Kronecker limit functions and an extension of the Rohrlich-Jensen formula

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Abstract

In [Ro84] Rohrlich proved a modular analogue of Jensen's formula. Under certain conditions, the Rohrlich-Jensen formula expresses an integral of the log-norm $\log \|f\|$ of a $\mathrm{PSL}(2,\mathbb{Z})$ modular form f in terms of the Dedekind Delta function evaluated at the divisor of f. In [BK20] the authors re-interpreted the Rohrlich-Jensen formula as evaluating a regularized inner product of $\log \|f\|$ and extended the result to compute a regularized inner product of $\log \|f\|$ with what amounts to powers of the Hauptmoduli of $\mathrm{PSL}(2,\mathbb{Z})$. In the present article, we revisit the Rohrlich-Jensen formula and prove that, in the case of any Fuchsian group of the first kind with one cusp it can be viewed as a regularized inner product of special values of two Poincaré series, one of which is the Niebur-Poincaré series and the other is the resolvent kernel of the Laplacian. The regularized inner product can be seen as a type of Maass-Selberg relation. In this form, we develop a Rohrlich-Jensen formula associated to any Fuchsian group Γ of the first kind with one cusp by employing a type of Kronecker limit formula associated to the resolvent kernel. We present two examples of our main result: First, when Γ is the full modular group $\mathrm{PSL}(2,\mathbb{Z})$, thus reproving the theorems from [BK20]; and second when Γ is an Atkin-Lehner group $\Gamma_0(N)^+$, where explicit computations are given for certain genus zero, one and two levels.

1 Introduction and statement of results

1.1 The Poisson-Jensen formula

Let $D_R = \{z = x + iy \in \mathbb{C} : |z| < R\}$ be the disc of radius R centered at the origin in the complex plane $\mathbb{C}$. Let F be a non-constant meromorphic function on the closure $\overline{D_R}$ of D_R. Denote by c_F the leading non-zero coefficient of F at zero, meaning that for some integer m we have that $F(z) = c_F z^m + O(z^{m+1})$ as z approaches zero. For any $a \in D_R$, let $n_F(a)$ denote the order of F at a; there are a finite number of points a for which $n_F(a) \neq 0$. With this, Jensen's formula, as stated on page 341 of [La99], asserts that

$$\frac{1}{2\pi}\int\limits_0^{2\pi} \log|F(Re^{i\theta})|d\theta + \sum_{a\in D_R} n_F(a)\log(|a|/R) + n_F(0)\log(1/R) = \log|c_F|. \tag{1}$$

One can consider the action of a Möbius transformation which preserves D_R and seek to determine the resulting expression from (1). Such a consideration leads to the Poisson-Jensen formula, and we refer the reader to page 161 of [La87] for a statement and proof.

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On their own, the Jensen formula and the Poisson-Jensen formula paved the way toward Nevanlinna theory, which in its most elementary interpretation establishes subtle growth estimates for meromorphic functions; see Chapter VI of [La99]. Going further, Nevanlinna theory provided motivation for Vojta's conjectures whose insight into arithmetic algebraic geometry is profound. In particular, page 34 of [Vo87] contains a type of "dictionary" which translates between Nevalinna theory and number theory where Vojta asserts that Jensen's formula should be viewed as analogous to the Artin-Whaples product formula from class field theory.

1.2 A modular generalization

In [Ro84] Rohrlich proved what he aptly called a modular version of Jensen's formula. We now shall describe Rohrlich's result.

Let f be a meromorphic function on the upper half plane $\mathbb{H}$ which is invariant with respect to the action of the full modular group $\mathrm{PSL}(2,\mathbb{Z})$. Set $\mathcal{F}$ to be the "usual" fundamental domain of the quotient $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$, and let $d\mu$ denote the area form of the hyperbolic metric. Assume that f does not have a pole at the cusp ∞ of $\mathcal{F}$, and assume further that the Fourier expansion of f at ∞ has its constant term equal to one. Let $P(w)$ be the Kronecker limit function associated to the parabolic Eisenstein series associated to $\mathrm{PSL}(2,\mathbb{Z})$; below we will write $P(w)$ in terms of the Dedekind Delta function, but for now we want to keep the concept of a Kronecker limit function in the conversation. With all this, the Rohrlich-Jensen formula is the statement that

$$\frac{1}{2\pi}\int\limits_{\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}}\log|f(z)|d\mu(z)+\sum_{w\in\mathcal{F}}\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}P(w)=0. \tag{2}$$

In this expression, $\mathrm{ord}_w(f)$ denotes the order of f at w as a meromorphic function, and $\mathrm{ord}(w)$ denotes the order of the action of $\mathrm{PSL}(2,\mathbb{Z})$ on $\mathbb{H}$. As a means by which one can see beyond the above setting, one can view (2) as evaluating the inner product

$$\langle 1,\log|f(z)|\rangle=\int\limits_{\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}}1\cdot\log|f(z)|d\mu(z)$$

within the Hilbert space of L^2 functions on $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$.

There are various directions in which (2) has been extended. In [Ro84], Rohrlich described the analogue of (2) for general Fuchsian groups of the first kind and for meromorphic modular forms f of non-zero weight; see page 19 of [Ro84]. In [HIvPT19] the authors studied the quotient of hyperbolic three space when acted upon by the discrete group $\mathrm{PSL}(2,\mathcal{O}_K)$ where $\mathcal{O}_K$ denotes the ring of integers of an imaginary quadratic field K. In that setting, the function $\log|f|$ is replaced by a function which is harmonic at all but a finite number of points and at those points the function has prescribed singularities. As in [Ro84], the analogue of (2) involves a function P which is constructed from a type of Kronecker limit formula.

In [BK20] the authors returned to the setting of $\mathrm{PSL}(2,\mathbb{Z})$ acting on $\mathbb{H}$. Let $q_z=e^{2\pi iz}$ be the standard local coordinate near ∞ of $\mathrm{PSL}(2,\mathbb{Z})\backslash\mathbb{H}$. The Hauptmodul $j(z)$ is the unique $\mathrm{PSL}(2,\mathbb{Z})$ invariant holomorphic function on $\mathbb{H}$ whose expansion near ∞ is $j(z)=q_z^{-1}+o(q_z^{-1})$ as z approaches ∞. Let T_n denote the n-th Hecke operator and set $j_n(z)=j|T_n(z)$. The main results of [BK20] are the derivation of formulas for the regularized scalar product $\langle j_n(z),\log((\mathrm{Im}(z))^k|f(z)|)\rangle$ where f is a weight $2k$ meromorphic modular form with respect to PSL$(2,\mathbb{Z})$. Below we will discuss further the formulas from [BK20] and describe the way in which their results are natural extensions of (2).

1.3 Revisiting Rohrlich's theorem

The purpose of this article is to extend the point of view that the Rohlrich-Jensen formula is the evaluation of a particular type of inner product and to prove the extension of this formula in the setting of an arbitrary, not necessarily arithmetic, Fuchsian group of the first kind with one cusp. To do so, we shall revisit the role of each of the two terms $j|T_n(z)$ and $\log((\mathrm{Im}(z))^k|f(z)|)$.

The function $j|T_n(z)$ can be characterized as the unique holomorphic function which is PSL$(2,\mathbb{Z})$ invariant on $\mathbb{H}$ and whose expansion near ∞ is $q_z^{-n}+o(q_z^{-1})$. These properties hold for the special value $s=1$ of the Niebur-Poincaré series $F^{\Gamma}_{-n}(z,s)$, which is defined in [Ni73] for any Fuchsian group Γ of the first kind with one cusp and discussed in section 3.1 below. As proved in [Ni73], for any $m\in\mathbb{N}$, the Niebur-Poincaré series $F^{\Gamma}_{m}(z,s)$ is an eigenfunction of the hyperbolic Laplacian Δ_{hyp}; specifically, we have that

$$\Delta_{\mathrm{hyp}}F^{\Gamma}_{m}(z,s)=s(s-1)F^{\Gamma}_{m}(z,s).$$

Also, $F^{\Gamma}_{m}(z,s)$ is orthogonal to constant functions.

Furthermore, if $\Gamma=\mathrm{PSL}(2,\mathbb{Z})$, then for any positive integer n there is an explicitly computable constant c_n such that

$$F^{\mathrm{PSL}(2,\mathbb{Z})}_{-n}(z,1)=\frac{1}{2\pi\sqrt{n}}j_n(z)+c_n. \tag{3}$$

As a result, the Rohrlich-Jensen formula proved in [BK20], when combined with Rohrlich's formula from [Ro84], reduces to computing the regularized inner product of $F^{\mathrm{PSL}(2,\mathbb{Z})}_{-n}(z,1)$ with $\log((\mathrm{Im}(z))^k|f(z)|)$.

As for the term $\log((\mathrm{Im}(z))^k|f(z)|)$, we begin by recalling Proposition 12 from [JvPS19]. Let $2k\geq 4$ be any even positive integer, and let f be a weight $2k$ meromorphic form f which is Γ invariant and with q-expansion at ∞ that is normalized so its constant term is equal to one. Set $\|f\|(z)=y^k|f(z)|$, where $z=x+iy$. Let $\mathcal{E}^{\mathrm{ell}}_{\Gamma,w}(z,s)$ be the elliptic Eisenstein series associated to the aforementioned data; a summary of the relevant properties of $\mathcal{E}^{\mathrm{ell}}_{\Gamma,w}(z,s)$ is given in section 4.3 below. Then, in [JvPS19] it is proved that one has the asymptotic relation

$$\sum_{w\in\mathcal{F}_\Gamma}\mathrm{ord}_w(f)\mathcal{E}^{\mathrm{ell}}_{\Gamma,w}(z,s)=-s\log\left(|f(z)||\eta^4_{\Gamma,\infty}(z)|^{-k}\right)+O(s^2)\quad\text{as } s\to 0 \tag{4}$$

where $\mathcal{F}_\Gamma$ is the fundamental domain for the action of Γ on $\mathbb{H}$ and $\eta_{\Gamma,\infty}(z)$ is the analogue of the classical eta function for the modular group, see the Kronecker limit formula (24) for the parabolic Eisenstein series. With this, formula (4) can be written as

$$\log\left(\|f\|(z)\right)=kP_\Gamma(z)-\sum_{w\in\mathcal{F}_\Gamma}\mathrm{ord}_w(f)\lim_{s\to 0}\frac{1}{s}\mathcal{E}^{\mathrm{ell}}_{\Gamma,w}(z,s), \tag{5}$$

where $P_\Gamma(z)=\log(|\eta^4_{\Gamma,\infty}(z)|\mathrm{Im}(z))$ is the Kronecker limit function associated to the parabolic Eisenstein series $\mathcal{E}^{\mathrm{par}}_{\Gamma,\infty}(z,s)$; the precise normalizations and expressions defining $\mathcal{E}^{\mathrm{par}}_{\Gamma,\infty}(z,s)$ will be clarified below.

Following [CJS20], one can recast (5) in terms of the resolvent kernel, which we now shall undertake.

The resolvent kernel, also called the automorphic Green's function, $G_s^\Gamma(z,w)$ is the integral kernel which for almost all $s \in \mathbb{C}$ inverts the operator $\Delta_{\mathrm{hyp}} + s(s-1)$. In other words,

$$\Delta_{\mathrm{hyp}} G_s^\Gamma(z,w) = s(1-s) G_s^\Gamma(z,w).$$

The resolvent kernel is closely related to the elliptic Eisenstein series; see [vP16] as well as [CJS20]. Specifically, from Corollary 7.4 of [vP16], after taking into account a sign difference in our normalization, we have that

$$\mathrm{ord}(w)\mathcal{E}^{\mathrm{ell}}_{\Gamma,w}(z,s) = -\frac{2^{s+1}\sqrt{\pi}\,\Gamma(s+1/2)}{\Gamma(s)} G_s^\Gamma(z,w) + O(s^2) \quad \text{as } s \to 0 \tag{6}$$

for all $z, w \in \mathbb{H}$ with $z \neq \gamma w$ when $\gamma \in \Gamma$. It is now evident that one can express $\log(\|f\|(z))$ as a type of Kronecker limit function. Indeed, upon using the functional equation for the Green's function, we will prove below the following result. Under certain general conditions the form f, as described above, can be realized through a type of factorization theorem, namely that

$$\begin{aligned} \log(\|f\|(z)) &= -2k + 2\pi \sum_{w \in \mathcal{F}_\Gamma} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \lim_{s\to 1} \left(G_s^\Gamma(z,w) + \mathcal{E}^{\mathrm{par}}_{\Gamma,\infty}(z,s) \right) \\ &= 2\pi \sum_{w \in \mathcal{F}_\Gamma} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \left[\lim_{s\to 1} \left(G_s^\Gamma(z,w) + \mathcal{E}^{\mathrm{par}}_{\Gamma,\infty}(z,s) \right) - \frac{2}{\mathrm{vol}_{\mathrm{hyp}}(\Gamma\backslash\mathbb{H})} \right]. \end{aligned} \tag{7}$$

With all this, it is evident that one can view the inner product realization of the Rohrlich-Jensen formula as a special value of the inner product of the Niebur-Poincaré series $F_m^\Gamma(z,s)$ and the resolvent kernel $G_s^\Gamma(z,w)$ plus the parabolic Eisenstein series $\mathcal{E}^{\mathrm{par}}_{\Gamma,\infty}(z,s)$. Furthermore, because all terms are eigenfunctions of the Laplacian, one can seek to compute the inner product in hand in a manner similar to that which yields the Maass-Selberg formula.

1.4 Our main results

Unless otherwise explicitly stated, we will assume for the remainder of this article that Γ is *any Fuchsian group of the first kind with one cusp*. By conjugating Γ, if necessary, we may assume that the cusp is at ∞, with the cuspidal width equal to one. The group Γ will be arbitrary, but fixed, throughout this article, so, for the sake of brevity, in the sequel, we will suppress the index Γ in the notation for Eisenstein series, the Niebur-Poincaré series, the Kronecker limit function, the fundamental domain and the resolvent kernel. When Γ is taken to be the modular group or the Atkin-Lehner group, that will be indicated in the notation.

With the above discussion, we have established that one manner in which the Rohrlich-Jensen formula can be understood is through the study of the regularized inner product

$$\langle F_{-n}(\cdot,1), \overline{\lim_{s\to 1}\left(G_s(\cdot,w) + \mathcal{E}^{\mathrm{par}}_\infty(\cdot,s)\right)}\rangle, \tag{8}$$

which is defined as follows. Since Γ has one cusp at ∞, one can construct a (Ford) fundamental domain $\mathcal{F}$ of the action of Γ on $\mathbb{H}$. Let $M = \Gamma\backslash\mathbb{H}$. A cuspidal neighborhood $\mathcal{F}_\infty(Y)$ of ∞ is given by $0 < x \leq 1$ and $y \geq Y$, where $z = x + iy$ and some $Y \in \mathbb{R}$ sufficiently large. (We recall that we have normalized the cusp to be of width one.) Let $\mathcal{F}(Y) = \mathcal{F} \setminus \mathcal{F}_\infty(Y)$. Then, we define (8) to be

$$\lim_{Y\to\infty} \int\limits_{\mathcal{F}(Y)} F_{-n}(z,1) \lim_{s\to 1}\left(G_s(z,w) + \mathcal{E}^{\mathrm{par}}_\infty(z,s)\right) d\mu_{\mathrm{hyp}}(z)$$

where $d\mu_{\mathrm{hyp}}(z)$ denotes the hyperbolic volume element. The function $G_s(z,w) + \mathcal{E}_\infty^{\mathrm{par}}(z,s)$ is unbounded as $z \to w$. However, the asymptotic growth of the function is logarithmic thus integrable, hence it is not necessary to regularize the integral in (8) in a neighborhood containing w. The need to regularize the inner product (8) stems solely the from the exponential growth behavior of the factor $F_{-n}(z,1)$ as $z \to \infty$.

Our first main result of this article is the following theorem.

Theorem 1. *For any positive integer n and any point $w \in \mathcal{F}$*

$$\langle F_{-n}(\cdot,1), \overline{\lim_{s\to 1} \left(G_s(\cdot,w) + \mathcal{E}_\infty^{\mathrm{par}}(\cdot,s)\right)} \rangle = -\frac{\partial}{\partial s} F_{-n}(w,s)\Big|_{s=1}. \tag{9}$$

We can combine Theorem 1 with the factorization theorem (7) and properties of $F_{-n}(z,1)$ proved in [Ni73] and obtain the following extension of the Rohrlich-Jensen formula.

Corollary 1. *In addition to the notation above, assume that the even weight $2k \geq 0$ meromorphic form f has been normalized so its q-expansion at ∞ has constant term equal to 1. Then we have that*

$$\langle F_{-n}(\cdot,1), \log \|f\| \rangle = -2\pi \sum_{w \in \mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \frac{\partial}{\partial s} \left. F_{-n}(w,s)\right|_{s=1}. \tag{10}$$

Let g be a Γ invariant analytic function which necessarily has a pole at ∞. As such, there is a positive integer K and set of complex numbers $\{a_n\}_{n=1}^K$ such that

$$g(z) = \sum_{n=1}^{K} a_n q_z^{-n} + O(1) \quad \text{as } z \to \infty.$$

It is proved in [Ni73] that

$$g(z) = \sum_{n=1}^{K} 2\pi\sqrt{n} a_n F_{-n}(z,1) + c(g) \tag{11}$$

for some constant depending only upon g. With this, we can combine Corollary 1 and the Theorem on page 19 of [Ro84] to obtain the following result.

Corollary 2. *With notation as above, there is a constant β, defined by the Laurent expansion of $\mathcal{E}_\infty^{\mathrm{par}}(z,s)$ near $s=1$, such that*

$$\langle g, \log\|f\| \rangle = -2\pi \sum_{w\in\mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \left(2\pi \sum_{n=1}^{K} \sqrt{n} a_n \frac{\partial}{\partial s} F_{-n}(w,s)\Big|_{s=1} + c(g)(P(w) - \beta\, \mathrm{vol}_{\mathrm{hyp}}(M) + 2) \right). \tag{12}$$

The constant β is given in (24). We refer the reader to equation (24) for further details regarding the normalizations which define β and the parabolic Kronecker limit function P.

Finally, we will consider the generating function of the normalized series constructed from the right-hand side of (9). Specifically, we will prove the following identity.

Theorem 2. *With notation as above, the generating series*

$$\sum_{n\geq 1} 2\pi\sqrt{n}\frac{\partial}{\partial s}F_{-n}(w,s)\Big|_{s=1} q_z^n$$

is, in the z variable, the holomorphic part of the weight two biharmonic Maass form

$$\mathcal{G}_w(z) := i\frac{\partial}{\partial z}\left(\frac{\partial}{\partial s}\left(G_s(z,w) + \mathcal{E}_\infty^{\mathrm{par}}(w,s)\right)\Big|_{s=1}\right).$$

Note that the weight two biharmonic Maas form is a function which satisfies the weight two modularity in z and which is annihilated by $\Delta_2^2 = (\xi_0 \circ \xi_2)^2$, where, classically $\xi_\kappa := 2iy^\kappa \overline{\frac{\partial}{\partial \bar{z}}}$. It is clear from the definition that $\mathcal{G}_w(z)$ satisfies the weight two modularity in the z variable. In section 5.4 we will prove that $(\xi_0 \circ \xi_2)^2 \mathcal{G}_w(z) = 0$.

In the case $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, our results yield the main theorems from [BK20], as we will discuss below.

1.5 Outline of the paper

In section 2 we will establish notation and recall certain results from the literature. There are two specific examples of Poincaré series which are particularly important for our study, the Niebur-Poincaré series and the resolvent kernel. Both series are defined, and basic properties are presented in section 3. In section 4 we state the Kronecker limit formulas associated to parabolic and elliptic Eisenstein series, and we prove the factorization theorem (7). The proofs of the main results listed above will be given in section 5.

To illustrate our results, various examples are given in section 6. Our first example is when $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ where, as claimed above, our results yield the main theorems of [BK20]. We then turn to the case when Γ is an Atkin-Lehner group $\Gamma_0(N)^+$ for square-free level N. The first examples are when the genus of $\Gamma_0(N)^+$ is zero and when the function g in Corollary 2 is the Hauptmodul $j_N^+(z)$. The next two examples we present are for levels $N = 37$ and $N = 103$. For these levels the genus of the quotient by $\Gamma_0(N)^+$ is one and two, respectively. In these cases, certain generators of the corresponding function fields were constructed in [JST16]. Consequently, we are able to employ the results from [JST16] and fully develop Corollary 2.

2 Background material

2.1 Basic notation

Let $\Gamma \subset \mathrm{PSL}(2,\mathbb{R})$ denote a Fuchsian group of the first kind acting by fractional linear transformations on the hyperbolic upper half-plane $\mathbb{H} := \{z = x+iy \in \mathbb{C} \,|\, x,y \in \mathbb{R};\, y > 0\}$. We let $M := \Gamma\backslash\mathbb{H}$, which is a finite volume hyperbolic Riemann surface, and denote by $p : \mathbb{H} \longrightarrow M$ the natural projection. We assume that M has e_Γ elliptic fixed points and one cusp at ∞ of width one. By an abuse of notation, we also say that Γ has a cusp at ∞ of width one, meaning that the stabilizer Γ_∞ of ∞ is generated by the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. We identify M locally with its universal cover $\mathbb{H}$. By $\mathcal{F}$ we denote the "usual" (Ford) fundamental domain for Γ acting on $\mathbb{H}$.

We let μ_{hyp} denote the hyperbolic metric on M, which is compatible with the complex structure of M, and has constant negative curvature equal to minus one. The hyperbolic line element ds^2_{hyp}, resp. the hyperbolic Laplacian Δ_{hyp} acting on functions, are given in the coordinate $z = x + iy$ on $\mathbb{H}$ by

$$ds^2_{\text{hyp}} := \frac{dx^2 + dy^2}{y^2}, \quad \text{resp.} \quad \Delta_{\text{hyp}} := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

By $d_{\text{hyp}}(z, w)$ we denote the hyperbolic distance between to the two points $z \in \mathbb{H}$ and $w \in \mathbb{H}$. Our normalization of the hyperbolic Laplacian is different from the one considered in [Ni73] and [He83] where the Laplacian is taken with the plus sign.

2.2 Modular forms

Following [Se73], we define a weakly modular form f of even weight $2k$ for $k \geq 0$ associated to Γ to be a function f which is meromorphic on $\mathbb{H}$ and satisfies the transformation property

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^{2k} f(z), \quad \text{for any} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma. \tag{13}$$

In the setting of this paper, any weakly modular form f will satisfy the relation $f(z+1) = f(z)$, so that for some positive integer N we can write

$$f(z) = \sum_{n=-N}^{\infty} a_n q_z^n, \qquad \text{where } q_z = e(z) = e^{2\pi i z}.$$

If $a_n = 0$ for all $n < 0$, then f is said to be holomorphic at the cusp at ∞. A holomorphic modular form with respect to Γ is a weakly modular form which is holomorphic on $\mathbb{H}$ and at all the cusps of Γ.

When the weight k is zero, the transformation property (13) indicates that the function f is invariant with respect to the action of elements of the group Γ, so it may be viewed as a meromorphic function on the surface $M = \Gamma \backslash \mathbb{H}$. In other words, a meromorphic function on M is a weakly modular form of weight 0.

For any two weight $2k$ weakly modular forms f and g associated to Γ, with integrable singularities at finitely many points in $\mathcal{F}$, the generalized inner product $\langle \cdot, \cdot \rangle$ is defined as

$$\langle f, g \rangle = \lim_{Y \to \infty} \int\limits_{\mathcal{F}(Y)} f(z)\overline{g(z)}(\text{Im}(z))^{2k} d\mu_{\text{hyp}}(z) \tag{14}$$

where the integration is taken over the portion $\mathcal{F}(Y)$ of the fundamental domain $\mathcal{F}$ equal to $\mathcal{F} \setminus \mathcal{F}_\infty(Y)$.

2.3 Atkin-Lehner groups

Let $N = p_1 \cdot \ldots \cdot p_r$ be a square-free, non-negative integer including the case $N = 1$. The subset of $\text{SL}(2, \mathbb{R})$, defined by

$$\Gamma_0(N)^+ := \left\{ \frac{1}{\sqrt{e}} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}) : \, ad - bc = e, \, a, b, c, d, e \in \mathbb{Z}, \, e \mid N, \; e \mid a, \; e \mid d, \; N \mid c \right\}$$

is an arithmetic subgroup of $\text{SL}(2, \mathbb{R})$. We use the terminology Atkin-Lehner groups of level N to describe $\Gamma_0(N)^+$ in part because these groups are obtained by adding all Atkin-Lehner involutions to the congruence group $\Gamma_0(N)$, see [AL70]. Let $\{\pm \mathrm{Id}\}$ denote the set of two elements where Id is the identity matrix. In general, if Γ is a subgroup of $\mathrm{SL}(2,\mathbb{R})$, we let $\overline{\Gamma} := \Gamma/\{\pm \mathrm{Id}\}$ denote its projection into $\mathrm{PSL}(2,\mathbb{R})$.

Set $Y_N^+ := \overline{\Gamma_0(N)^+}\backslash\mathbb{H}$. According to [Cum04], for any square-free N the quotient space Y_N^+ has one cusp at ∞ with the cusp width equal to one. The spaces Y_N^+ will be used in the last section where we give examples of our results for generators of function fields of meromorphic functions on Y_N^+.

2.4 Generators of function fields of Atkin-Lehner groups of small genus

An explicit construction of generators of function fields of all meromorphic functions on Y_N^+ with genus $g_{N,+} \leq 3$ was given in [JST16].

When $g_{N,+} = 0$, the function field of meromorphic functions on Y_N^+ is generated by a single function, the Hauptmodul $j_N^+(z)$, which is normalized so that its q-expansion is of the form $q_z^{-1} + O(q_z)$. The Hauptmodul $j_N^+(z)$ appears in the "Monstrous Moonshine" and was investigated in many papers, starting with Conway and Norton [CN79]. The action of the m-th Hecke operator T_m on $j_N^+(z)$ produces a meromorphic form on Y_N^+ with the q-expansion $j_N^+|T_m(z) = q_z^{-m} + O(q_z)$.

When $g_{N,+} \geq 1$, the function field associated to Y_N^+ is generated by two functions $x_N^+(z)$ and $y_N^+(z)$. Stemming from the results in [JST16], we have that for $g_{N,+} \leq 3$ the generators $x_N^+(z)$ and $y_N^+(z)$ such that their q-expansions are of the form

$$x_N^+(z) = q_z^{-a} + \sum_{j=1}^{a-1} a_j q_z^{-j} + O(q_z) \quad \text{and} \quad y_N^+(z) = q_z^{-b} + \sum_{j=1}^{b-1} b_j q_z^{-j} + O(q_z)$$

where a, b are positive integers with $a \leq 1 + g_{N,+}$, and $b \leq 2 + g_{N,+}$. Furthermore, for $g_{N,+} \leq 3$, it is shown in [JST16] that all coefficients in the q-expansion for $x_N^+(z)$ and $y_N^+(z)$ are integers. For all such N, the precise values of these coefficients out to large order were computed, and the results are available at [JSTurl].

3 Two Poincaré series

In this section we will define the Niebur-Poincaré series $F_m(z,s)$ and the resolvent kernel, also referred to as the automorphic Green's function $G_s(z,w)$. We refer the reader to [Ni73] for additional information regarding $F_m(z,s)$ and to [He83] and [Iwa02] and references therein for further details regarding $G_s(z,w)$. As said above, we will suppress the group Γ from the notation.

3.1 Niebur-Poincaré series

We start with the definition and properties of the Niebur-Poincaré series $F_m(z,s)$ associated to a co-finite Fuchsian group with one cusp; then we will specialize results to the setting of Atkin-Lehner groups.

3.1.1 Niebur-Poincaré series associated to a co-finite Fuchsian group with one cusp

Let m be a non-zero integer, $z = x + iy \in \mathbb{H}$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Recall the notation $e(x) := \exp(2\pi i x)$, and let $I_{s-1/2}$ denote the modified I-Bessel function of the first kind; see, for example Appendix B.4, formula (B.32) of [Iwa02]). The Niebur-Poincaré series $F_m(z, s)$ is defined by the series

$$F_m(z,s) = F_m^{\Gamma}(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} e(m\mathrm{Re}(\gamma z))(\mathrm{Im}(\gamma z))^{1/2} I_{s-1/2}(2\pi|m|\mathrm{Im}(\gamma z)). \qquad (15)$$

For fixed m and z, the series (15) converges absolutely and uniformly on any compact subset of the half plane $\mathrm{Re}(s) > 1$. Moreover, $\Delta_{\mathrm{hyp}} F_m(z,s) = s(1-s)F_m(z,s)$ for all $s \in \mathbb{C}$ in the half plane $\mathrm{Re}(s) > 1$. From Theorem 5 of [Ni73], we have that for any non-zero integer m, the function $F_m(z,s)$ admits a meromorphic continuation to the whole complex plane $s \in \mathbb{C}$. Moreover, $F_m(z,s)$ is holomorphic at $s = 1$ and, according to the spectral expansion given in Theorem 5 of [Ni73], $F_m(z,1)$ is orthogonal to constant functions, meaning that

$$\langle F_m(z,1), 1\rangle = 0.$$

For our purposes, it is necessary to employ the Fourier expansion of $F_m(z,s)$ in the cusp ∞. The Fourier expansion is proved in [Ni73] and involves Kloosterman sums $S(m,n;c)$, which we now define. For any integers m and n, and real number c, define

$$S(m,n;c) = S_\Gamma(m,n;c) := \sum_{\left(\begin{smallmatrix} a & * \\ c & d \end{smallmatrix}\right)\in\Gamma_\infty\backslash\Gamma/\Gamma_\infty} e\left(\frac{ma+nd}{c}\right).$$

For $\mathrm{Re}(s) > 1$ and $z = x + iy \in \mathbb{H}$, the Fourier expansion of $F_m(z,s)$ is given by

$$F_m(z,s) = e(mx)y^{1/2} I_{s-1/2}(2\pi|m|y) + \sum_{k=-\infty}^{\infty} b_k(y,s;m)e(kx), \qquad (16)$$

where

$$b_0(y,s;m) = \frac{y^{1-s}}{(2s-1)\Gamma(s)} 2\pi^s |m|^{s-1/2} \sum_{c>0} S(m,0;c)c^{-2s} = \frac{y^{1-s}}{(2s-1)} B_0(s;m)$$

and, for $k \neq 0$

$$b_k(y,s;m) = B_k(s;m)y^{1/2} K_{s-1/2}(2\pi|m|y),$$

with

$$B_k(s;m) = 2\sum_{c>0} S(m,k;c)c^{-1} \cdot \begin{cases} J_{2s-1}\left(\frac{4\pi}{c}\sqrt{mk}\right), & \text{if } mk > 0 \\ I_{2s-1}\left(\frac{4\pi}{c}\sqrt{|mk|}\right), & \text{if } mk < 0. \end{cases}$$

In the above expression, J_{2s-1} denotes the J-Bessel function and $K_{s-1/2}$ is the modified Bessel function; see, for example, formula (B.28) in [Iwa02] for J_{2s-1} and formula (B.34) of [Iwa02]) for $K_{s-1/2}$.

According to the proof of Theorem 6 from [Ni73], the Fourier expansion (16) extends by the principle of analytic continuation to the case when $s = 1$, hence putting $B_k(1;m) := \lim_{s\downarrow 1} B_k(s;m)$, we have

$$F_m(z,1) = \frac{\sinh(2\pi|m|y)}{\pi\sqrt{|m|}} e(mx) + B_0(1;m) + \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{1}{2\sqrt{|k|}} e^{-2\pi|k|y} B_k(1;m)e(kx). \qquad (17)$$

It is clear from (17) that for $n > 0$ one has that

$$F_{-n}(z,1) = \frac{1}{2\pi\sqrt{n}} q_z^{-n} + O(1) \quad \text{as } z \to \infty.$$

Moreover, applying $\frac{\partial}{\partial s}$ to the Fourier expansion (16), taking $s = 1$ and reasoning analogously as in the proof of Lemma 4.3. (1), p. 19 of [BK20] we immediately deduce the following crude bound

$$\frac{\partial}{\partial s} F_{-n}(z,s)\Big|_{s=1} \ll \exp\left(2\pi n \mathrm{Im}(z)\right), \quad \text{as} \quad \mathrm{Im}(z) \to \infty. \tag{18}$$

We note that the value of the derivative of the Niebur-Poincaré series at $s = 1$ satisfies a differential equation, namely that

$$\begin{aligned}\Delta_{\mathrm{hyp}} \left(\frac{\partial}{\partial s} \left. F_{-n}(z,s) \right|_{s=1} \right) &= \lim_{s\to 1} \Delta_{\mathrm{hyp}} \left(\frac{F_{-n}(z,s) - F_{-n}(z,1)}{(s-1)} \right) = \\ &= \lim_{s\to 1} \left(\frac{s(1-s)F_{-n}(z,s) - 0}{(s-1)} \right) = -F_{-n}(z,1). \end{aligned} \tag{19}$$

3.1.2 Fourier expansion when Γ is an Atkin-Lehner group

One can explicitly evaluate $B_0(1;m)$ for $m > 0$ when Γ is an Atkin-Lehner group. Set $\Gamma = \overline{\Gamma_0(N)^+}$ where N is a squarefree, which we express as $N = \prod_{\nu=1}^{r} p_\nu$. Let $B^+_{0,N}(1;m)$ denote the coefficient $B_0(1;m)$ for $\overline{\Gamma_0(N)^+}$.
From Theorem 8 and Proposition 9 of [JST16] we get that

$$B^+_{0,N}(1;m) = \frac{12\sigma(m)}{\pi\sqrt{m}} \prod_{\nu=1}^{r} \left(1 - \frac{p_\nu^{\alpha_{p_\nu}(m)+1}(p_\nu - 1)}{\left(p_\nu^{\alpha_{p_\nu}(m)+1} - 1\right)(p_\nu + 1)} \right), \tag{20}$$

where $\sigma(m)$ denotes the sum of divisors of a positive integer m and $\alpha_p(m)$ is the largest integer such that $p^{\alpha_p(m)}$ divides m. These expressions will be used in our explicit examples in section 6 below.

3.2 Automorphic Green's function

The automorphic Green's function, also called the resolvent kernel, for the Laplacian on M is defined on page 31 of [He83]. In the notation of [He83], let χ be the identity character, $z, w \in \mathcal{F}$ with $z \neq w$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Formally, consider the series

$$G_s(z,w) = \sum_{\gamma\in\Gamma} k_s(\gamma z, w)$$

with

$$k_s(z,w) := -\frac{\Gamma(s)^2}{4\pi\Gamma(2s)} \left[1 - \left| \frac{z-w}{z-\overline{w}} \right|^2 \right]^s F\left(s,s;2s;1 - \left| \frac{z-w}{z-\overline{w}} \right|^2 \right)$$

and where $F(\alpha,\beta;\gamma;u)$ is the classical hypergeometric function. We should point out that the normalization we are using, which follows [He83], differs from the normalization for the Green's function in Chapter 5 of [Iwa02]; the two normalizations differ by a minus sign. With this said, it is proved in [He83], Proposition 6.5. on p.33 that the series which defines $G_s(z,w)$ converges uniformly and absolutely on compact subsets of $(z,w,s) \in \mathcal{F}\times\mathcal{F}\times\{s\in\mathbb{C} : \mathrm{Re}(s) > 1\}$.

Furthermore, for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, and all $z, w \in \mathbb{H}$ with $z \neq \gamma w$ for $\gamma \in \Gamma$, the function $G_s(z,w)$ is the eigenfunction of Δ_{hyp} associated to the eigenvalue $s(1-s)$.

Combining formulas 9.134.1. and 8.703. from [GR07] and applying the identity

$$\cosh(d_{\mathrm{hyp}}(z,w)) = \left(2 - \left[1 - \left|\frac{z-w}{z-\overline{w}}\right|^2\right]\right)\left(1 - \left|\frac{z-w}{z-\overline{w}}\right|^2\right)^{-1}$$

we deduce that

$$k_s(z,w) = -\frac{1}{2\pi} Q^0_{s-1}(\cosh(d_{\mathrm{hyp}}(z,w))),$$

where Q^{μ}_{ν} is the associated Legendre function as defined by formula 8.703 in [GR07], with $\nu = s-1$ and $\mu = 0$.

Now, we can combine Theorem 4 of [Ni73] with Theorem 5.3 of [Iwa02], to deduce the Fourier expansion of the automorphic Green function in terms of the Niebur-Poincaré series. Specifically, let $w \in \mathcal{F}$ be fixed. Assume $z \in \mathcal{F}$ with $y = \mathrm{Im}(z) > \max\{\mathrm{Im}(\gamma w) : \gamma \in \Gamma\}$, and assume $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$. Then $G_s(z,w)$ admits the expansion

$$G_s(z,w) = -\frac{y^{1-s}}{2s-1}\mathcal{E}^{\mathrm{par}}_{\infty}(w,s) - \sum_{k\in\mathbb{Z}\smallsetminus\{0\}} y^{1/2}K_{s-1/2}(2\pi|k|y)F_{-k}(w,s)e(kx) \tag{21}$$

where $\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)$ is the parabolic Eisenstein series associated to the cusp at ∞ of Γ, see the next section for its full description.

Function $G_s(z,w)$ is unbounded as $z \to w$ and, according to Proposition 6.5. from [He83] we have the asymptotics

$$G_s(z,w) = \frac{\mathrm{ord}(w)}{2\pi}\log|z-w| + O(1), \quad \text{as} \quad z \to w.$$

4 Eisenstein series and their Kronecker limit formulas

The purpose of this section is two-fold. First, we state the definitions of parabolic and elliptic Eisenstein series as well as their associated Kronecker limit formulas. Specific examples of the parabolic Kronecker limit formulas are recalled from [JST16]. Second, we prove the factorization theorem for meromorphic forms in terms of elliptic Kronecker limit functions, as stated in (5).

4.1 Parabolic Kronecker limit functions

Associated to the cusp at ∞ of Γ one has a parabolic Eisenstein series $\mathcal{E}^{\mathrm{par}}_{\infty}(z,s)$. Let Γ_∞ denote the stabilizer subgroup within Γ of ∞. For $z \in \mathbb{H}$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, $\mathcal{E}^{\mathrm{par}}_{\infty}(z,s)$ is defined by the series

$$\mathcal{E}^{\mathrm{par}}_{\infty}(z,s) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \mathrm{Im}(\gamma z)^s.$$

It is well-known that $\mathcal{E}^{\mathrm{par}}_{\infty}(z,s)$ admits a meromorphic continuation to all $s \in \mathbb{C}$ and a functional equation in s.

For us, the Kronecker limit formula means the determination of the constant term in the Laurent expansion of $\mathcal{E}^{\mathrm{par}}_{\infty}(z,s)$ at $s=1$. Classically, Kronecker's limit formula is the assertion that for $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$ one has that

$$\mathcal{E}^{\mathrm{par}}_{\infty}(z,s) = \frac{3}{\pi(s-1)} - \frac{1}{2\pi}\log\bigl(|\Delta(z)|\mathrm{Im}(z)^6\bigr) + C + O(s-1) \ \text{ as } \ s \to 1. \tag{22}$$

where $C = 6(1-12\,\zeta'(-1) - \log(4\pi))/\pi$ and $\Delta(z)$ is Dedekind's Delta function which defined by

$$\Delta(z) = \left[q_z^{1/24}\prod_{n=1}^{\infty}(1-q_z^n)\right]^{24} = \eta(z)^{24}. \tag{23}$$

We refer to [Si80] for a proof of (22), though the above formulation follows the normalization from [JST16].

For general Fuchsian groups of the first kind, Goldstein [Go73] studied analogues of the Kronecker's limit formula associated to parabolic Eisenstein series. After a slight renormalization and trivial generalization, Theorem 3-1 from [Go73] asserts that the parabolic Eisenstein series $\mathcal{E}^{\mathrm{par}}_{\infty}(z,s)$ admits the Laurent expansion

$$\mathcal{E}^{\mathrm{par}}_{\infty}(z,s) = \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M)(s-1)} + \beta - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M)}\log(|\eta^4_{\infty}(z)|\mathrm{Im}(z)) + O(s-1), \tag{24}$$

as $s \to 1$ and where $\beta = \beta_\Gamma$ is a certain real constant depending only on the group Γ. As the notation suggests, the function $\eta_\infty(z)$ is a holomorphic form for Γ and can be viewed as a generalization of the eta function $\eta(z)$ which is defined in (23) for the full modular group.

By employing the functional equation for the parabolic Eisenstein series, as stated in Theorem 6.5 of [Iwa02], one can re-write the Kronecker limit formula as stating that

$$\mathcal{E}^{\mathrm{par}}_{\infty}(z,s) = 1 + \log(|\eta^4_{\infty}(z)|\mathrm{Im}(z)) \cdot s + O(s^2) \qquad \text{as } s \to 0, \tag{25}$$

see Corollary 3 of [JvPS19]. In this formulation, we will call the function

$$P(z) = P_\Gamma(z) := \log(|\eta^4_{\infty}(z)|\mathrm{Im}(z))$$

the parabolic Kronecker limit function of Γ.

4.2 Atkin-Lehner groups

Let $N = p_1 \cdot \ldots \cdot p_r$ be a positive squarefree number, which includes the possibility that $N=1$ and set

$$\ell_N = 2^{1-r}\mathrm{lcm}\Bigl(4,\ 2^{r-1}\frac{24}{(24,\sigma(N))}\Bigr)$$

where lcm stands for the least common multiple of two numbers. In [JST16], Theorem 16, it is proved that

$$\Delta_N(z) := \left(\prod_{v|N}\eta(vz)\right)^{\ell_N} \tag{26}$$

is a weight $k_N = 2^{r-1}\ell_N$ holomorphic form for $\Gamma_0(N)^+$ vanishing only at the cusp. By the valence formula, the order of vanishing of $\Delta_N(z)$ at the cusp is $\nu_N := k_N\,\mathrm{vol}_{\mathrm{hyp}}(Y_N^+)/(4\pi)$ where $\mathrm{vol}_{\mathrm{hyp}}(Y_N^+) = \pi\sigma(N)/(3\cdot 2^r)$ is the hyperbolic volume of the surface Y_N^+.

The Kronecker limit formula (24) for the parabolic Eisenstein series $\mathcal{E}_{\infty}^{\mathrm{par},N}(z,s)$ associated to Y_N^+ reads as

$$\mathcal{E}_{\infty}^{\mathrm{par},N}(z,s) = \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(Y_N^+)(s-1)} + \beta_N - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(Y_N^+)} P_N(z) + O((s-1)) \tag{27}$$

as $s \to 1$. From Example 7 and Example 4 of [JvPS19] we have the explicit evaluations of β_N and $P_N(z)$. Namely,

$$\beta_N = -\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(Y_N^+)} \left(\sum_{j=1}^{r} \frac{(p_j-1)\log p_j}{2(p_j+1)} - \log N + 2\log(4\pi) + 24\zeta'(-1) - 2 \right) \tag{28}$$

and the parabolic Kronecker limit function $P_N(z)$ is given by

$$P_N(z) = \log \left(\sqrt[2^r]{\prod_{v|N} |\eta(vz)|^4} \cdot \mathrm{Im}(z) \right).$$

4.3 Elliptic Kronecker limit functions

Elliptic subgroups of Γ have finite order and a unique fixed point within $\mathbb{H}$. For all but a finite number of $w \in \mathcal{F}$, the order of the elliptic subgroup Γ_w which fixes w is one. For $z \in \mathbb{H}$ with $z \neq w$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the elliptic Eisenstein series $\mathcal{E}_w^{\mathrm{ell}}(z,s)$ is defined by the series

$$\mathcal{E}_w^{\mathrm{ell}}(z,s) = \sum_{\gamma \in \Gamma_w \backslash \Gamma} \sinh(d_{\mathrm{hyp}}(\gamma z, w))^{-s} = \sum_{\gamma \in \Gamma_w \backslash \Gamma} \left(\frac{2\,\mathrm{Im}(w)\mathrm{Im}(\gamma z)}{|\gamma z - w|\,|\gamma z - \overline{w}|} \right)^s. \tag{29}$$

It was first shown in [vP10] that (29) admits a meromorphic continuation to all $s \in \mathbb{C}$.
The analogue of the Kronecker limit formula for $\mathcal{E}_w^{\mathrm{ell}}(z,s)$ was first proved in [vP10]; see also [JvPS19]. In the setting of this paper, it is shown in [vP10] that for any $w \in \mathcal{F}$ the series (29) admits the Laurent expansion

$$\begin{aligned}\mathrm{ord}(w)\, \mathcal{E}_w^{\mathrm{ell}}(z,s) - \frac{2^s\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)} \mathcal{E}_{\infty}^{\mathrm{par}}(w,1-s)\,\mathcal{E}_{\infty}^{\mathrm{par}}(z,s) = \\ = -\frac{2\pi}{\mathrm{vol}_{\mathrm{hyp}}(M)} - \frac{2\pi}{\mathrm{vol}_{\mathrm{hyp}}(M)} \log\bigl(|H_\Gamma(z,w)|^{\mathrm{ord}(w)}\mathrm{Im}(z)\bigr) \cdot s + O(s^2) \quad \text{as } s \to 0.\end{aligned} \tag{30}$$

As a function of z, $H(z,w) := H_\Gamma(z,w)$ is holomorphic on $\mathbb{H}$ and uniquely determined up to multiplication by a complex constant of absolute value one; in addition, $H(z,w)$ is an automorphic form with a non-trivial multiplier system, which depends on w, with respect to Γ acting on z. The function $H(z,w)$ vanishes if and only if $z = \gamma w$ for some $\gamma \in \Gamma$. We call the function

$$E_w(z) = E_{w,\Gamma}(z) := \log\bigl(|H(z,w)|^{\mathrm{ord}(w)}\mathrm{Im}(z)\bigr)$$

the elliptic Kronecker limit function of Γ at w.

4.4 A factorization theorem

We can now prove equation (5).

Proposition 1. *With notation as above, let f be a weight $2k$ meromorphic form on $\mathbb{H}$ with q-expansion at ∞ given by*

$$f(z) = 1 + \sum_{n=1}^{\infty} b_f(n) q_z^n, \tag{31}$$

Let $\mathrm{ord}_w(f)$ *denote the order f at w and define the function*

$$H_f(z) := \prod_{w \in \mathcal{F}} H(z,w)^{\mathrm{ord}_w(f)}$$

where $H(z,w) = H_\Gamma(z,w)$ is given in (30)*. Then there exists a complex constant c_f such that*

$$f(z) = c_f H_f(z). \tag{32}$$

Furthermore,

$$|c_f| = \exp\left(-\frac{2\pi}{\mathrm{vol}_{\mathrm{hyp}}(M)} \sum_{w \in \mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \left(2 - \log 2 + P(w) - \beta\, \mathrm{vol}_{\mathrm{hyp}}(M)\right) \right),$$

where $P(w)$ and β are defined through the parabolic Kronecker limit function (24)*.*

Proof. The proof closely follows the proof of Theorem 9 from [JvPS19]. Specifically, following the first part of the proof almost verbatim, we conclude that the quotient

$$F_f(z) := \frac{H_f(z)}{f(z)}$$

is a non-vanishing holomorphic function on M which is bounded and non-zero at the cusp at ∞. Hence, $\log |F_f(z)|$ is L^2 on M. From its spectral expansion and the fact that $\log |F_f(z)|$ is harmonic, one concludes $\log |F_f(z)|$ is constant, hence so is $F_f(z)$. The evaluation of the constant is obtained by considering the limiting behavior as z approaches ∞, which is obtained by using the asymptotic behavior of $H(z,w)$ as $\mathrm{Im}(z) \to \infty$, as given in Proposition 6 of [JvPS19]. □

By following the proof of Proposition 12 from [JvPS19] we obtain (4), and hence (5), for meromorphic forms f on $\mathbb{H}$ with q-expansion (31). We leave the verification of this simple argument to the reader.

5 Proofs of main results

5.1 Proof of Theorem 1

Let $Y > 1$ be sufficiently large so that the cuspidal neighborhood $\mathcal{F}_\infty(Y)$ of the cusp ∞ in $\mathcal{F}$ is of the form $\{z \in \mathbb{H} : 0 < x < 1, y > Y\}$. For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, and arbitrary, but fixed $w \in \mathcal{F}$, we then have that

$$\begin{aligned}
&\int\limits_{\mathcal{F}(Y)} \Delta_{\mathrm{hyp}}(F_{-n}(z,1)) \left(G_s(z,w) + \mathcal{E}_\infty^{\mathrm{par}}(w,s)\right) d\mu_{\mathrm{hyp}}(z) \\
&\qquad - \int\limits_{\mathcal{F}(Y)} F_{-n}(z,1) \Delta_{\mathrm{hyp}} \left(G_s(z,w) + \mathcal{E}_\infty^{\mathrm{par}}(w,s)\right) d\mu_{\mathrm{hyp}}(z) \\
&\qquad = -s(1-s) \int\limits_{\mathcal{F}(Y)} F_{-n}(z,1) \left(G_s(z,w) + \mathcal{E}_\infty^{\mathrm{par}}(w,s)\right) d\mu_{\mathrm{hyp}}(z).
\end{aligned}$$

Actually, the first summand on the left-hand side is zero since $F_{-n}(n,1)$ is holomorphic; however, this judicious form of the number zero is significant since we will use the method behind the Maass-Selberg theorem to study the left-hand side of the above equation. Before this, note that the integrand on the right-hand side of the above equation is holomorphic at $s=1$. As a result, we can write

$$\frac{\partial}{\partial s}\left(-s(1-s)\int\limits_{\mathcal{F}(Y)} F_{-n}(z,1)\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)\right)d\mu_{\mathrm{hyp}}(z)\right)\Bigg|_{s=1}$$
$$=\int\limits_{\mathcal{F}(Y)} F_{-n}(z,1)\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)\right)d\mu_{\mathrm{hyp}}(z).$$

Therefore,

$$\begin{aligned}\langle F_{-n}(z,1), &\overline{\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)\right)}\rangle \\ &=\lim_{Y\to\infty}\int\limits_{\mathcal{F}(Y)} F_{-n}(z,1)\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(z,s)\right)d\mu_{\mathrm{hyp}}(z) \\ &=\lim_{Y\to\infty}\left[\frac{\partial}{\partial s}\left(\int\limits_{\mathcal{F}(Y)}\Delta_{\mathrm{hyp}}(F_{-n}(z,1))\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)\right)d\mu_{\mathrm{hyp}}(z)\right.\right. \\ &\left.\left.-\int\limits_{\mathcal{F}(Y)} F_{-n}(z,1)\Delta_{\mathrm{hyp}}\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)\right)d\mu_{\mathrm{hyp}}(z)\right)\Bigg|_{s=1}\right]\end{aligned}\tag{33}$$

The quantity on the right-hand side of (33) is setup for an application of Green's theorem as in the proof of the Maass-Selberg relations for the Eisenstein series. As described on page 89 of [Iwa02], when applying Green's theorem to each term on the right-side of (33) for fixed Y, the resulting boundary terms on the sides of the fundamental domain, which are identified by Γ, will sum to zero. As such, we get that

$$\begin{aligned}\langle F_{-n}(z,1), &\overline{\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)\right)}\rangle \\ &=\lim_{Y\to\infty}\left[\frac{\partial}{\partial s}\left(\int_0^1\frac{\partial}{\partial y}F_{-n}(z,1)\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)\right)dx\right.\right. \\ &\left.\left.-\int_0^1 F_{-n}(z,1)\frac{\partial}{\partial y}\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)\right)dx\right)\Bigg|_{s=1}\right],\end{aligned}\tag{34}$$

where functions of z and its derivatives with respect to $y=\mathrm{Im}(z)$ are evaluated at $z=x+iY$. In order to compute the difference of the two integrals of the right-hand side of (34), we will use the Fourier expansions (17) and (21) of the series $F_{-n}(z,1)$ and $G_s(z,w)$ respectively. It will be more convenient to write the first coefficient in the expansion (17) as $e(-nx)\sqrt{y}I_{\frac{1}{2}}(2\pi ny)$, as in (16).

Specifically, since the exponential functions $e(-nx)$ are orthogonal for different values of n, we get that (34) is equal to

$$- F_{-n}(w,s)\sqrt{Y}\left(\frac{\partial}{\partial y}\left(\sqrt{y}I_{\frac{1}{2}}(2\pi ny)\right)\Big|_{y=Y}\cdot K_{s-\frac{1}{2}}(2\pi nY) - I_{\frac{1}{2}}(2\pi nY)\cdot\frac{\partial}{\partial y}\left(\sqrt{y}K_{s-\frac{1}{2}}(2\pi ny)\right)\Big|_{y=Y}\right)$$

$$+ B_0(1;-n)(1-s)\frac{Y^{-s}}{2s-1}\mathcal{E}_{\infty}^{\mathrm{par}}(w,s) + \sum_{j\in\mathbb{Z}\smallsetminus\{0\}} F_j(w,s)\left(b_j(Y,1;-n)\cdot\frac{\partial}{\partial y}\left(\sqrt{y}K_{s-\frac{1}{2}}(2\pi|j|y)\right)\Big|_{y=Y} - \frac{\partial}{\partial y}b_j(y,1;-n)\big|_{y=Y}\cdot\sqrt{Y}K_{s-\frac{1}{2}}(2\pi|j|Y)\right) = T_1(Y,s;w)+T_2(Y,s;w)+T_3(Y,s;w),$$

where the last equality above provides the definitions of the functions T_1, T_2 and T_3. Therefore, from (34) we conclude that

$$\langle F_{-n}(z,1), \overline{\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}_{\infty}^{\mathrm{par}}(w,s)\right)}\rangle = = \lim_{Y\to\infty}\left[\frac{\partial}{\partial s}\left(T_1(Y,s;w)+T_2(Y,s;w)+T_3(Y,s;w)\right)\Big|_{s=1}\right] \quad (35)$$

We will treat each of the three terms on the right-hand side of (35) separately.
To evaluate the term T_1 in (35), we apply formulas 8.486.2 and 8.486.11 of [GR07] in order to compute derivatives of the Bessel functions. In doing so, we conclude that

$$T_1(Y,s;w) = -\frac{X}{2}F_{-n}(w,s)\left[K_{s-\frac{1}{2}}(X)(I_{-\frac{1}{2}}(X)+I_{\frac{3}{2}}(X)) + I_{\frac{1}{2}}(X)(K_{s-\frac{3}{2}}(X)+K_{s+\frac{1}{2}}(X))\right],$$

where we set $X = 2\pi nY$. Next, we express $K_{s+\frac{1}{2}}(X)$ in terms of $K_{s-\frac{1}{2}}(X)$ and $K_{s-\frac{3}{2}}(X)$, using formula 8.485.10 from [GR07] to get

$$K_{s+\frac{1}{2}}(X) = K_{s-\frac{3}{2}}(X) + \frac{2s-1}{X}K_{s-\frac{1}{2}}(X).$$

Then, applying formula 8.486.21 from [GR07], we deduce that

$$\frac{\partial}{\partial s}\left[K_{s-\frac{1}{2}}(X)(I_{-\frac{1}{2}}(X)+I_{\frac{3}{2}}(X)) + I_{\frac{1}{2}}(X)(K_{s-\frac{3}{2}}(X)+K_{s+\frac{1}{2}}(X))\right]\Big|_{s=1}$$
$$= \sqrt{\frac{\pi}{2X}}e^{X}\mathrm{Ei}(-2X)\left[-(I_{-\frac{1}{2}}(X)+I_{\frac{3}{2}}(X)) + \sqrt{\frac{2}{\pi X}}(2-1/X)\sinh(X)\right] + \frac{2}{X^2}e^{-X}\sinh(X),$$

where $\mathrm{Ei}(x)$ denotes the exponential integral; see section 8.2 of [GR07]. Continuing, we now employ formula (B.36) from [Iwa02] which asserts certain asymptotic behavior of the I-Bessel function as $X\to\infty$; we are interested in the cases when $\nu = -1/2$ and when $\nu = 3/2$. This result, together with the bound $\mathrm{Ei}(-2X) \leq e^{-2X}/(2X)$, which follows from the expression 8.212.10 from [GR07] for $\mathrm{Ei}(-x)$ with $x>0$, yields that

$$\lim_{X\to\infty}\frac{X}{2}\frac{\partial}{\partial s}\left[K_{s-\frac{1}{2}}(X)(I_{-\frac{1}{2}}(X)+I_{\frac{3}{2}}(X)) + I_{\frac{1}{2}}(X)(K_{s-\frac{3}{2}}(X)+K_{s+\frac{1}{2}}(X))\right]\Big|_{s=1} = 0.$$

Therefore,

$$\lim_{Y\to\infty} \frac{\partial}{\partial s} T_1(Y,s;w)|_{s=1} = -\frac{\partial}{\partial s} F_{-n}(w,s)|_{s=1} \cdot$$
$$\cdot \lim_{X\to\infty} \frac{X}{2}\left[K_{s-\frac{1}{2}}(X)(I_{-\frac{1}{2}}(X)+I_{\frac{3}{2}}(X)) + I_{\frac{1}{2}}(X)(K_{s-\frac{3}{2}}(X)+K_{s+\frac{1}{2}}(X))\right].$$

Finally, by applying (B.36) from [Iwa02] again, we deduce that

$$\lim_{X\to\infty} \frac{X}{2}\left[K_{s-\frac{1}{2}}(X)(I_{-\frac{1}{2}}(X)+I_{\frac{3}{2}}(X)) + I_{\frac{1}{2}}(X)(K_{s-\frac{3}{2}}(X)+K_{s+\frac{1}{2}}(X))\right] = 1.$$

Hence

$$\lim_{Y\to\infty} \frac{\partial}{\partial s} T_1(Y,s;w)|_{s=1} = -\frac{\partial}{\partial s} F_{-n}(w,s)|_{s=1}. \tag{36}$$

As for the term T_2 in (35), let us use the Laurent series expansion (24) of $\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)$, from which one easily deduces that

$$\frac{\partial}{\partial s}\left.(s-1)\frac{Y^{-s}}{2s-1}\mathcal{E}^{\mathrm{par}}_{\infty}(w,s)\right|_{s=1} = \frac{1}{Y}\left(\beta - \frac{P(w)+2+\log Y}{\mathrm{vol}_{\mathrm{hyp}}(M)}\right).$$

Therefore

$$\lim_{Y\to\infty} \frac{\partial}{\partial s} T_2(Y,s;w)|_{s=1} = 0. \tag{37}$$

It remains to study the term T_3 in (35). Let us set $g(s,y,k) := \sqrt{y}K_{s-\frac{1}{2}}(2\pi k y)$ for some positive integers k. Then $b_j(y,1;-n) = B_j(1;-n)g(1,y,|n|)$ and

$$T_3(Y,s;w) = \sum_{j\in\mathbb{Z}\smallsetminus\{0\}} B_j(1;-n)F_j(w,s)\left(g(1,Y,|n|)\frac{\partial}{\partial y}\, g(s,y,|j|)|_{y=Y}\right.$$
$$\left. - \, g(s,Y;|j|)\frac{\partial}{\partial y}\, g(1,y,|n|)|_{y=Y}\right).$$

For positive integers m and ℓ let us define

$$G(s,Y,m,\ell) := g(1,Y,m)\frac{\partial}{\partial y}\, g(s,y,\ell)|_{y=Y} - g(s,Y;\ell)\frac{\partial}{\partial y}\, g(1,y,m)|_{y=Y}.$$

Applying the formula 8.486.11 from [GR07] to differentiate the $K-$Bessel function, together with formula 8.486.10 to express $K_{s+\frac{1}{2}}(2\pi|j|Y)$ we arrive at

$$G(s,Y,|n|,|j|) = \frac{\pi Y}{2}K_{s-\frac{1}{2}}(2\pi|j|Y)K_{\frac{1}{2}}(2\pi|n|Y)\cdot$$
$$\cdot\left(|n|(K_{-\frac{1}{2}}(2\pi|n|Y)+K_{\frac{3}{2}}(2\pi|n|Y)) - |j|\left(2K_{s-\frac{3}{2}}(2\pi|j|Y) + \frac{2s-1}{2\pi|j|Y}K_{s-\frac{1}{2}}(2\pi|j|Y)\right)\right).$$

Now, we combine the bound (B.36) from [Iwa02] with evaluation of the derivative $\frac{\partial}{\partial\nu}K_\nu$ at $\nu=\pm 1/2$ (formula 8.486.21 of [GR07]) and the bound $\mathrm{Ei}(-4\pi|j|Y) \leq \exp(-4\pi|j|Y)/(4\pi|j|Y)$ for the exponential integral function to deduce the following crude bounds

$$\max\left\{G(s,Y,|n|,|j|),\ \left.\frac{\partial}{\partial s}G(s,Y,|n|,|j|)\right|_{s=1}\right\} \ll (|n|+|j|)\exp(-2\pi Y(|n|+|j|)),\ \text{as } Y\to+\infty,$$

where the implied constant is independent of $Y, |j|$.
This, together with the bound (18) and the Fourier expansion (17) yields

$$\frac{\partial}{\partial s}T_3(Y,s;w)\Big|_{s=1} \ll \sum_{j\in\mathbb{Z}\smallsetminus\{0\}}(|n|+|j|)|B_j(1;-n)|\exp\left(-2\pi Y(|n|+|j|)+2\pi|j|\mathrm{Im}(w)\right)$$

It remains to estimate the sum on the right hand side of the above equation as $Y\to\infty$. The bounds for the Kloosterman sum zeta function as stated on page 75 of [Iwa02]) yield bounds for $B_j(1;-n)$ for $j\neq 0$. Specifically, one has that

$$B_j(1;-n) \ll \exp\left(\frac{4\pi\sqrt{|jn|}}{c_\Gamma}\right)$$

where c_Γ is a certain positive constant depending on the group Γ; in fact, c_Γ is equal to the minimal positive left-lower entry of a matrix from Γ. Also, the implied constant in the bound for $B_j(1;-n)$ is independent of j. Therefore

$$\frac{\partial}{\partial s}\,T_3(Y,s;w)|_{s=1} \ll \sum_{j\in\mathbb{Z}\smallsetminus\{0\}}(|n|+|j|)\exp\left(-2\pi\left((|j|+|n|)Y-2\sqrt{|jn|}/c_\Gamma-|j|\mathrm{Im}(w)\right)\right).$$

For $Y>2\mathrm{Im}(w)+2\sqrt{n}/c_\Gamma$, this series over j is uniformly convergent and is $o(1)$ as $Y\to\infty$. In other words,

$$\lim_{Y\to\infty}\frac{\partial}{\partial s}\,T_3(Y,s;w)|_{s=1}=0. \tag{38}$$

When combining (38) with (35) (36) and (37), we have that

$$\langle F_{-n}(z,1),\overline{\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}^{\mathrm{par}}_\infty(w,s)\right)}\rangle=\frac{\partial}{\partial s}\,F_{-n}(w,s)|_{s=1}\,,$$

which completes the proof of (9).

5.2 Proof of Corollary 1

The proof of Corollary 1 is a combination of Theorem 1 and the factorization theorem as stated in Proposition 1. The details are as follows.
To begin we shall prove formula (7). Starting with (5), which is written as

$$\log\left(y^k|f(z)|\right)=kP(z)-\sum_{w\in\mathcal{F}}\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\lim_{s\to 0}\frac{1}{s}\mathrm{ord}(w)\mathcal{E}^{\mathrm{ell}}_w(z,s),$$

we can express $\lim_{s\to 0}\frac{1}{s}\mathrm{ord}(w)\mathcal{E}^{\mathrm{ell}}_w(z,s)$ in terms of the resolvent kernel. Specifically, using (6), we have that

$$\log\left(y^k|f(z)|\right)=kP(z)+\sum_{w\in\mathcal{F}}\frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)}\lim_{s\to 0}\left(\frac{2^s\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s+1)}(2s-1)G_s(z,w)\right). \tag{39}$$

By applying the functional equation for the Green's function, see Theorem 3.5 of [He83] on pages 250–251, we get

$$\lim_{s\to 0}\frac{2^s\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s+1)}(2s-1)G_s(z,w)=\lim_{s\to 1}\left(\frac{2^{1-s}\sqrt{\pi}\Gamma(1/2-s)}{\Gamma(2-s)}\left((1-2s)G_s(z,w)\right.\right.$$
$$\left.\left.-\ \mathcal{E}^{\mathrm{par}}_\infty(z,1-s)\mathcal{E}^{\mathrm{par}}_\infty(w,s)\right)\right).$$

From the Kronecker limit formula (25) and standard Taylor series expansion of the gamma function we immediately deduce that

$$\lim_{s\to 0}\frac{2^s\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s+1)}(2s-1)G_s(z,w)=\lim_{s\to 1}2\pi(-1+(s-1)(2-\log 2))\cdot \\ \cdot\left[2(1-s)G_s(z,w)-(G_s(z,w)+\mathcal{E}_\infty^{\mathrm{par}}(w,s))-P(z)(1-s)\mathcal{E}_\infty^{\mathrm{par}}(w,s)\right].$$

According to [Iwa02], p. 106, the point $s=1$ is the simple pole of $G_s(z,w)$ with the residue $-1/\operatorname{vol}_{\mathrm{hyp}}(M)$ (note that our $G_s(z,w)$ differs from the automorphic Green's function from [Iwa02] by a factor of -1). Therefore, the Kronecker limit formula (24) yields the following equation

$$\lim_{s\to 0}\frac{2^s\sqrt{\pi}\Gamma(s-1/2)}{\Gamma(s+1)}(2s-1)G_s(z,w)=-\frac{2\pi}{\operatorname{vol}_{\mathrm{hyp}}(M)}P(z)-\frac{4\pi}{\operatorname{vol}_{\mathrm{hyp}}(M)} \\ +2\pi\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}_\infty^{\mathrm{par}}(w,s)\right). \tag{40}$$

Recall that the classical the Riemann-Roch theorem implies that

$$k\frac{\operatorname{vol}_{\mathrm{hyp}}(M)}{2\pi}=\sum_{w\in\mathcal{F}}\frac{\operatorname{ord}_w(f)}{\operatorname{ord}(w)};$$

hence, after multiplying (40) by $\frac{\operatorname{ord}_w(f)}{\operatorname{ord}(w)}$ and taking the sum over all $w\in\mathcal{F}$ from (39), we arrive at (7), as claimed.

Having proved (7), observe that the left-hand side of (7) is real valued. As proved in [Ni73], $F_{-n}(z,1)$ is orthogonal to constant functions. Therefore, in order to prove (10) one simply applies (9), which was established above.

5.3 Proof of Corollary 2

In order to prove (12), it suffices to compute $\langle 1,\overline{\lim_{s\to 1}(G_s(z,w)+\mathcal{E}_\infty^{\mathrm{par}}(w,s))}\rangle$, which we will write as

$$\int\limits_{\mathcal{F}}\lim_{s\to 1}\left(G_s(z,w)+\frac{1}{\operatorname{vol}_{\mathrm{hyp}}(M)(s-1)}+\mathcal{E}_\infty^{\mathrm{par}}(w,s)-\frac{1}{\operatorname{vol}_{\mathrm{hyp}}(M)(s-1)}\right)d\mu_{\mathrm{hyp}}(z).$$

From its spectral expansion, the function $\lim_{s\to 1}\left(G_s(z,w)+\frac{1}{\operatorname{vol}_{\mathrm{hyp}}(M)(s-1)}\right)$ is L^2 on $\mathcal{F}$ and orthogonal to constant functions. Therefore, by using the Laurent series expansion (24), we get that

$$\langle 1,\overline{\lim_{s\to 1}(G_s(z,w)+\mathcal{E}_\infty^{\mathrm{par}}(w,s))}\rangle=\operatorname{vol}_{\mathrm{hyp}}(M)\left(\beta-\frac{P(w)}{\operatorname{vol}_{\mathrm{hyp}}(M)}\right),$$

which completes the proof.

5.4 Proof of Theorem 2

Our starting point is the Fourier expansion of the sum $G_s(z,w)+\mathcal{E}_\infty^{\mathrm{par}}(w,s)$. Namley, for $\operatorname{Re}(s)>1$ and $\operatorname{Im}(w)$ sufficiently large we have that

$$G_s(z,w)+\mathcal{E}_\infty^{\mathrm{par}}(w,s)=\left(1-\frac{y^{1-s}}{2s-1}\right)\mathcal{E}_\infty^{\mathrm{par}}(w,s) \\ -\sum_{k\in\mathbb{Z}\setminus\{0\}}\sqrt{y}K_{s-\frac{1}{2}}(2\pi|k|y)F_{-k}(w,s)e(kx). \tag{41}$$

If $\mathrm{Im}(z)$ is sufficiently large, exponential decay of $K_{s-\frac{1}{2}}(2\pi|k|y)$ is sufficient to ensure that the right-hand side of (41) is holomorphic at $s=1$. The Laurent series expansion of $\mathcal{E}_{\infty}^{\mathrm{par}}(w,s)$, combined with the expansions $y^{1-s}=1+(1-s)\log y+\frac{1}{2}(1-s)^2\log^2 y+O((1-s)^3)$ and $(2s-1)^{-1}=(1-2(s-1))^{-1}=1-2(s-1)+4(s-1)^2+O((s-1)^3)$ yields

$$\frac{\partial}{\partial s}\left(1-\frac{y^{1-s}}{2s-1}\right)\mathcal{E}_{\infty}^{\mathrm{par}}(w,s)\Big|_{s=1}=\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M)}\left[-4+2\beta\,\mathrm{vol}_{\mathrm{hyp}}(M)-2P(w)\right.$$
$$\left.+\log y\,(\beta\mathrm{vol}_{\mathrm{hyp}}(M)-P(w)-2)-\tfrac{1}{2}\log^2 y\right].$$

Additionally, for $\mathrm{Im}(z)$ sufficiently large, the series on the right-hand side of (41) is a uniformly convergent series of functions which are holomorphic at $s=1$. As such, we may differentiate the series term by term. By employing formulas 8.469.3 and 8.486.21 of [GR07], we deduce for $k\neq 0$ that

$$\frac{\partial}{\partial s}\left(\sqrt{y}K_{s-\frac{1}{2}}(2\pi|k|y)F_{-k}(w,s)\right)\Big|_{s=1}=\frac{e^{-2\pi|k|y}}{2\sqrt{|k|}}\cdot$$
$$\cdot\left[\frac{\partial}{\partial s}\,F_{-k}(w,s)|_{s=1}-F_{-k}(w,1)e^{4\pi|k|y}\mathrm{Ei}(-4\pi|k|y)\right],$$

where $\mathrm{Ei}(x)$ denotes the exponential integral function; see section 8.21 of [GR07]. From this, we get the expression that

$$\frac{\partial}{\partial s}\left(G_s(z,w)+\mathcal{E}_{\infty}^{\mathrm{par}}(w,s)\right)\Big|_{s=1}=(\log y+2)\left(\beta-\frac{P(w)+2}{\mathrm{vol}_{\mathrm{hyp}}(M)}\right)-\frac{\log^2 y}{2\mathrm{vol}_{\mathrm{hyp}}(M)}$$
$$-\sum_{k\in\mathbb{Z}\setminus\{0\}}\frac{1}{2\sqrt{|k|}}\left[\frac{\partial}{\partial s}\,F_{-k}(w,s)|_{s=1}-F_{-k}(w,1)e^{4\pi|k|y}\mathrm{Ei}(-4\pi|k|y)\right]e^{2\pi ikx-2\pi|k|y}.$$

Let us now compute the derivative $\frac{\partial}{\partial z}$ of the above expression. After multiplying by $i=\sqrt{-1}$, we get that

$$\mathcal{G}_w(z)=\frac{1}{y}\left(\beta-\frac{P(w)+2}{\mathrm{vol}_{\mathrm{hyp}}(M)}\right)-\frac{\log y}{y\mathrm{vol}_{\mathrm{hyp}}(M)}+\sum_{k\geq 1}2\pi\sqrt{k}\frac{\partial}{\partial s}\,F_{-k}(w,s)|_{s=1}\,q_z^k$$
$$+\sum_{k\geq 1}\frac{F_{-k}(w,1)}{2\sqrt{k}y}q_z^k-\sum_{k\leq -1}2\pi\sqrt{|k|}F_{-k}(w,1)\mathrm{Ei}(4\pi ky)q_z^k+\sum_{k\leq -1}\frac{F_{-k}(w,1)}{2\sqrt{|k|}y}e^{2\pi ik(x-iy)}.$$

The proof of the assertion that $\sum_{k\geq 1}2\pi\sqrt{k}\frac{\partial}{\partial s}\,F_{-k}(w,s)|_{s=1}\,q_z^k$ is the holomorphic part of $\mathcal{G}_w(z)$ follows by citing the uniqueness of the analytic continuation in z.

It is left to prove that $\mathcal{G}_w(z)$ is weight two biharmonic Maass form. Since $\mathcal{G}_w(z)$ is obtained by taking the derivative $\frac{\partial}{\partial z}$ of a $\Gamma-$invariant function, it is obvious that $\mathcal{G}_w(z)$ is weight two in z. Moreover, the straightforward computation that

$$iy^2\frac{\partial}{\partial\bar{z}}\mathcal{G}_w(z)=\Delta_{\mathrm{hyp}}\left(\frac{\partial}{\partial s}\left(G_s(z,w)+\mathcal{E}_{\infty}^{\mathrm{par}}(w,s)\right)\Big|_{s=1}\right)=-\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}_{\infty}^{\mathrm{par}}(w,s)\right),$$

combined with the fact that $\Delta_{\mathrm{hyp}}\left(\lim_{s\to 1}\left(G_s(z,w)+\mathcal{E}_{\infty}^{\mathrm{par}}(w,s)\right)\right)=0$ proves that $\mathcal{G}_w(z)$ is biharmonic.

6 Examples

6.1 The full modular group

Throughout this subsection, let $\Gamma = \mathrm{PSL}(2,\mathbb{Z})$, in which case the the parabolic Kronecker limit function, $P(w)$ can be expressed, in the notation of [BK20], as

$$P(w) = P_{\mathrm{PSL}(2,\mathbb{Z})}(w) = \log(|\eta(w)|^4 \cdot \mathrm{Im}(w)) = \mathbb{j}(w) - 1,$$

where $\eta(w)$ is Dedekind's eta function and the last equality follows from the definition of $\mathbb{j}_0(w) = \mathbb{j}(w)$ given on p. 1 of [BK20].

In this setting, Corollary 1, when combined with (3) and Rohrlich's theorem (2) yields that

$$\langle j_n, \log ||f|| \rangle = 2\pi\sqrt{n}\left(-2\pi \sum_{w\in\mathcal{F}} \frac{\mathrm{ord}_w(f)}{\mathrm{ord}(w)} \left(\frac{\partial}{\partial s}\, F_{-n}(w,s)\big|_{s=1} - c_n P(w)\right)\right). \tag{42}$$

Moreover, equating the constant terms in the Fourier series expansions for $F_{-n}(z,1)$ and $j_n(z)$, one easily deduces that $2\pi\sqrt{n}c_n = 24\sigma(n)$. This proves Theorem 1.2 of [BK20] and shows that, in the notation of [BK20] one has

$$\mathbb{j}_n(w) = 2\pi\sqrt{n}\frac{\partial}{\partial s}\, F_{-n}(w,s)\big|_{s=1} - 24\sigma(n)P(w), \tag{43}$$

an identity which provides a description of $\mathbb{j}_n(w)$, for $n \geq 1$ different from the one given by formula (3.10) of [BK20]. Furthermore, from the identity (19), combined with the fact that $\Delta_{\mathrm{hyp}}P(w) = 1$, which is a straightforward implication of the Kronecker limit formula (24), it follows that

$$\Delta_{\mathrm{hyp}}\mathbb{j}_n(w) = 2\pi\sqrt{n}\,(F_{-n}(w,1) - c_n) = j_n(w),$$

which agrees with formula (3.10) of [BK20].

Reasoning as above, we easily see that Theorem 1.3. of [BK20] follows from Corollary 2 with $g(z) = j_n(z)$.

Finally, in view of (42), Theorem 2 is closely related to the first part of Theorem 1.4 of [BK20]. Namely, for large enough $\mathrm{Im}(z)$, in the notation of [BK20]

$$\begin{aligned}\mathbb{H}_w(z) &= \sum_{n\geq 0} \mathbb{j}_n(w) q_z^n = \mathbb{j}_0(w) + \sum_{n\geq 1}\left(2\pi\sqrt{n}\frac{\partial}{\partial s}\, F_{-n}(w,s)\big|_{s=1} - 24\sigma(n)P(w)\right) q_z^n \\ &= 1 + P(w)\left(1 - 24\sum_{n\geq 1}\sigma(n)q_z^n\right) + \sum_{n\geq 1} 2\pi\sqrt{n}\frac{\partial}{\partial s}\, F_{-n}(w,s)\big|_{s=1}\, q_z^n.\end{aligned}$$

Theorem 2 implies that the function $\mathbb{H}_w(z)$ is the holomorphic part of the weight two biharmonic Maass form

$$\widehat{\mathbb{H}}_w(z) = P(w)\widehat{E}_2(z) + \mathcal{G}_w(z),$$

where

$$\widehat{E}_2(z) = 1 - 24\sum_{n\geq 1}\sigma(n)q_z^n - \frac{3}{\pi y}$$

is the weight two completed Eisenstein series for the full modular group.

Remark 3. The identity (43) also appears on p. 99 of [JKK14]. Further, it is observed in [JKK14] that $h_n^*(w) = \xi_0\left(\frac{\partial}{\partial s} F_{-n}(w,s)|_{s=1}\right)$ is a harmonic weak Maass form of weight 2 for which $\xi_2(h_n^*(w)) = j_n(w) + 24\sigma(n)$ and where $\xi_\kappa := 2iy^\kappa \overline{\frac{\partial}{\partial \bar{z}}}$. Moreover, in Section 4 of [JKK14] it is proved that each $h_n(w)$ is a harmonic Maass forms with bounded holomorphic parts. Additionally, it is shown in [JKK14] that for each $n > 0$ one has that $h_n^*(w) = 4\pi h_n(w)$, where the set $\{h_m(w)\}_{m\in\mathbb{Z}}$ is a basis for the space $\mathcal{V}$ of weight 2 harmonic weak Maass forms; the basis was constructed in [DIT16].

6.2 Genus zero Atkin-Lehner groups

Let $N = \prod_{\nu=1}^r p_\nu$ be a positive square-free integer which is one of the 44 possible values for which the quotient space $Y_N^+ = \overline{\Gamma_0^+(N)}\backslash\mathbb{H}$ has genus zero; see [Cum04] for a list of such N as well as [JST16b]. Let $\Delta_N(z)$ be the Kronecker limit function on Y_N^+ associated to the parabolic Eisenstein series; it is given by formula (26) above.

In the notation of Section 4.2,the function $\Delta_N(z)(j_N^+(z) - j_N^+(w))^{\nu_N}$, is the weight $k_N = 2^{r-1}\ell_N$ holomorphic modular form which possesses the constant term 1 in its q-expansion. Furthermore, this function vanishes only at the point $z = w$, and, by the Riemann-Roch formula, its order of vanishing is equal to $k_N \operatorname{vol}_{\text{hyp}}(Y_N^+) \cdot \operatorname{ord}(w)/(4\pi)$.

When $N = 1$, one has $k_1 = 12$, $\ell_1 = 24$, $\nu_1 = 1$ and $\operatorname{vol}_{\text{hyp}}(Y_N^+) = \pi/3$, hence $\Delta_1(z)(j_1^+(z) - j_1^+(w))^{\nu_1}$ equals the prime form $(\Delta(z)(j(z) - j(w)))^{1/\operatorname{ord}(w)}$ taken to the power $\operatorname{ord}(w)$; see page 3 of [BK20].

For any integer $m > 1$ the q-expansion of the form $j_N^+|T_m(z)$ is $q_z^{-m} + O(q_z)$; hence there exists a constant $C_{m,N}$ such that $j_N^+|T_m(z) = 2\pi\sqrt{m}F_{-m}(z,1) + C_{m,N}$. The constant $C_{m,N}$ can be explicitly evaluated in terms of m and N by equating the constant terms in the q-expansions. Upon doing so, one obtains, using equation (20), that

$$C_{m,N} = -2\pi\sqrt{m}B_{0,N}^+(1;-m) = -24\sigma(m)\prod_{\nu=1}^r\left(1 - \frac{p_\nu^{\alpha_{p_\nu}(m)+1}(p_\nu - 1)}{\left(p_\nu^{\alpha_{p_\nu}(m)+1} - 1\right)(p_\nu + 1)}\right)$$
$$= -24\sigma(m)\prod_{\nu=1}^r\left(1 - \kappa_m(p_\nu)\right),$$

where we simplified the notation by denoting the second term in the product over ν by $\kappa_m(p_\nu)$. We now can apply Corollary 2 with

$$g(z) = j_N^+|T_m(z) = 2\pi\sqrt{m}F_{-m}(z,1) - 24\sigma(m)\prod_{\nu=1}^r\left(1 - \kappa_m(p_\nu)\right)$$

and $f(z) = \Delta_N(z)(j_N^+(z) - j_N^+(w))^{\nu_N}$. Corollary 2 becomes the statement that

$$\langle j_N^+|T_m(z), \log(y^{\frac{k_N}{2}}|\Delta_N(z)(j_N^+(z) - j_N^+(w))^{\nu_N}|)\rangle$$
$$= -k_N \operatorname{vol}_{\text{hyp}}(Y_N^+)\left[\pi\sqrt{m}\,\frac{\partial}{\partial s}F_{-m}(w,s)\Big|_{s=1}\right.$$
$$\left.+12\sigma(m)\prod_{\nu=1}^r\left(1 - \kappa_m(p_\nu)\right)\left(\beta_N \operatorname{vol}_{\text{hyp}}(Y_N^+) - \log\left(|\Delta_N(w)|^{2/k_N}\cdot\operatorname{Im}(w)\right) - 2\right)\right],$$

where β_N is given by (28). In this form, we have obtained an alternate proof and generalization of formula (1.2) from [BK20], which is the special case $N = 1$.

6.3 A genus one example

Let us consider the case when $\Gamma = \overline{\Gamma_0(37)^+}$. The choice of $N = 37$ is significant since this level corresponds to the smallest square-free integer N such that Y_N^+ is genus one. From Proposition 11 of [JST16], we have that $\mathrm{vol}_{\mathrm{hyp}}(Y_{37}^+) = 19\pi/3$ and

$$\beta_{37} = \frac{3}{19\pi}\left(\frac{10}{19}\log 37 + 2 - 2\log(4\pi) - 24\zeta'(-1)\right).$$

The function field generators are $x_{37}^+(z) = q_z^{-2} + 2q_z^{-1} + O(q_z)$ and $y_{37}^+(z) = q_z^{-3} + 3q_z^{-1} + O(q_z)$, as displayed in Table 5 of [JST16]. The generators $x_{37}^+(z)$ and $y_{37}^+(z)$ satisfy the cubic relation $y^2 - x^3 + 6xy - 6x^2 + 41y + 49x + 300 = 0$.

The functions $x_{37}^+(z)$ and $y_{37}^+(z)$ can be expressed in in terms of the Niebur-Poincaré series by comparing their q-expansions. The resulting expressions are that

$$\begin{aligned} x_{37}^+(z) &= 2\pi[\sqrt{2}F_{-2}(z,1) + 2F_{-1}(z,1)] - 2\pi(\sqrt{2}B_{0,37}^+(1;-2) + 2B_{0,37}^+(1;-1)) \\ &= 2\pi[\sqrt{2}F_{-2}(z,1) + 2F_{-1}(z,1)] - \frac{60}{19} \end{aligned}$$

and

$$\begin{aligned} y_{37}^+(z) &= 2\pi[\sqrt{3}F_{-3}(z,1) + 3F_{-1}(z,1)] - 2\pi(\sqrt{3}B_{0,37}^+(1;-3) + 3B_{0,37}^+(1;-1)) \\ &= 2\pi[\sqrt{3}F_{-3}(z,1) + +3F_{-1}(z,1)] - \frac{84}{19}. \end{aligned}$$

It is important to note that $x_{37}^+(z)$ has a pole of order two at $z = \infty$, i.e., its q-expansion begins with q_z^{-2}. As such, $x_{37}^+(z)$ is a linear transformation of the Weierstrass $\wp$-function, in the coordinates of the upper half plane, associated to the elliptic curve obtained by compactifying the space Y_{37}^+. Hence, there are three distinct points $\{w\}$ on Y_{37}^+, corresponding to the two torsion points under the group law, such that $x_{37}^+(z) - x_{37}^+(w)$ vanishes as a function of z only when $z = w$. The order of vanishing necessarily is equal to two. The cusp form $\Delta_{37}(z)$ vanishes at ∞ to order 19. Therefore, for such w, the form

$$f_{37,w}(z) = \Delta_{37}^2(z)(x_{37}^+(z) - x_{37}^+(w))^{19}$$

is a weight $2k_{37} = 24$ holomorphic form. The constant term in its q-expansion is equal to 1, and $f_{37,w}(z)$ vanishes for points $z \in \mathcal{F}$ only when $z = w$. The order of vanishing of $f_{37,w}(z)$ at $z = w$ is $38 \cdot \mathrm{ord}(w)$.

With all this, we can apply Corollary 2. The resulting formulas are that

$$\begin{aligned} \langle x_{37}^+, \log(\|f_{37,w}\|)\rangle &= -152\pi^2\left(\frac{\partial}{\partial s}\left(\sqrt{2}F_{-2}(w,s) + 2F_{-1}(w,s)\right)\Big|_{s=1}\right) \\ &\quad + 240\pi\left(\log\left(|\eta(w)\eta(37w)|^2 \cdot \mathrm{Im}(w)\right) - \frac{10}{19}\log 37 + 2\log(4\pi) + 24\zeta'(-1)\right) \end{aligned}$$

and

$$\begin{aligned} \langle y_{37}^+, \log(\|f_{37,w}\|)\rangle &= -152\pi^2\left(\frac{\partial}{\partial s}\left(\sqrt{3}F_{-3}(w,s) + 3F_{-1}(w,s)\right)\Big|_{s=1}\right) \\ &\quad + 336\pi\left(\log\left(|\eta(w)\eta(37w)|^2 \cdot \mathrm{Im}(w)\right) - \frac{10}{19}\log 37 + 2\log(4\pi) + 24\zeta'(-1)\right). \end{aligned}$$

Of course, one does not need to assume that w corresponds to a two torsion point. In general, Corollary 2 yields an expression where the right-hand side is a sum of two terms, and the corresponding factor in front would be one-half of the factors above.

6.4 A genus two example

Consider the level $N = 103$. In this case, $\mathrm{vol}_{\mathrm{hyp}}(Y_{103}^+) = 52\pi/3$ and the function field generators are $x_{103}^+(z) = q_z^{-3} + q_z^{-1} + O(q_z)$ and $y_{103}^+(z) = q_z^{-4} + 3q_z^{-2} + 3q_z^{-1} + O(q_z)$, as displayed in Table 7 of [JST16]. The generators $x_{103}^+(z)$ and $y_{103}^+(z)$ satisfy the polynomial relation $y^3 - x^4 - 5yx^2 - 9x^3 + 16y^2 - 21yx - 60x^2 + 65y - 164x + 18 = 0$. The surface Y_{103}^+ has genus two.

From Theorem 6 of [Ni73], we can write $x_{103}^+(z)$ and $y_{103}^+(z)$ in terms of the Niebur-Poincaré series. Explictly, we have that

$$\begin{aligned} x_{103}^+(z) &= 2\pi[\sqrt{3}F_{-3}(z,1) + F_{-1}(z,1)] - 2\pi(\sqrt{3}B_{0,103}^+(1;-3) + B_{0,103}^+(1;-1)) \\ &= 2\pi[\sqrt{3}F_{-3}(z,1) + F_{-1}(z,1)] - \frac{15}{13} \end{aligned}$$

and

$$\begin{aligned} y_{103}^+(z) &= 2\pi[\sqrt{4}F_{-4}(z,1) + 3\sqrt{2}F_{-2}(z,1) + 3F_{-1}(z,1)] \\ &\quad - 2\pi(\sqrt{4}B_{0,103}^+(1;-4) + 3\sqrt{2}B_{0,103}^+(1;-2) + 3B_{0,103}^+(1;-1)) \\ &\quad = 2\pi[2F_{-4}(z,1) + 3\sqrt{2}F_{-2}(z,1) + 3F_{-1}(z,1)] - \frac{57}{13}. \end{aligned}$$

The order of vanishing of $\Delta_{103}(z)$ at the cusp is $\nu_{103} = (12 \cdot 52\pi/3)/(4\pi) = 52$. Therefore, for an arbitrary, fixed $w \in \mathbb{H}$, the form

$$f_{103,w}(z) = \Delta_{103}^3(z)(x_{103}^+(z) - x_{103}^+(w))^{52}$$

is the weight $3k_{103} = 36$ holomorphic form which has constant term in the q-expansion equal to 1. Let $\{w_1, w_2, w_3\}$ be the three, not necessarily distinct, points in the fundamental domain $\mathcal{F}$ where $(x_{103}^+(z) - x_{103}^+(w))$ vanishes. One of the points w_j is equal to w. The form $f_{103,w_j}(z)$ vanishes at $z = w_j$ to order $52 \cdot \mathrm{ord}(w_j)$, $j = 1, 2, 3$.

From Section 4.2, we have that

$$\beta_{103} = \frac{3}{52\pi}\left(\frac{53}{104}\log 103 + 2 - 2\log(4\pi) - 24\zeta'(-1)\right)$$

and $P_{103}(z) = \log\left(|\eta(z)\eta(103z)|^2 \cdot \mathrm{Im}(z)\right)$. Let us now apply Corollary 2 with $g(z) = x_{103}^+(z)$, in which case $c(g) = -15/13$. In doing so, we get that

$$\begin{aligned} \langle x_{103}^+, \log(\|f_{103,w}\|)\rangle &= -208\pi^2 \sum_{j=1}^{3}\left(\frac{\partial}{\partial s}\left(\sqrt{3}F_{-3}(w_j,s) + F_{-1}(w_j,s)\right)\Big|_{s=1}\right) \\ &\quad + 120\pi \sum_{j=1}^{3}\left(\log\left(|\eta(w_j)\eta(103w_j)|^2 \cdot \mathrm{Im}(w_j)\right)\right) \\ &\quad - 360\pi\left(\frac{53}{104}\log 103 - 2\log(4\pi) - 24\zeta'(-1)\right). \end{aligned}$$

Similarly, we can take $g(z) = y^+_{103}(z)$, in which case $c(g) = -57/13$ and we get that

$$\begin{aligned}\langle y^+_{103}, \log(\|f_{103,w}\|)\rangle = -208\pi^2 \sum_{j=1}^{3} \left(\frac{\partial}{\partial s} \left(2F_{-4}(w_j,s) + 3\sqrt{2}F_{-2}(w_j,s) + 3F_{-1}(w_j,s)\right)\Big|_{s=1} \right) \\ + 456\pi \sum_{j=1}^{3} \left(\log\left(|\eta(w_j)\eta(103w_j)|^2 \cdot \mathrm{Im}(w_j)\right)\right) \\ - 1368\pi \left(\frac{53}{104} \log 103 - 2\log(4\pi) - 24\zeta'(-1) \right).\end{aligned}$$

6.5 An alternative formulation

In the above discussion, we have written the constant β and the Kronecker limit function P separately. However, it should be pointed out that in all instances the appearance of these terms are in the combination $\beta\,\mathrm{vol}_{\mathrm{hyp}}(M) - P(z)$. From (24), we can write

$$\beta\,\mathrm{vol}_{\mathrm{hyp}}(M) - P(z) = \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(M)} \mathrm{CT}_{s=1} \mathcal{E}^{\mathrm{par}}_{\infty}(z,s),$$

where $\mathrm{CT}_{s=1}$ denotes the constant term in the Laurent expansion at $s = 1$. It may be possible that such notational change can provide additional insight concerning the formulas presented above.

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