distance is computed by the formula

$$\cos(d_{\mathrm{FS}}(z, w)) = \frac{|\langle z, w\rangle|}{\sqrt{\langle z, z\rangle}\sqrt{\langle w, w\rangle}}$$

where, if $z = (\mathcal{Z}_0, \ldots, \mathcal{Z}_n)$ and $w = (\mathcal{W}_0, \ldots, \mathcal{W}_n)$, then $\langle z, w\rangle = z \cdot {}^t\overline{w} = \mathcal{Z}_0\overline{\mathcal{W}_0} + \cdots \mathcal{Z}_n\overline{\mathcal{W}_n}$.

Occasionally, our computations will be on the affine chart where $\mathcal{Z}_0 \neq 0$, so then we will consider the affine coordinates $(z_1, \ldots, z_n)$. Then the local Kähler potential takes the form $\rho_0(z) = \log(1 + |z_1|^2 + \ldots + |z_n|^2)$.

Let $P_D(z)$ denote any homogenous polynomial with divisor $\mathcal{D}$. We denote by $\|P_D(z)\|^2_\mu$ the log-norm of the polynomial P_D with respect to μ. The formula for $\|P_D(z)\|^2_\mu$ is

$$\log \|P_D(z)\|^2_\mu = \log |P_D(z)|^2 - \deg(P_D)\rho(z) \tag{19}$$

for $z \in \mathbb{CP}^n \setminus \mathcal{D}$. If z approaches $\mathcal{D}$ transversally, then $\log \|P_D(z)\|^2_\mu$ has a logarithmic singularity which is L^1 integrable. For the sake of brevity, we may omit the subscript μ.

Let $\Delta_{\mathbb{CP}^n}$ signify the corresponding Laplacian $\Delta_{\mathbb{CP}^n}$ which acts on smooth functions on $\mathbb{CP}^n$. An eigenfunction of the Laplacian $\Delta_{\mathbb{CP}^n}$ is an *a priori* C^2 function ψ_j which satisfies the equation

$$\Delta_{\mathbb{CP}^n}\psi_j + \lambda_j\psi_j = 0$$

for some constant λ_j, which is the eigenvalue associated to ψ_j. The spectrum $\{\lambda_j\}_{j\geq 0}$ of $\Delta_{\mathbb{CP}^n}$ is well known; see, for example, [BGM71] or [Lu98]. Classically, $\lambda_0 = 0$ where the eigenfunction is an appropriately normalized positive constant function.

Let $\mathrm{vol}_\mu(\mathbb{CP}^n)$ denote the volume of $\mathbb{CP}^n$, meaning the integral over $\mathbb{CP}^n$ of the volume form μ^n. With our normalizations, we have that

$$\mathrm{vol}_\mu(\mathbb{CP}^n) = \frac{\pi^n}{n!}.$$

Additionally, we have that $\lambda_j = 4j(j + n)$ for all $j \geq 1$. Let $H_{j,j}(n + 1)$ be the vector space of eigenfunctions with eigenvalue λ_j, Then the dimension N_j of $H_{j,j}$ is

$$N_j = \binom{n+j}{j}^2 - \binom{n+j-1}{j-1}^2 = \frac{(n+2j)((n+j-1)!)^2}{n!(n-1)!(j!)^2}.$$

Moreover, as discussed in section 1 of [Gr83], the Hilbert space $L^2(\mathbb{CP}^n)$ of all square integrable functions on $\mathbb{CP}^n$, with respect to the volume form μ^n, has the orthogonal decomposition

$$L^2(\mathbb{CP}^n) = \bigoplus_{j=0}^{\infty} H_{j,j}(n+1)$$

into finite dimensional subspaces $H_{j,j}(n + 1)$, which as stated consist of eigenfunctions of the Laplacian with the corresponding eigenvalue $4j(j+n)$. Each subspace $H_{j,j}(n+1)$ is an irreducible representation of the unitary group $\mathbf{U}(n+1)$, and the spaces are distinct. More precisely, elements of $H_{j,j}(n+1)$ are homogeneous harmonic polynomials of degree j in the variables $\mathcal{Z}_0, ..., \mathcal{Z}_n$ and $\overline{\mathcal{Z}}_0, ..., \overline{\mathcal{Z}}_n$. As is standard, we may assume that coefficients of those harmonic polynomials are real so then any eigenfunction evaluated at real values of its variables is itself real-valued.

2.3 The Radon transform

Let f be a continuous function on $\mathbb{CP}^n$, and let H be any hyperplane in $\mathbb{CP}^n$. The Radon transform of f evaluated at H, which we denote by $Rf(H)$, is defined by

$$Rf(H) = \int\limits_H f(w)\mu_H(w)$$

where $\mu_H(w)$ is the Fubini-Study volume element induced on H from the Fubini-Study metric on $\mathbb{CP}^n$. Denote the Grassmannian of hyperplanes in $\mathbb{CP}^n$ by $(\mathbb{CP}^n)^*$. Recall that $(\mathbb{CP}^n)^*$ is non-canonically isomorphic to $\mathbb{CP}^n$. Let us make a choice regarding this isomorphism. Quite simply, the point $(\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_n) \in \mathbb{CP}^n$ is identified with the hyperplane

$$\{(\mathcal{Z}_0, \mathcal{Z}_1, \ldots, \mathcal{Z}_n) \in \mathbb{CP}^n : \mathcal{Z}_0\mathcal{W}_0 + \mathcal{Z}_1\mathcal{W}_1 + \ldots + \mathcal{Z}_n\mathcal{W}_n = 0\}.$$

As such, we can view the Radon transform $Rf(H)$ of f as a function on $\mathbb{CP}^n$.

As proved in [Gr83], by Schur's Lemma the Radon transform R acts on $H_{j,j}(n+1)$ by scalar multiplication by $c(j,n) = c_n \cdot \frac{(-1)^j j!}{(j+n-1)!}$ where c_n is certain normalizing factor depending solely on the dimension n. In our setting, the normalizing factor can be easily computed by evaluating the Radon transform of the L^2-normalized constant eigenfunction $\psi_0(w) = \frac{1}{\sqrt{\mathrm{vol}_\mu(\mathbb{CP}^n)}}$ and taking H to be the (affine) hyperplane $z_1 = 0$. In this case $c(0,n) = c_n \cdot \frac{1}{(n-1)!}$, so then

$$\int\limits_H \frac{1}{\sqrt{\mathrm{vol}_\mu(\mathbb{CP}^n)}}\mu_H(w) = c_n \cdot \frac{1}{(n-1)!} \cdot \frac{1}{\sqrt{\mathrm{vol}_\mu(\mathbb{CP}^n)}}.$$

Therefore, $c_n = (n-1)! \cdot \mathrm{vol}_\mu(\mathbb{CP}^{n-1}) = \pi^{n-1}$. If $\psi_j \in H_{j,j}(n+1)$, where R acts by the scalar $c(j,n)$, we will simply have $R\psi_j(H) = c(j,n)\psi_j(H)$ where we identify $(\mathbb{CP}^n)^*$ with $\mathbb{CP}^n$ as above.

In summary, we have the following formula. With the notation and normalizations set above, for any hyperplane H in $\mathbb{CP}^n$ and any eigenfunction $\psi_j \in H_{j,j}(n+1)$, one has that

$$\int\limits_H \psi_j(w)\mu_H(w) = \pi^{n-1} \cdot \frac{(-1)^j j!}{(j+n-1)!}\psi_j(H). \tag{20}$$

Again, it is necessary to note that we have chosen an identification between the hyperplane H, which is a point in $(\mathbb{CP}^n)^*$, and a point in $\mathbb{CP}^n$, which through a slight abuse of notation we also write as H.

2.4 Heat kernel and Green's function

The heat kernel $K_{\mathbb{CP}^n}(z, w; t)$ associated to the Laplacian $\Delta_{\mathbb{CP}^n}$ on $\mathbb{CP}^n$ is the unique solution to the heat-equation

$$\frac{\partial}{\partial t}K_{\mathbb{CP}^n}(z, w; t) = \Delta_{\mathbb{CP}^n}K_{\mathbb{CP}^n}(z, w; t) \qquad \text{for} \quad t > 0 \quad \text{and} \quad z, w \in \mathbb{CP}^n$$

such that for any continuous function f on $\mathbb{CP}^n$, one has

$$\lim_{t\to 0}\int_{\mathbb{CP}^n} K_{\mathbb{CP}^n}(z,w;t)f(w)\mu_{\mathbb{CP}^n}(w)=f(z).$$

As can be shown, $K_{\mathbb{CP}^n}(z,w;t)$ depends only on $t>0$ and the Fubini-Study distance between z and w; see, for example, [Lu98]. The heat kernel admits a spectral expansion in terms of the eigenfunctions $\psi_j\in H_{j,j}(n+1)$. Namely, one has the formula that

$$K_{\mathbb{CP}^n}(z,w;t)=\sum_{\lambda_j\geq 0}\psi_j(z)\overline{\psi_j(w)}e^{-\lambda_j t}=\frac{1}{\mathrm{vol}_\mu(\mathbb{CP}^n)}+\sum_{\lambda_j>0}\psi_j(z)\overline{\psi_j(w)}e^{-\lambda_j t}. \tag{21}$$

In [HI02] it is proved that

$$K_{\mathbb{CP}^n}(z,w;t)=\frac{1}{\mathrm{vol}_\mu(\mathbb{CP}^n)}+\frac{1}{\pi^n}\sum_{j=1}^{\infty}(2j+n)\frac{(j+n-1)!}{j!}P_j^{(n-1,0)}(\cos(2r))e^{-4j(j+n)t}, \tag{22}$$

where $r=d_{\mathrm{FS}}(z,w)$ is the Fubini-Study distance and $P_j^{(\alpha,\beta)}$ is the Jacobi polynomial defined in (15). Actually, the transition from (21) to (22) is based on stronger results. Indeed, it is proved that

$$\sum_{\lambda_j=4j(n+j)}\psi_j(z)\overline{\psi_j(w)}=\frac{(2j+n)}{\pi^n}\frac{(j+n-1)!}{j!}P_j^{(n-1,0)}(\cos(2r)); \tag{23}$$

see Theorem 1 of [Lu98] as well as Theorem 1 and the preceding discussion in [HI02], keeping in mind the notation conventions employed in the present article.

The Green's function $G_{\mathbb{CP}^n}(z,w;s)$ on $\mathbb{CP}^n$ is the integeral kernel of the right inverse to the operator $\Delta_{\mathbb{CP}^n}+s(1-s)$ on $L^2(\mathbb{CP}^n)$ for $s\in\mathbb{C}$. In order for such an inverse to exist, it is necessary to assume that $s(1-s)$ is not equal to an eigenvalue of $\Delta_{\mathbb{CP}^n}$. As discussed in Section 5 of [CJS20] and references therein, the Green's function $G_{\mathbb{CP}^n}(z,w;s)$ and the heat kernel $K_{\mathbb{CP}^n}(z,w;t)$ are related by the formula

$$G_{\mathbb{CP}^n}(z,w;s)-\frac{1}{\mathrm{vol}_\mu(\mathbb{CP}^n)}\frac{1}{s^2}=\int_0^\infty\left(K_{\mathbb{CP}^n}(z,w;t)-\frac{1}{\mathrm{vol}_\mu(\mathbb{CP}^n)}\right)e^{-s^2t}dt. \tag{24}$$

The identity (24) holds for $s\in\mathbb{C}$ with $\mathrm{Re}(s^2)>0$ and for all distinct points $z,w\in\mathbb{CP}^n$. However, one can use the spectral expansion of the heat kernel in order to obtain a meromorphic continuation of (24) to all $s\in\mathbb{C}$.

3 A Kronecker limit formula

The following result from [CJS20] is an analogue of the classical Kronecker's limit formula, which is stated here in the setting of projective space.

Theorem 4 *Let $\mathcal{D}$ be the divisor of a polynomial P_D on $\mathbb{CP}^n$, and assume that $\mathcal{D}$ is smooth up to codimension two in $\mathbb{CP}^n$. Then there exist constants c_0 and c_1 such that for $z\notin\mathcal{D}$ we have that*

$$\int_{\mathcal{D}} G_{\mathbb{CP}^n}(z,w;s)\mu_{\mathcal{D}}(w)=\frac{\mathrm{vol}_\mu(\mathcal{D})}{\mathrm{vol}_\mu(\mathbb{CP}^n)}\frac{1}{s^2}+c_0\log\|P_D(z)\|_\mu^2+c_1+O(s)\quad \textit{as } s\to 0. \tag{25}$$

Let us now consider the case when P_D is a linear polynomial in $n+1$ projective coordinates $(\mathcal{Z}_0, \ldots, \mathcal{Z}_n)$ of $\mathbb{CP}^n$. With this, Theorem 4 becomes the following result.

Proposition 1 *Let $P_D(z) = P_D(\mathcal{Z}_0, \ldots, \mathcal{Z}_n) = \mathcal{W}_0\mathcal{Z}_0 + \mathcal{W}_1\mathcal{Z}_1 + \ldots + \mathcal{W}_n\mathcal{Z}_n$ be the linear polynomial in $n+1$ complex projective coordinate variables with the divisor $\mathcal{D}$. Let H_n denote the n-th harmonic number. Then, for $z \notin \mathcal{D}$ we have that*

$$\int\limits_{\mathcal{D}} G_{\mathbb{CP}^n}(z,w;s)\mu_{\mathcal{D}}(w) = \frac{\mathrm{vol}_\mu(\mathcal{D})}{\mathrm{vol}_\mu(\mathbb{CP}^n)}\frac{1}{s^2} - \frac{1}{4\pi}\log|P_D(z)|^2$$
$$+ \frac{1}{4\pi}\left(\log||D||^2 + \rho(z) - H_n\right) + O(s) \quad \text{as } s \to 0. \tag{26}$$

Proof: Set
$$G_{\mathbb{CP}^n}(z,w) = \lim_{s\to 0}\left(G_{\mathbb{CP}^n}(z,w;s)\mu_{\mathcal{D}}(w) - \frac{1}{\mathrm{vol}_\mu(\mathbb{CP}^n)}\frac{1}{s^2}\right).$$

Then $G_{\mathbb{CP}^n}(z,w)$ is the integral kernel which inverts the action of the Laplacian when restricted to the subspace of $L^2(\mathbb{CP}^n)$ that is orthogonal to the constant functions. Formula (25) in Theorem 4 can be written as

$$\int\limits_{\mathcal{D}} G_{\mathbb{CP}^n}(z,w)\mu_{\mathcal{D}}(w) = c_0 \log\|P_D(z)\|^2_\mu + c_1.$$

If we now integrate over $\mathbb{CP}^n$ with respect to $\mu_{\mathbb{CP}^n}(z)$, we get that

$$0 = c_0 \int\limits_{\mathbb{CP}^n} \log\|P_D(z)\|^2_\mu \mu_{\mathbb{CP}^n}(z) + c_1 \mathrm{vol}_\mu(\mathbb{CP}^n).$$

In our normalizations, we have that $c_0 = -1/4\pi$; see, for example, page 94 of [Fo76] and page 10 of [La88] as well as page 338 of [JK98]. As a result, we have

$$c_1 = \frac{1}{4\pi}\cdot\frac{1}{\mathrm{vol}_\mu(\mathbb{CP}^n)}\int\limits_{\mathbb{CP}^n}\log\|P_D(z)\|^2_\mu \mu_{\mathbb{CP}^n}(z). \tag{27}$$

It is left to evaluate the integral on the right-hand side of (27).

Set $D = (\mathcal{W}_0, \mathcal{W}_1, \ldots, \mathcal{W}_n)$, so then $P_D(z) = D \cdot {}^t z$. One can choose an element $\gamma \in \mathbf{U}(n+1)$ so that $D\gamma = (c, 0, \ldots, 0)$. Through a rescaling of w by a complex number of absolute value one, we may assume that

$$c = \|D\|.$$

Let $z = \tilde{z}^t\gamma$, where $\tilde{z} = (\tilde{\mathcal{Z}}_0, \tilde{\mathcal{Z}}_1, \ldots, \tilde{\mathcal{Z}}_n)$. On the affine chart $\tilde{\mathcal{Z}}_0 \neq 0$, the polynomial $P_D(z)$ is simply $P_D(z) = P_D(\tilde{z}^t\gamma) = D\gamma^t\tilde{z} = c\tilde{\mathcal{Z}}_0$.

By the $\mathbf{U}(n+1)$ invariance of the Fubini-Study volume form

$$\int\limits_{\mathbb{CP}^n}\log\|P_D(z)\|^2_\mu \mu_{\mathbb{CP}^n}(z) = \int\limits_{\mathbb{CP}^n}\log\|P_D(\tilde{z}^t\gamma)\|^2_\mu \mu_{\mathbb{CP}^n}(\tilde{z}).$$

By (19) we have
$$\log\|P_D(\tilde{z}^t\gamma)\|^2_\mu = \log|P_D(\tilde{z}^t\gamma)|^2 - \deg(P_D)\rho(\tilde{z}^t\gamma).$$

Here we have $\log|P_D(\tilde{z}^t\gamma)|^2 = \log|c\tilde{\mathcal{Z}}_0|^2 = \log(c^2) + \log|\tilde{\mathcal{Z}}_0|^2$, $\deg(P_D) = 1$, and $\rho(\tilde{z}^t\gamma) = \log\|\tilde{z}^t\gamma\|^2 = \log\|\tilde{z}\|^2 = \log(|\tilde{\mathcal{Z}}_0|^2 + \cdots + |\tilde{\mathcal{Z}}_n|^2)$. Therefore

$$\log\|P_D(\tilde{z}^t\gamma)\|_\mu^2 = \log(c^2) + \log|\tilde{\mathcal{Z}}_0|^2 - \log(|\tilde{\mathcal{Z}}_0|^2 + \cdots + |\tilde{\mathcal{Z}}_n|^2) = \log\|D\|^2 - \rho_0(\tilde{z})$$

and

$$\int\limits_{\mathbb{CP}^n} \log\|P_D(z)\|_\mu^2 \mu_{\mathbb{CP}^n}(z) = \log\|D\|^2 \mathrm{vol}_\mu(\mathbb{CP}^n) - \int\limits_{\mathbb{CP}^n} \rho_0(\tilde{z})\mu_{\mathbb{CP}^n}(\tilde{z}). \tag{28}$$

The last integral can be evaluated using polar coordinates on the affine chart $\tilde{\mathcal{Z}}_0 \neq 0$, which can be viewed as $\mathbb{C}^n$. Indeed, if we let S^{2n-1} denote the unit sphere in $\mathbb{C}^n$, then

$$\begin{aligned}\int\limits_{\mathbb{CP}^n} \rho_0(\tilde{z})\mu_{\mathbb{CP}^n}(\tilde{z}) &= \mathrm{vol}(S^{2n-1}) \int\limits_0^\infty \log(1+\rho^2)\frac{\rho^{2n-1}}{(\rho^2+1)^{n+1}} d\rho \\ &= n\mathrm{vol}_\mu(\mathbb{CP}^n) \int\limits_1^\infty \log(t) \frac{(1-t)^{n-1}}{t^{n+1}} dt.\end{aligned}$$

The last integral can be evaluated by using formula 4.253.3 from [GR07] with $u = 1$, $\mu = n$ and $\lambda = n+1$. With this, we obtain that

$$\int\limits_{\mathbb{CP}^n} \rho_0(\tilde{z})\mu_{\mathbb{CP}^n}(\tilde{z}) = \mathrm{vol}_\mu(\mathbb{CP}^n)\left(\frac{\Gamma'}{\Gamma}(n+1) - \frac{\Gamma'}{\Gamma}(1)\right) = \mathrm{vol}_\mu(\mathbb{CP}^n) H_n.$$

Combining this evaluation with (28) and (27) we arrive at the formula that

$$c_1 = \frac{1}{4\pi}\left(\log||D||^2 - H_n\right).$$

By substituting this expression together with $c_0 = -1/(4\pi)$ into (25), and employing (19) we have completed our proof of (26). □

Remark 1 An anonymous referee pointed out that some of the above calculations, such as the evaluation of c_1, are related to the proof of Proposition 5.3 from [GS90]. We will elaborate on this observation in Section 7.5.

4 The log-norm of a linear polynomial

In this section we will refine further the results of Theorem 4 and Proposition 1 by expressing the integral of the Green's function in terms of a certain series involving Jacobi polynomials. The convergence issues will be addressed by appealing to the bounds (16) and (17).

Proposition 2 *With the above notation, assume that* $\mathrm{Re}(s^2) > 0$, $z \notin \mathcal{D} \cup \{D\}$. *Then, we have the relation*

$$\begin{aligned}&\int\limits_{\mathcal{D}} \left(G_{\mathbb{CP}^n}(z,w;s) - \frac{1}{\mathrm{vol}_\mu(\mathbb{CP}^n)}\frac{1}{s^2}\right)\mu_{\mathcal{D}}(w) \\ &\qquad = \frac{1}{\pi}\sum_{j=1}^\infty \frac{(2j+n)(-1)^j}{s^2+4j(j+n)} P_j^{(n-1,0)}(\cos(2d_{FS}(z,D))).\end{aligned} \tag{29}$$

The series in (29) *converges uniformly and absolutely when* z *and* s *lie in compact subsets of the above specified regions.*

Proof: We start with the expression (24) for the Green's function in terms of the heat kernel $K_{\mathbb{CP}^n}(z,w;t)$ on $\mathbb{CP}^n$ for $\mathrm{Re}(s^2)>0$. Using the equations (21) and (22) the function under the integral sign in (24) can be written as

$$\begin{aligned} K_{\mathbb{CP}^n}(z,w;t) - \frac{1}{\mathrm{vol}_\mu(\mathbb{CP}^n)} &= \sum_{\lambda_j>0} \psi_j(z)\overline{\psi_j(w)}e^{-\lambda_j t} \\ &= \frac{1}{\pi^n}\sum_{j=1}^{\infty}(2j+n)\frac{(j+n-1)!}{j!}P_j^{(n-1,0)}(\cos(2r))e^{-4j(j+n)t}, \end{aligned}$$

where we have set $r=d_{\mathrm{FS}}(z,w)$. The above series is uniformly bounded provided the distance between z and w is bounded away from zero, which is satisfied for $z\notin\mathcal{D}$ since we will consider $w\in\mathcal{D}$.

Therefore, we can apply the Fubini-Tonelli theorem to get that

$$\int_{\mathcal{D}}\left(G_{\mathbb{CP}^n}(z,w;s) - \frac{1}{\mathrm{vol}_\mu(\mathbb{CP}^n)}\frac{1}{s^2}\right)\mu_{\mathcal{D}}(w) = \int_0^\infty\int_{\mathcal{D}}\left(\sum_{\lambda_j>0}\psi_j(z)\overline{\psi_j(w)}e^{-\lambda_j t}\right)\mu_{\mathcal{D}}(w)e^{-s^2t}dt.$$

As stated above, we may assume that the eigenfunctions ψ_j are homogeneous polynomials with real coefficients, so then

$$\int_{\mathcal{D}}\overline{\psi_j(w)}\mu_{\mathcal{D}}(w) = \int_{\mathcal{D}}\psi_j(\overline{w})\mu_{\mathcal{D}}(w) = \int_{\overline{\mathcal{D}}}\psi_j(w)\mu_{\overline{\mathcal{D}}}(w).$$

In other words, integrating $\overline{\psi_j(w)}$ over $\mathcal{D}$ amounts to taking the Radon transform of the function ψ_j which belongs to the subspace $H_{j,j}(n+1)$, so the integral equals the integral of $\psi_j(w)$ over the conjugate hyperplane $\overline{\mathcal{D}}$. On the other hand, under our isomorphism $(\mathbb{CP}^n)^*\simeq\mathbb{CP}^n$ we have that $\overline{\mathcal{D}}$ corresponds to $\overline{D}$. Let us make this precise. The divisor $\overline{\mathcal{D}}$ is given by $\overline{\mathcal{D}}=\{\overline{v}=(\overline{\mathcal{V}}_0,\ldots,\overline{\mathcal{V}}_n)\mid P_D(\overline{v})=0\}$. Now $P_D(\overline{v})=D\cdot{}^t\overline{v}$ so $P_D(\overline{v})=0$ is equivalent to $D\cdot{}^t\overline{v}=0=\overline{D}\cdot{}^tv$. So under the isomorphism $(\mathbb{CP}^n)^*\simeq\mathbb{CP}^n$ we have that $\overline{\mathcal{D}}$ corresponds to $\overline{D}$.

For fixed and positive t, we can use equation (20), as applied to the hyperplane $\overline{\mathcal{D}}$, when combined with equation (23) to get the formula

$$\begin{aligned} \int_{\mathcal{D}}\sum_{\lambda_j>0}\psi_j(z)\overline{\psi_j(w)}e^{-\lambda_j t}\mu_{\mathcal{D}}(w) &= \sum_{j=1}^{\infty}\frac{\pi^{n-1}(-1)^j j!}{(j+n-1)!}\sum_{\lambda_j=4j(n+j)}\psi_j(z)\psi_j(\overline{\mathcal{D}})e^{-\lambda_j t} \\ &= \sum_{j=1}^{\infty}\frac{\pi^{n-1}(-1)^j j!}{(j+n-1)!}\sum_{\lambda_j=4j(n+j)}\psi_j(z)\psi_j(\overline{D})e^{-\lambda_j t} \\ &= \sum_{j=1}^{\infty}\frac{\pi^{n-1}(-1)^j j!}{(j+n-1)!}\sum_{\lambda_j=4j(n+j)}\psi_j(z)\overline{\psi_j(D)}e^{-\lambda_j t} \\ &= \sum_{j=1}^{\infty}\frac{\pi^{n-1}(-1)^j j!}{(j+n-1)!}\frac{(2j+n)}{\pi^n}\frac{(j+n-1)!}{j!}P_j^{(n-1,0)}(\cos(2d_{\mathrm{FS}}(z,D)))e^{-\lambda_j t} \\ &= \frac{1}{\pi}\sum_{j=1}^{\infty}(2j+n)(-1)^jP_j^{(n-1,0)}(\cos(2d_{\mathrm{FS}}(z,D)))e^{-4j(j+n)t}. \end{aligned} \tag{30}$$

As described above, the eigenfunctions are appropriately scaled so that $\psi_j(\overline{D})=\overline{\psi_j(D)}$. Now, multiply (30) by e^{-s^2t} and then integrate with respect to t for $t>0$. After interchanging the sum and the integral over t, we deduce that

$$\int_0^\infty \int_{\mathcal{D}} \left(\sum_{\lambda_j > 0} \psi_j(z)\overline{\psi_j(w)} e^{-\lambda_j t} \right) \mu_{\mathcal{D}}(w) e^{-s^2 t} dt = \frac{1}{\pi} \sum_{j=1}^{\infty} \frac{(2j+n)(-1)^j}{s^2 + 4j(j+n)} P_j^{(n-1,0)}(\cos(2d_{\mathrm{FS}}(z,D))).$$

The absolute convergence of each series in the above discussion is confirmed using the bound (17). With all this, the proof of (29) is complete. □

By combining Proposition 1 and Proposition 2 we arrive at the identity

$$\frac{1}{\pi} \sum_{j=1}^{\infty} \frac{(2j+n)(-1)^j}{s^2+4j(j+n)} P_j^{(n-1,0)}(\cos(2d_{\mathrm{FS}}(z,D))) = -\frac{1}{4\pi} \log |P_D(z)|^2 + \frac{1}{4\pi}\left(\log ||D||^2 + \rho(z) - H_n\right) + O(s) \quad \text{as } s \to 0 \text{ with } \mathrm{Re}(s^2) > 0.$$

The series of the left-hand side of the above equation is a holomorphic function of s in the region $\mathrm{Re}(s^2) > 0$ which can be analytically continued to $s = 0$, since the integral on the left-hand side of the equation (29) is analytic at $s = 0$. The uniqueness of Taylor series representation of analytic function implies the identity

$$\log |P_D(z)|^2 = \log ||D||^2 + \rho(z) - H_n - \sum_{j=1}^{\infty} \frac{(2j+n)(-1)^j}{j(j+n)} P_j^{(n-1,0)}(\cos(2d_{\mathrm{FS}}(z,D))). \quad (31)$$

The bound (17) immediately implies that the series in (31) is uniformly and absolutely convergent for $w \in \mathcal{D}$ and z in compact subsets of $\mathbb{CP}^n \setminus (\mathcal{D} \cup \{D\})$.

The next proposition considers the sum of Jacobi polynomials.

Proposition 3 *With the notation as above, for $r = d_{FS}(z,D) \neq 0$ and any positive integer ℓ, one has that*

$$\begin{aligned}\sum_{j=1}^{\infty} \frac{(2j+\ell)(-1)^j}{j(j+\ell)} P_j^{(\ell-1,0)}(\cos(2r)) + H_\ell &= -\frac{d}{d\nu} P_\nu(\cos(2r))\Big|_{\nu=0} \\ &= -\frac{d}{d\nu} F(-\nu, \nu+1; 1; 1-\cos^2 r)\Big|_{\nu=0} \\ &= -\int_0^\infty \left(\pi Y_1(x) + \frac{2}{x} J_0(x)\right) J_0(x \sin r) dx,\end{aligned}$$

where $P_\nu(x)$ is the Legendre function, $F(-\nu, \nu+1; 1; 1-\cos^2 r)$ is the classical hypergeometric function and Y_1 and J_0 are the Bessel functions.

Proof: By starting with formula 8.961.8 from [GR07] we see that for any $\ell \geq 2$ and $x \in [-1,1)$

$$\begin{aligned}\sum_{j=1}^{\infty} (-1)^j \frac{2j+\ell}{j(j+\ell)} P_j^{(\ell-1,0)}(x) &= \sum_{j=1}^{\infty} \frac{(-1)^j}{j} P_j^{(\ell,0)}(x) - \sum_{j=1}^{\infty} \frac{(-1)^j}{(j+\ell)} P_{j-1}^{(\ell,0)}(x) \\ &= \sum_{j=1}^{\infty} (-1)^j \frac{2j+\ell+1}{j(j+\ell+1)} P_j^{(\ell,0)}(x) + \frac{1}{\ell+1}.\end{aligned}$$

By replacing ℓ with $\ell-1$, we get that

$$\sum_{j=1}^{\infty}(-1)^j\frac{2j+\ell}{j(j+\ell)}P_j^{(\ell-1,0)}(x)=\sum_{j=1}^{\infty}(-1)^j\frac{2j+\ell-1}{j(j+\ell-1)}P_j^{(\ell-2,0)}(x)-\frac{1}{\ell}. \tag{32}$$

Let us now repeat this process $\ell-1$ times, after which we get for any positive integer ℓ and $r\neq 0$ the formula

$$\sum_{j=1}^{\infty}(-1)^j\frac{2j+\ell}{j(j+\ell)}P_j^{(\ell-1,0)}(\cos(2r))+H_\ell=\sum_{j=1}^{\infty}(-1)^j\frac{2j+1}{j(j+1)}P_j(\cos(2r))+1, \tag{33}$$

where P_n denotes the Legendre polynomial.

From formula 8.793 of [GR07] we have, for any $\nu\notin\mathbb{Z}$ and $x\in(-1,1]$, the expression

$$\sum_{k=0}^{\infty}(-1)^k\left(\frac{1}{\nu-k}-\frac{1}{k+\nu+1}\right)P_k(x)=\frac{\pi P_\nu(x)}{\sin(\nu x)}. \tag{34}$$

From (34), we can write

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{\nu-k}-\frac{1}{k+\nu+1}\right)P_k(x)=\frac{\pi P_\nu(x)}{\sin(\nu x)}-\left(\frac{1}{\nu}-\frac{1}{\nu+1}\right).$$

By letting ν approach zero, one obtains

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{-k}-\frac{1}{k+1}\right)P_k(x)=\frac{d}{d\nu}P_\nu(x)\Big|_{\nu=0}+1,$$

or

$$\sum_{k=1}^{\infty}(-1)^k\left(\frac{1}{k}+\frac{1}{k+1}\right)P_k(x)=-1-\frac{d}{d\nu}P_\nu(x)\Big|_{\nu=0}.$$

By letting $x=\cos(2r)$, where $r=d_{\mathrm{FS}}(z,D)$, and combining the last equation with (33) we arrive at the identity

$$\sum_{j=1}^{\infty}(-1)^j\frac{2j+\ell}{j(j+\ell)}P_j^{(\ell-1,0)}(\cos(2r))+H_\ell=-\frac{d}{d\nu}P_\nu(\cos(2r))\Big|_{\nu=0},$$

which holds for any $\ell\geq 1$ and $r\neq 0$. With this, we have proved the first equality which was claimed in the statement of the Proposition.

For convenience, let us recall that $r=\mathrm{d_{FS}}(z,D)$ and $\cos^2(r)=\frac{1}{2}(\cos(2r)+1)$. By applying formula 8.751.1 with $m=0$ from [GR07] we obtain that

$$P_\nu(\cos(2r))=P_\nu(2\cos^2 r-1)=F(-\nu,\nu+1;1;1-\cos^2 r).$$

Further, if we employ formula 6.512.1 from [GR07] (where, in their notation, we take $\nu=0$, μ to be equal to (our) $2\nu+1$, $a=1$ and $b=\sin r$) we get that

$$F(-\nu,\nu+1;1;1-\cos^2 r)=F(-\nu,\nu+1;1;\sin^2 r)=\int_0^\infty J_{2\nu+1}(x)J_0(x\sin r)dx.$$

Finally, upon differentiating with respect to ν and applying the formula 8.486(1), part 6, of [GR07] with $n=1$, the proof is complete. □

The following corollary summarizes different representations of the log-norm of a linear polynomial which are obtained by combining Proposition 3 with (31).

Corollary 2 *Assuming the notation as above and $z \notin \mathcal{D} \cup \{D\}$, one has that*

$$2\log|P_D(z)| = \rho(z) + \log||D||^2 + \frac{d}{d\nu}F(-\nu, \nu+1; 1; 1-\cos^2 r)\Big|_{\nu=0}. \tag{35}$$

Additionally, for any integer $\ell \geq 1$, one has that

$$2\log|P_D(z)| = \rho(z) + \log||D||^2 - H_\ell - \sum_{j=1}^{\infty} \frac{(2j+\ell)(-1)^j}{j(j+\ell)} P_j^{(\ell-1,0)}(\cos(2r)). \tag{36}$$

Finally, one also has that

$$2\log|P_D(z)| = \rho(z) + \log||D||^2 + \int_0^\infty \left(\pi Y_1(x) + \frac{2}{x}J_0(x)\right) J_0(x \sin r)dx. \tag{37}$$

The identities (35) and (36) will serve as a starting point for the computation of the equivalent expressions for the Mahler measure of the linear polynomial $P_D(z)$. Actually, we will not use equation (37) and present the formula only for possible future interest.

5 A change of variables formula

Let S denote the domain of integration in (4), meaning that S is the subset in the affine chart $\mathcal{Z}_0 \neq 0$ of $\mathbb{CP}^n$ consisting of n-tuples of affine coordinates $(z_1, ..., z_n)$ such that

$$(z_1, ..., z_n) = (e^{i\theta_1}, \ldots, e^{i\theta_n}) \quad \text{with} \quad (\theta_1, \ldots, \theta_n) \in [0, 2\pi]^n.$$

We assume that S is equipped with the measure

$$\mu_S(z) = \frac{1}{(2\pi i)^n}\frac{dz_1}{z_1}\cdots\frac{dz_n}{z_n} = \frac{1}{(2\pi)^n}d\theta_1\cdots d\theta_n.$$

The following discussion is based on the material from pages 419-422 of [Wa66] which is summarized here for the convenience of the reader.

Let $h \in L^1([0,1])$, meaning $h(x)$ is an absolutely integrable function for $x \in [0,1]$. We will view x as a function of $z, D \in \mathbb{CP}^n$ by

$$x = x(z, D) := (\cos(d_{\mathrm{FS}}(z, D)))^2 = \cos^2 r,$$

in the notation of the previous section. Consider the integral

$$I(D; h) = \int_S h(x(z, D))\mu_S(z).$$

Let us write $D = (\mathcal{W}_0, \ldots, \mathcal{W}_n) = (r_0 e^{i\varphi_0}, \ldots, r_n e^{i\varphi_n})$. Set $\mathcal{Z}_0 = 1$ and $\mathcal{Z}_m = e^{i\theta_m}$ for each integer m from 1 to n to be a choice of coordinates for $z \in S$. With this, set

$$\mathcal{X} := \sum_{m=0}^{n} \mathcal{Z}_m \overline{\mathcal{W}}_m = \sum_{m=0}^{n} r_m e^{i(\theta_m - \varphi_m)}.$$

Following the discussion on pages 419-422 of [Wa66], we can view $\mathcal{X}$ as the endpoint of an $(n+1)-$step random walk in two dimensions. Step number m is of length r_m, and the walk occurs in the direction with angle $(\theta_m - \varphi_m) \in [-\pi, \pi]$. The directions are viewed as independent and identically distributed random variables, and the probability distribution of each is uniform on the interval $[-\pi, \pi]$. Let

$$d(D) = |\mathcal{W}_0| + \cdots + |\mathcal{W}_n|$$

be the L^1 norm of D. Let $\mathcal{Y}$ be the random variable which is the distance of $\mathcal{X}$ to the origin. Observe that for $z \in S$ we can write x as

$$x = (\cos(d_{\mathrm{FS}}(z, D)))^2 = \frac{1}{c(D)^2} \left| \sum_{m=0}^{n} \mathcal{Z}_m \overline{\mathcal{W}}_m \right|^2 = \frac{\mathcal{Y}^2}{c(D)^2}$$

where $c(D)^2 = (n+1)\|D\|^2$. It is proved on page 420 of [Wa66] that for any $u \in [0, d(D)]$ the cumulative distribution $F_D(u)$ of $\mathcal{Y}$ is given by

$$\mathrm{Prob}(\mathcal{Y} \leq u) = F_D(u) = u \int_0^\infty J_1(ut) \prod_{m=0}^{n} J_0(r_m t) dt.$$

Of course, $F_D(u) = 0$ for $u < 0$ and $F_D(u) = 1$ for $u > d(D)$, and J_0 and J_1 are the classical J-Bessel functions. The probability density function $f_D(u)$ of $\mathcal{Y}$ is obtained by differentiating $F_D(u)$ with respect to u. Using formula 8.472.1 of [GR07], we deduce that for $u \in [0, d(D)]$ the function $f_D(u)$ is given by

$$f_D(u) = \int_0^\infty ut J_0(ut) \prod_{m=0}^{n} J_0(r_m t) dt;$$

also, $f_D(u)$ is equal to zero for $u \notin [0, d(D)]$. When $r_m = 1$ for all m, the above formula is a classical result of Kluyver [Kl05]; see also formula (2.1) of [BSWZ12].

With all this, we can re-write the integral $I(D; h)$ as

$$\begin{aligned} I(D; h) = \int_S h(x(z, D)) \mu_S(z) &= \int_0^{d(D)} h\left(u^2/c(D)^2\right) f_D(u) du \\ &= \int_0^{d(D)} h\left(u^2/c(D)^2\right) \left(\int_0^\infty ut J_0(ut) \prod_{m=0}^{n} J_0(r_m t) dt \right) du. \end{aligned} \tag{38}$$

Finally, if we let $v = u/c(D)$, we arrive at a general change of variables formula, namely that

$$\int_S h(x(z, D)) \mu_S(z) = c(D)^2 \int_0^{d(D)/c(D)} h(v^2) \left(\int_0^\infty vt J_0(c(D)vt) \prod_{m=0}^{n} J_0(r_m t) dt \right) dv. \tag{39}$$

Recall that the Cauchy-Schwarz inequality implies that $d(D) \leq c(D)$, so then the above stated assumption on the function h indeed is sufficient for the above identity to hold. Furthermore, only in the case when D is a multiple of $(1, \cdots, 1)$ do we have $d(D) = c(D)$, otherwise $d(D) < c(D)$. This well-known aspect of the Cauchy-Schwarz inequality will be important when we apply (39) to prove our results.

6 Proof of the main results

6.1 Proof of Theorem 1

We begin with equation (35) and integrate along S with respect to the measure $\mu_S(z)$. Recall that we derived (35) under the condition that $z \notin \mathcal{D} \cup \{D\}$ where $\mathcal{D}$ is the divisor of the polynomial P_D on $\mathbb{CP}^n$. The left-hand-side of the resulting formula is $2m(P_D)$, so it remains to compute the integral of the right-hand side. Recall that $r = d_{\mathrm{FS}}(z, D)$, in the notation of Section 4 and $x = x(z, D) = \cos^2 r$, in the notation of the previous section.

When $(\mathcal{D} \cup \{D\}) \cap S = \emptyset$, there is an $\epsilon > 0$ such that for all $z \in S$ we have the bound $d_{\mathrm{FS}}(z, D) \geq \epsilon > 0$. Hence, the series which defines the hypergeometric function $F(-\nu, \nu + 1; 1; 1 - \cos^2 r)$ converges absolutely and uniformly in ν since f is positive and bounded away from 0. Therefore, we may differentiate the series expansion for $F(-\nu, \nu + 1; 1; 1 - \cos^2 r)$ term by term to get that

$$\frac{d}{d\nu} F(-\nu, \nu + 1; 1; 1 - \cos^2 r)\Big|_{\nu=0} = -\sum_{j=1}^{\infty} \frac{1}{j}(1 - \cos^2 r)^j. \tag{40}$$

Each term in the series (40) is non-negative, so the monotone convergence theorem applies to give that

$$2m(P_D) = 2\log c(D) - \int\limits_S \left(\sum_{j=1}^{\infty} \frac{1}{j}(1 - \cos^2 r)^j \right) \mu_S(z). \tag{41}$$

Suppose that $(\mathcal{D} \cup \{D\}) \cap S \neq \emptyset$. Choose an $\epsilon > 0$ and set

$$S_\epsilon := \{z \in S : d_{\mathrm{FS}}(z, w) \geq \epsilon \text{ for all } w \in \mathcal{D} \cup \{D\}\}.$$

By preceding as above, we get arrive at the formula that

$$2m(P_D; S_\epsilon) = 2\frac{\mathrm{vol}(S_\varepsilon)}{(2\pi)^n} \log c(D) - \int\limits_{S_\epsilon} \left(\sum_{j=1}^{\infty} \frac{1}{j}(1 - \cos^2 r)^j \right) \mu_S(z). \tag{42}$$

where

$$m(P_D; S_\epsilon) := \frac{1}{(2\pi)^n} \int\limits_{S_\epsilon} \log \left| P_D(1, e^{i\theta_1}, e^{i\theta_2}, \ldots, e^{i\theta_n}) \right| d\theta_1\, d\theta_2 \cdots d\theta_n$$

If $(\mathcal{D} \cup \{D\}) \cap S \neq \emptyset$, then $(\mathcal{D} \cup \{D\}) \cap S$ has μ_S measure zero, as also noted on page 270 of [De97]. Hence, the function $\log |P_D|$ lies in $L^1(\mu_S)$. Therefore, by letting ϵ approach zero, we have, by the monotone convergence theorem, that (42) becomes (41). In other words, in both cases when $(\mathcal{D} \cup \{D\}) \cap S = \emptyset$ and when $(\mathcal{D} \cup \{D\}) \cap S \neq \emptyset$, we arrive at (41), which we now study.

The series on the right-hand side of (41) is a series of non-negative functions; hence, we can interchange the sum and the integral to get, for any integer $N \geq 1$

$$2m(P_D) - 2\log c(D) + \sum_{j=1}^{N} \frac{1}{j} \int\limits_S (1 - \cos^2 r)^j \mu_S(z) = -\sum_{j=N+1}^{\infty} \frac{1}{j} \int\limits_S (1 - \cos^2 r)^j \mu_S(z). \tag{43}$$

In order to complete the proof of Theorem 1 we will evaluate the integral over S and show that the right-hand side of (43) is dominated by $2\Gamma(3/4)G(n, D)/(3N^{3/4})$ for a certain constant

$G(n,D)$ which depends solely on n and D. We will now do so, and the formula for $G(n,D)$ is given in (47) below.

As before, D is identified with $(w_1,...,w_n)=(\mathcal{W}_0,...,\mathcal{W}_n)\in\mathbb{CP}^n$ (recall that the affine chart is chosen so that $\mathcal{W}_0\neq 0$). For $z\in S$, we have that

$$x^k(z,D)=(\cos r)^{2k}=\frac{\left|1+\sum_{\ell=1}^n \overline{w}_\ell e^{i\theta_\ell}\right|^{2k}}{(1+n)^k\left(1+\sum_{\ell=1}^n|w_\ell|^2\right)^k},$$

so then

$$\frac{1}{(2\pi)^n}\int_0^{2\pi}\cdots\int_0^{2\pi}x^k(z,D)d\theta_1\cdots d\theta_n=\frac{a_1(n,k,D)}{(1+n)^k\left(1+\sum_{\ell=1}^n|w_\ell|^2\right)^k},\tag{44}$$

where $a_1(n,k,D)$ denotes the constant term in the expression

$$x^k(z,D)=\left|1+\sum_{\ell=1}^n\overline{w}_\ell e^{i\theta_\ell}\right|^{2k}=\left(1+\sum_{\ell=1}^n w_\ell e^{-i\theta_\ell}\right)^k\left(1+\sum_{\ell=1}^n\overline{w}_\ell e^{i\theta_\ell}\right)^k.$$

The multinomial theorem implies that

$$a_1(n,k,D)=\sum_{\ell_0+\ldots+\ell_n=k,\ell_m\geq 0,m=1,\ldots n}\binom{k}{\ell_0,\ell_1,\ldots,\ell_n}^2|w_1|^{2\ell_1}\cdots|w_n|^{2\ell_n}=\frac{a(n,k,D)}{|\mathcal{W}_0|^{2k}}.$$

Therefore,

$$\int_S(1-x(z,D))^j\mu_S(z)=\sum_{k=0}^j\binom{j}{k}(-1)^k\frac{a(n,k,D)}{c(D)^{2k}}.$$

Inserting this into (43) we get

$$\left|2m(P_D)-2\log c(D)+\sum_{j=1}^N\frac{1}{j}\sum_{k=0}^j\binom{j}{k}\frac{(-1)^k a(n,k,D)}{c(D)^{2k}}\right|\leq\sum_{j=N+1}^\infty\frac{1}{j}\int_S(1-x(z,D))^j\mu_S(z).\tag{45}$$

To complete the proof of the theorem, it remains to deduce a uniform bound for the series in (45).

Let us apply the change of variables formula (39) and write, for any $j\geq 1$,

$$\int_S(1-x(z,D))^j\mu_S(z)=c(D)^2\int_0^{d(D)/c(D)}(1-v^2)^j\left(\int_0^\infty vtJ_0(c(D)vt)\prod_{m=0}^n J_0(r_mt)dt\right)dv.\tag{46}$$

Since $d(D)/c(D)\leq 1$, we have that $v\leq 1$, hence $\max\{1,\frac{\pi}{2}c(D)vt\}\geq v\max\{1,\frac{\pi}{2}c(D)t\}$. By using the bound (18), we get that

$$|J_0(c(D)vt)|\leq v^{-1/2}\max\{1,\frac{\pi}{2}c(D)t\}^{-1/2}.$$

Therefore,

$$\int_0^{d(D)/c(D)} (1-v^2)^j v|J_0(c(D)vt)|dv \le \left(\int_0^1 (1-v^2)^j v^{1/2} dv\right) \max\{1, \frac{\pi}{2}c(D)t\}^{-1/2}$$
$$= \frac{1}{2}\left(\int_0^1 (1-u)^j u^{-1/4} du\right) \max\{1, \frac{\pi}{2}c(D)t\}^{-1/2}$$
$$= \frac{\Gamma(j+1)\Gamma(3/4)}{2\Gamma(j+1+3/4)} \max\{1, \frac{\pi}{2}c(D)t\}^{-1/2}$$
$$\le \frac{\Gamma(3/4)}{2j^{3/4}} \max\{1, \frac{\pi}{2}c(D)t\}^{-1/2},$$

where we have applied [GR07], formula 3.196.3 with $a = 0$, $b = 1$, $\mu = 3/4$ and $\nu = j+1$ in order to evaluate the integral with respect to u. Trivially, $\sum_{j=N+1}^{\infty} j^{-7/4} \le \frac{4}{3}N^{-3/4}$. Hence, after multiplying by $1/j$ and summing over $j \ge N+1$ in (46), we arrive at the bound

$$\sum_{j=N+1}^{\infty} \frac{1}{j} \int_S (1-x(z,D))^j \mu_S(z) \le \frac{2\Gamma(3/4)}{3} \frac{G(n,D)}{N^{3/4}}$$

where

$$G(n,D) = c(D)^2 \int_0^{\infty} t \left(\max\{1, \frac{\pi}{2}c(D)t\}\right)^{-\frac{1}{2}} \prod_{m=0}^{n} |J_0(r_m t)| dt. \tag{47}$$

By combining with (45) we obtain the inequality

$$|2m(P_D) - 2E_1(N;n,D)| \le \frac{2\Gamma(3/4)}{3} \frac{G(n,D)}{N^{3/4}},$$

where $E_1(N;n,D)$ is defined by (13). We have now completed the proof of the first inequality in (12). When dividing by 2 and letting $N \to \infty$, we also have completed the proof of Theorem 1.

□

As a concluding comment, let us point out a further refinement which will yield an elementary bound for $G(n,D)$. By using (18), we arrive at the inequality

$$G(n,D) \le c(D)^2 \int_0^{\infty} t \left(\max\{1, \frac{\pi}{2}c(D)t\}\right)^{-\frac{1}{2}} \prod_{m=0}^{n} \left(\max\{1, \frac{\pi}{2}r_m t\}\right)^{-\frac{1}{2}} dt. \tag{48}$$

Without loss of generality, assume that $r_0 \le \ldots \le r_n$, so then $r_n \le c(D)$. If $n \ge 3$, the integral is convergent since the integral is $O(t^{-n/2})$ for $t > 2/(\pi r_0)$. In order to evaluate (48), we can write the domain of integration as

$$\int_0^{\infty} = \int_0^{2/(\pi c(D))} + \int_{2/(\pi c(D))}^{2/(\pi r_n)} + \cdots + \int_{2/(\pi r_{j+1})}^{2/(\pi r_j)} + \cdots + \int_{2/(\pi r_0)}^{\infty} .$$

All integrals are elementary, so then the evaluation of each integral will yield an explicit and elementary upper bound for $G(n,D)$. In the case when not all r_j's are distinct, some of these integrals are zero, and the exponents for each integrand need to take into account the multiplicity of r_j in set of components of $\mathcal{D}$. The computations are straightforward.

6.2 Proof of Theorem 2 when $\ell = 1$

Choose any integer $N \geq 1$. When $\ell = 1$ we can write (36) for $z \notin (\mathcal{D} \cup \{D\})$ as

$$2\log|P_D(z)| = \rho(z) + \log||D||^2 - 1 - \sum_{j=1}^{N}(-1)^j \frac{2j+1}{j(j+1)}P_j(\cos(2r)) - \sum_{j=N+1}^{\infty}(-1)^j \frac{2j+1}{j(j+1)}P_j(\cos(2r)), \tag{49}$$

where, as before $r = d_{\mathrm{FS}}(z, D)$. We now will utilize the following three points: The identity $\cos(2r) = 2\cos^2 r - 1$; formula 8.962.1 from [GR07] for the Jacobi polynomial with $\alpha = \beta = 0$; and that hypergeometric function $F(j+\ell, -j; 1; \cos^2 r)$ at these values as a finite sum. In doing so, we arrive at the formula

$$P_j^{(\ell-1,0)}(2\cos^2 r - 1) = (-1)^j \sum_{k=0}^{j} \binom{j+\ell+k-1}{k}\binom{j}{k}(-1)^k(\cos r)^{2k},$$

which holds for all $z \in S$ when $(\mathcal{D} \cup \{D\}) \cap S = \emptyset$. For now, we will assume $(\mathcal{D} \cup \{D\}) \cap S = \emptyset$. By integrating this equation with respect to S and employing (44) we get

$$(-1)^j \int_S P_j^{(\ell-1,0)}(\cos(2r))\mu_S(z) = \sum_{k=0}^{j} \binom{j+\ell+k-1}{k}\binom{j}{k}(-1)^k \frac{a(n,k,D)}{c(D)^{2k}}. \tag{50}$$

By integrating (49) with respect to S and applying (50) with $\ell = 1$ we arrive at

$$|m(P_D) - E_2(N; n, D)| \leq \frac{1}{2}\sum_{j=N+1}^{\infty} \frac{2j+1}{j(j+1)} \int_S |P_j(\cos(2d_{\mathrm{FS}}(z,D)))|\, \mu_S(z), \tag{51}$$

where $E_2(N; n, D)$ is defined by (14).

Before studying the right-hand-side of (51), let us address the setting when $(\mathcal{D}\cup\{D\})\cap S \neq \emptyset$. Indeed, the extension of (51) to the case when $(\mathcal{D} \cup \{D\}) \cap S \neq \emptyset$ follows the method of proof of (41) in that case. By integrating over S_ϵ rather than all of S, one gets an analogue of (51) for $\epsilon > 0$. At that point, one lets ϵ approach zero. The function $\log|P_D|$ and each Jacobi polynomial $P_j^{(\ell-1,0)}$ is in $L^1(S)$, so one obtains the left-hand-side of (51) as ϵ approaches zero. As for the right-hand-side of (51), one uses the monotone convergence theorem, as in the proof of (43). With this, one shows that (51) also holds when $(\mathcal{D} \cup \{D\}) \cap S \neq \emptyset$.

It is left to study the series on the right-hand side of (51). We will use inequality (16) with $\cos(2r) = \cos(2d_{\mathrm{FS}}(z, D))$ instead of x. Recall the notation $x(z, D) = \cos^2(d_{\mathrm{FS}}(z, D))$, so then the inequality (16) gives that

$$\int_S |P_j(\cos(2d_{\mathrm{FS}}(z,D)))|\, \mu_S(z) \leq \frac{2}{\sqrt{\pi}\sqrt[4]{2}}\frac{1}{\sqrt{2j+1}} \int_S (x(z,D)(1-x(z,D)))^{-1/4}\, \mu_S(z). \tag{52}$$

Therefore, the right-hand-side of (51) can be bounded from above by

$$\frac{1}{\sqrt{\pi}\sqrt[4]{2}}\left(\sum_{j=N+1}^{\infty} \frac{\sqrt{2j+1}}{j(j+1)}\right) \int_S (x(z,D)(1-x(z,D)))^{-1/4}\, \mu_S(z).$$

The goal is to make all bounds effective and explicit, so we shall. Trivially, we have that

$$\sum_{j=N+1}^{\infty} \frac{\sqrt{2j+1}}{j(j+1)} \leq \sum_{j=N+1}^{\infty} \frac{2j^{-1/2}}{j^2} \leq \frac{4}{\sqrt{N}},$$

which, when combined with (51), yields the inequality

$$|m(P_D) - E_2(N; n, D)| \leq \frac{C}{\sqrt{N}} \int\limits_S (x(z,D)(1-x(z,D)))^{-1/4} \mu_S(z) \text{ for } C = \frac{4}{\sqrt{\pi}\sqrt[4]{2}}. \qquad (53)$$

The integral in (53) can be re-written using the change of variables formula (39). In doing so, it becomes

$$H(n,D) = c(D)^2 \int\limits_0^{d(D)/c(D)} v^{-1/2}(1-v^2)^{-1/4} \left(\int\limits_0^{\infty} vtJ_0(c(D)vt) \prod_{m=0}^{n} J_0(r_m t) dt \right) dv. \qquad (54)$$

The Fubini-Tonelli theorem then implies that

$$H(n,D) \leq c(D)^2 \int\limits_0^{\infty} t \prod_{m=0}^{n} |J_0(r_m t)| \cdot \left(\int\limits_0^1 v^{\frac{1}{2}} \left(1-v^2\right)^{-\frac{1}{4}} |J_0(c(D)tv)| dv \right) dt. \qquad (55)$$

The Cauchy-Schwarz inequality together with the elementary inequality (18) for the J-Bessel function gives the inequality

$$\int\limits_0^1 v^{\frac{1}{2}} \left(1-v^2\right)^{-\frac{1}{4}} |J_0(c(D)tv)| dv \leq \sqrt{\frac{\pi}{2}} \left(\int\limits_0^1 vJ_0^2(c(D)tu) dv \right)^{\frac{1}{2}} \leq \sqrt{\frac{\pi}{2}} \left(\max\{1, \frac{\pi}{2}c(D)t\} \right)^{-\frac{1}{2}}.$$

Finally, by substituting this inequality into (55), we arrive at the bound

$$H(n,D) \leq c(D)^2 \sqrt{\frac{\pi}{2}} \int\limits_0^{\infty} t \left(\max\{1, \frac{\pi}{2}c(D)t\} \right)^{-\frac{1}{2}} \prod_{m=0}^{n} |J_0(r_m t)| dt = \sqrt{\frac{\pi}{2}} G(n,D).$$

Therefore,

$$|m(P_D) - E_2(N; n, D)| \leq \frac{2\sqrt[4]{2}}{\sqrt{N}} G(n,D),$$

which proves the second inequality in (12). Formula (8) follows by letting N tend to infinity. □

6.3 Proof of equation (10)

Equation (10) follows by a direct manipulation of the inner sum appearing in (8). To ease the notation, we will set $b(n,k,D) = (-1)^k a(n,k,D)/c(D)^{2k}$. Then, we can write

$$\sum_{j=1}^{\infty} \frac{2j+1}{j(j+1)} \sum_{k=0}^{j} \binom{j+k}{k} \binom{j}{k} b(n,k,D) = 1 - \frac{2}{n+1}$$
$$+ \sum_{j=2}^{\infty} \frac{1}{j} \left(\sum_{k=0}^{j} \binom{j+k}{k} \binom{j}{k} b(n,k,D) + \sum_{k=0}^{j-1} \binom{j+k-1}{k} \binom{j-1}{k} b(n,k,D) \right).$$

It is elementary to prove that

$$\sum_{k=0}^{j}\binom{j+k}{k}\binom{j}{k}b(n,k,D)+\sum_{k=0}^{j-1}\binom{j+k-1}{k}\binom{j-1}{k}b(n,k,D) = 2\sum_{k=0}^{j}\binom{j+k-1}{k}\binom{j}{k}b(n,k,D),$$

which completes the proof of equation (10). □

6.4 Proof of Theorem 2 when $\ell \geq 2$

Assume $\ell \geq 2$ and $D \neq r(1,1,\ldots,1)$ for some $r \neq 0$. Proceeding as above, we integrate (36) along S with respect to $\mu_S(z)$ and employ (50) to arrive at

$$\left| m(P_D) - \log c(D) - \frac{H_\ell}{2} - \frac{1}{2}\sum_{j=1}^{N}\frac{2j+\ell}{j(j+\ell)}\sum_{k=0}^{j}\binom{j+\ell+k-1}{k}\binom{j}{k}\frac{(-1)^k a(n,k,D)}{c(D)^{2k}} \right| \leq \frac{1}{2}\sum_{j=N+1}^{\infty}\frac{2j+\ell}{j(j+\ell)}\int_S \left| P_j^{(\ell-1,0)}(\cos(2d_{\mathrm{FS}}(z,D)))\right| \mu_S(z). \quad (56)$$

In the notation as above, set

$$H(j;\ell,D) := \int_S \left|P_j^{(\ell-1,0)}(\cos(2d_{\mathrm{FS}}(z,D)))\right|\mu_S(z) = \int_S \left|P_j^{(\ell-1,0)}(2x(z,D)-1)\right|\mu_S(z). \quad (57)$$

Equation (39) applies, so then

$$H(j;\ell,D) = c(D)^2 \int_0^{d(D)/c(D)} \left|P_N^{(\ell-1,0)}(2v^2-1)\right| \left(\int_0^{\infty} vtJ_0(c(D)vt)\prod_{m=0}^{n}J_0(r_m t)dt\right)dv.$$

We now apply the bound in (17) with $x = 2v^2-1 \in [-1, 2(d(D)/c(D))^2-1] \subset [-1,1)$ to get

$$(1-v^2)^{1/4}v^{1/2}\left(1-v^2\right)^{(\ell-1)/2}|P_N^{(\ell-1,0)}(2v^2-1)| \leq \frac{6\sqrt{2}}{\sqrt{2j+\ell}}.$$

Recall that $d(D)$ is the L^1 norm of D, and $c(D)^2 = (n+1)\|D\|^2$. By the Cauchy-Schwarz inequality, $d(D)/c(D) \leq 1$ with equality if and only if $D = r(1,1,\cdots,1)$ for some non-zero r. Since we assume that $D \neq r(1,1,\cdots,1)$, it follows that $d(D)/c(D) < 1$, hence

$$|P_N^{(\ell-1,0)}(2v^2-1)| \leq \frac{A(D,\ell)}{\sqrt{2j+\ell}}v^{-1/2}(1-v^2)^{-1/4},$$

where

$$A(D,\ell) = 6\sqrt{2}\left(1-\frac{d(D)^2}{c(D)^2}\right)^{-(\ell-1)/2}.$$

Therefore

$$H(j;\ell,D) \leq \frac{c(D)^2A(D,\ell)}{\sqrt{2j+\ell}}\int_0^{\infty} t\prod_{m=0}^{n}|J_0(r_m t)| \cdot \left(\int_0^1 v^{\frac{1}{2}}\left(1-v^2\right)^{-\frac{1}{4}}|J_0(c(D)tv)|dv\right)dt$$

The integral on the right-hand side of the above equation is the same as the integral which appears in (55). So then the argument following (55) applies and gives the inequality

$$H(j;\ell,D) \le \sqrt{\frac{\pi}{2}}\frac{A(D,\ell)}{\sqrt{2j+\ell}}G(n,D).$$

Consequently, we have shown that

$$\sum_{j=N+1}^{\infty}\frac{2j+\ell}{j(j+\ell)}\int\limits_{S}\left|P_j^{(\ell-1,0)}(\cos(2\mathrm{dist}_{\mathrm{FS}}(z,D)))\right|\mu_S(z) \le \sqrt{\frac{\pi}{2}}A(D,\ell)G(n,D)\sum_{j=N+1}^{\infty}\frac{\sqrt{2j+\ell}}{j(j+\ell)}.$$

This proves that the right-hand side of (56) is bounded by $\sqrt{\pi}A(D,\ell)G(n,D)N^{-1/2}$.

With this, the proof of Theorem 2 when $\ell \ge 2$ follows upon letting $N \to \infty$ in (56). □

7 Concluding remarks

7.1 Proof of Corollary 1

Let us rewrite a result from [R-VTV04] in our notation. Specifically, equation (4.7) from [R-VTV04] becomes the formula that

$$2m(P_D) = \log ||D||^2 - \gamma - \sum_{m=1}^{\infty}\frac{1}{m}\sum_{\ell=0}^{m}\binom{m}{\ell}\frac{(-1)^\ell a(n,\ell,D)}{l!||D||^{2l}}. \tag{58}$$

Trivially,

$$\sum_{\ell=0}^{1}\binom{1}{\ell}\frac{(-1)^\ell a(n,\ell,D)}{l!||D||^{2l}} = 0,$$

from which (11) follows immediately by comparing (7) with (58). □

7.2 Additional formulas for Mahler measures

For any $n \ge 3$ choose any D and one of the formulas we have proved, say Theorem 1. For any integer $\tilde{n} > n$, let $\widetilde{D}$ be the vector of coefficients whose first n components is D and whose last $\tilde{n} - n$ coordinates are zero. The normalization in (1) is such that $m(P_D) = m(P_{\widetilde{D}})$. Also, for any k we have that $a(\tilde{n},k,\widetilde{D}) = a(n,k,D)$. However,

$$c(\widetilde{D})^2 = (\tilde{n}+1)\|\widetilde{D}\|^2 = (\tilde{n}+1)\|D\|^2 = \frac{\tilde{n}+1}{n+1}c(D)^2.$$

Let us set $m = \tilde{n} - n$. With this, the main formula in Theorem 1 becomes the statement that for any $m \ge 0$, one has that

$$m(P_D) = \log c(D) + \frac{1}{2}\log\left(\frac{n+m+1}{n+1}\right) - \frac{1}{2}\sum_{j=1}^{\infty}\frac{1}{j}\sum_{k=0}^{j}\binom{j}{k}\frac{(-1)^k a(n,k,D)(n+1)^{2k}}{c(D)^{2k}(n+m+1)^{2k}}. \tag{59}$$

Similar identities can be proved by the by the same means from Theorem 2.

Equation (59) with $m = 0$ and $m \geq 1$ yields the following curious combinatorial identity, similar to (11)

$$\log\left(\frac{n+m+1}{n+1}\right) = \sum_{j=1}^{\infty} \frac{1}{j} \sum_{k=0}^{j} \binom{j}{k} \frac{(-1)^k a(n,k,D)}{c(D)^{2k}} \left[\left(\frac{n+1}{n+m+1}\right)^{2k} - 1\right], \tag{60}$$

which holds true for any $m \geq 1$.

Note that all estimates we have derived for $G(n, D)$ grow for fixed D as n increases. As such, the above considerations do not seem to aid with convergence issues when applying our results to numerical estimations.

7.3 The excluded cases when $n = 2$ and $D = r(1, \cdots, 1)$

In the case $n = 2$, it may be possible to revisit our computations and obtain bounds. For example, rather than using (18), one could use the asymptotic formula

$$J_0(x) = \sqrt{\frac{2}{\pi x}} \cos(z - \pi/4) + O(x^{-3/2}) \quad \text{as } x \to \infty.$$

The oscillatory bounds may be such that one could derive a finite bound for $G(2, D)$ and possibly improved bounds for $G(n, D)$ for $n \geq 3$ as well. These considerations are undertaken in [AMS20].

The exclusion of the point $D = r(1, \cdots, 1)$ in Theorem 2 comes from the problem of deriving bounds for the $P_j^{(\ell-1,0)}(r)$ near $r = 1$; see (56). For this, one can seek to employ Hilb-type formulas; see page 197 of [Sz74], page 6 of [BZ07] or page 980 of [FW85]. We will leave these questions for future consideration.

7.4 Other choices for S

We shall now describe how the approach taken in this paper can be generalized. In doing so, we will be somewhat vague in our discussion.

For this section let S be a "nice" set in $\mathbb{CP}^n$ with a "nice" measure μ_S. One example is the product of circles in an affine chart of $\mathbb{CP}^n$ and μ_S is the translation invariant metric on each circle. Let us define

$$m_S(F_{\mathcal{D}}) = \int_S \log \|F_{\mathcal{D}}\|_\mu^2(z)\mu_S(z),$$

where $F_{\mathcal{D}}$ is a holomorphic form on $\mathbb{CP}^n$ with divisor $\mathcal{D}$. The invariant $m_{\mathbb{CP}^n}(F_{\mathcal{D}})$ is obtained by integrating with respect to the Fubini-Study metric. By using the spectral expansion of the Green's function and the proof of Proposition 1 we obtain a general formula, namely

$$4\pi \mathrm{vol}_\mu(\mathbb{CP}^n) \sum_{\lambda_j > 0} \frac{1}{\lambda_j} \left(\int_S \psi_j \mu_S\right) \left(\int_{\mathcal{D}} \overline{\psi}_j \mu_{\mathcal{D}}\right) = \mathrm{vol}_\mu(\mathbb{CP}^n) m_S(F_{\mathcal{D}}) - \mathrm{vol}_\mu(S) m_{\mathbb{CP}^n}(F_{\mathcal{D}}). \tag{61}$$

In the above calculations, we were able to express the series

$$\sum_{\lambda_j > 0} \frac{1}{\lambda_j} \left(\int_S \psi_j \mu_S\right) \left(\int_{\mathcal{D}} \overline{\psi}_j \mu_{\mathcal{D}}\right)$$

as a sum of integrals of Jacobi polynomials by using the Radon transform. When this method applies, one then obtains a series of Legendre or Jacobi polynomials at various arguments which are then integrated over S. It certainly seems plausible that our approach, as described in this manner, will apply in other settings than developed in the present article.

In particular, let us note that in [LM18] the authors considered the case when S is a product of circles with different radii. It seems as if the methodology developed in this article will apply to give analogues of Theorem 1, Theorem 2 and Theorem 3 in that setting.

7.5 Estimates for canonical heights

The contents of this section are based on comments from an anonymous referee; we gratefully acknowledge them for sharing their mathematical insight.

First, let us rephrase our study in the context of Arakelov theory. Following the work in [Ma00], the calculation of Mahler measures is manifest within the study of arithmetic intersection theory. As such, one is naturally led to determine suitable Green's currents associated to a divisor $\mathcal{D}$. There are two immediate possibilities. First, if P_D is a polynomial whose divisor is $\mathcal{D}$, then the function

$$g_{\mathcal{D}}(z) = -\log(\|P_D(z)\|^2_{\mathrm{FS}})$$

is one such Green's function. In this expression, we have used $z = (\mathcal{Z}_0, \ldots, \mathcal{Z}_n)$ and the subscript "FS" to denote the Fubini-Study metric. Second, from above, one also can consider the function

$$\tilde{g}_{\mathcal{D}}(z) = \int\limits_{\mathcal{D}} G_{\mathbb{CP}^n}(z, w)\mu_{\mathcal{D}}(w).$$

The difference $g_{\mathcal{D}}(z) - \tilde{g}_{\mathcal{D}}(z)$ admits a smooth extension across the divisor $\mathcal{D}$, from which one can prove that

$$\mathrm{d}_z\mathrm{d}^c_z\left(g_{\mathcal{D}}(z) - \tilde{g}_{\mathcal{D}}(z)\right) = 0.$$

Therefore, there is a constant $B(D, n)$, which depends on the coefficients D of P_D and the dimension n, such that

$$g_{\mathcal{D}}(z) - \tilde{g}_{\mathcal{D}}(z) = B(D, n).$$

At this point, we have arrived at the second displayed line in the proof of Proposition 1. Our subsequent analysis addresses the details of establishing normalizations and evaluation of the constant $B(D, n)$, as well as the study of $\tilde{g}_{\mathcal{D}}(z)$ which we undertake via analytic continuation through the generalization of Kronecker limit formula from [CJS20].

Second, we now have an opportunity to restate the contents of Theorem 3 as follows. The bounds in (12) provide estimates for the Mahler measure in terms of either $E_1(N; n, D)$ or $E_2(N; n, D)$. Furthermore, the constant $G(n, D)$ can be explicitly computed when, for example, one combines (47) and (18). As such, Theorem 3 provides a means by which one can effectively and efficiently estimate the canonical height which was computed on page 107 of [Ma00].

7.6 Reinterpreting Mahler measures

The results in the present paper follow from the Kronecker-type limit formula derived in [CJS20]. The setting of [CJS20] was that of a general Kähler manifold X and $\mathcal{D}$ is a divisor which is smooth up to codimension two. In this article we took X to be $\mathbb{CP}^n$ and $\mathcal{D}$ to be a hyperplane. From this initial point, we then delved into detailed computations and identities involving the Legendre polynomials, Jacobi polynomials and J-Bessel functions. However, the foray into special function theory was expected. After all, in many instances one knows that heat kernels can be expressed in terms of spherical functions, and the Green's function can be computed from the heat kernel; see page 436 of [CJS20] and references therein. Additionally, Jacobi polynomials and Jacobi functions are known to be present in such aspects of harmonic analysis; see, for example, [Ko84].

We find it quite interesting that (61) can be viewed in the setting of harmonic analysis, and possibly beyond.

References

[AMS20] Anton, G., Malathu, J. A., Stinson, S.: *On an approximation of a J-Bessel integral and its applications (with an appendix by J.S. Friedman)* arxiv.org 2012.04165.

[BZ07] Bao, X.-X., Zhao, Y.-Q.: *A uniform asymptotic expansion for Jacobi polynomials via uniform treatment of Darboux's method*, J. of Approx. Theory **148** (2007), 1–11.

[BGM71] Berger, M., Gauduchon P., Mazet E.: *Le spectre d'une variété Riemanniane*, Lecture Notes in Math. **194**, Springer-Verlag, Berlin and New York, 1971.

[BG06] Bombieri, E., Gubler, W.: *Heights in Diophantine geometry*, New Mathematical Monographs **4**, Cambridge University Press, Cambridge, 2006.

[BSWZ12] Borwein, J. M., Straub, A., Wan, J., Zudilin, W.: *Densities of short uniform random walks (with an Appendix by D. Zagier)*, Canad. J. Math. **64** (2012), 961–990.

[Bo81] Boyd, D.: *Speculations concerning the range of Mahler's measure*, Can. Math. Bull **24** (1981), 453–469.

[Bo98] Boyd, D.: *Mahler's measure and special values of $L-$functions*, Exp. Math. **7** (1998), 37–82.

[CJS20] Cogdell, J., Jorgenson, J., Smajlović, L.: *Spectral construction of non-holomorphic Eisenstein-type series and their Kronecker limit formula*, in: *Integrability systems and algebraic geometry,* London Math. Soc. Lecture Note Ser., **459,** Cambridge Univ. Press, Cambridge, 2020, 393–427.

[De97] Deninger, C.: *Deligne periods of mixed motives, K-theory and the entropy of certain $\mathbb{Z}^n$-actions*, J. Amer. Math. Soc. **10** (1997), 259—281.

[De09] Deninger, C.: *Mahler measures and Fuglede-Kadison determinants*, Münster J. Math. **2** (2009), 45—63.

[De12] Deninger, C.: *Regulators, entropy and infinite determinants*, in: *Proceedings of the Regulators III Conference,* Contemp. Math., **571** Amer. Math. Soc., Providence, RI, 2012, 117-134.

[Fo76] Folland, G.: *Introduction to Partial Differential Equations*, Princeton University Press, Princeton, NJ, 1976.

[FW85] Frenzen, C. L., Wong, R.: *A uniform asymptotic expansion of the Jacobi polynomials with error bounds*, Canad. J. Math. **37** (1985), 979–1007.

[GS90] Gillet, H., Soulé, C.: *Characteristic classes for algebraic vector bundles with hermitian metric, II*, Annals of Math. **131** (1990), 205–238.

[GR07] Gradshteyn, I. S., Ryzhik, I. M.: *Table of Integrals, Series and Products.* Elsevier Academic Press, Amsterdam, 2007.

[Gr83] Grinberg, E. L.: *Spherical harmonics and integral geometry on projective spaces*, Trans. AMS **279** No. 1 (1983), 187–203.

[HS14] Haagerup, H., Schlichtkrull, H.: *Inequalities for Jacobi polynomials*, Ramanujan J. **33** (2014), 227–246.

[HI02] Hafoud, A., Intissar, A.: *Représentation intégrale de noyau de la chaleur sur l'espace projectif complexe $P^n(C)$, $n \geq 1$.* C. R. Math. Acad. Sci. Paris **335** No. 11 (2002), 871–876.

[JK98] Jorgenson, J., Kramer, J.: *Towards the arithmetic degree of line bundles on abelian varieties*, Manuscripta Math. **96** (1998), 335–370.

[Kl05] Kluyver, J. C.: *A local probability problem*, in: *Royal Netherlands Academy of Arts and Sciences*, Proceedings, **8**, 1905, 341-350.

[Ko84] Koornwinder, T.: *Jacobi functions and analysis on noncompact semisimple Lie groups*, in: *Special functions: group theoretical aspects and applications,* Math. Appl., Reidel, Dordrecht, 1984, 1-85.

[Kr06] Krasikov, I.: *Uniform bounds for Bessel functions*, J. Appl. Anal. **12** (2006), 83-91.

[LM18] Lalín, M., Mittal, T. *The Mahler measure for arbitrary tori*, Res. Number Theory **4** (2018), no. 2, Paper No. 16, 23 pp.

[La88] Lang, S.: *Introduction to Arakelov Theory*, Springer, New York, 1988.

[Lo82/83] Lorch, L.: *Alternative proof of a sharpened form of Bernstein's inequality for Legendre polynomials*, Appl. Anal. **14** (1982/1983), 237-240 Corrigendum in Appl. Anal. **50**, (1993), 47.

[Lu98] Lu, Q.: *The eigen functions of the complex projective space*, Acta Math. Sinica N.S. **14** no 1. (1998), 1–8.

[Ma00] Maillot, V.: *Géométrie d'Arakelov des variétés toriques et fibrés en droites intégrables*, Mém. Soc. Math. Fr. (N.S.) No. **80** (2000), vi+129 pp.

[R-VTV04] Rodriguez-Villegas, F., Toledano, R., Vaaler, J. D. : *Estimates for Mahler's measure of a linear form*, Proc. Edinb. Math. Soc. **47** (2004), 473–494.

[Sm81] Smyth, C. J.: *On measures of polynomials in several variables*, Bull. Austral. Math. Soc. **23** (1981), 49–63.

[Sm08] Smyth, C. J.: *The Mahler measure of algebraic numbers: A survey*, in: *Number theory and polynomials,* London Math. Soc. Lecture Note Ser., **352,** Cambridge Univ. Press, Cambridge, 2008, 322–349.

[Sz74] Szegö, G.: *Orthogonal Polynomials*, American Mathematical Society, Providence, RI, 1975.

[Wa66] Watson, G. N.: *A Treatise on the Theory of Bessel Functions* (2nd.ed.), Cambridge University Press, Cambrige, 1966.

James W. Cogdell
Department of Mathematics
Ohio State University
231 W. 18th Ave.
Columbus, OH 43210
U.S.A.
e-mail: cogdell@math.ohio-state.edu

Jay Jorgenson
Department of Mathematics
The City College of New York
Convent Avenue at 138th Street
New York, NY 10031
U.S.A.
e-mail: jjorgenson@mindspring.com

Lejla Smajlović
Department of Mathematics
University of Sarajevo
Zmaja od Bosne 35, 71 000 Sarajevo
Bosnia and Herzegovina
e-mail: lejlas@pmf.unsa.ba