L-FUNCTIONS AND FUNCTORIALITY

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The principle of functoriality is one of the central tenets of the Langlands program. It is a purely automorphic avatar of Langlands' vision of a non-abelian class field theory. One can find an outline of this in Section 5 below. There are two main approaches to functoriality. The one envisioned by Langlands is through the Arthur-Selberg trace formula. With the recent proof of the Fundamental Lemma by Ngô, Waldspurger, and others this method is now available and will be the subject of a forthcoming book of Jim Arthur [1]. The second method is that of L-functions.

The method of L-functions was pioneered by Piatetski-Shapiro. It primarily deals with functoriality in the case where the target group is GL_n. The fundamental tool here is the converse theorem for GL_n, as explained in Section 6 below. As Piatetski-Shapiro said "Arthur's approach is more general, but this approach is much easier." (see Shahidi's contribution to [6]). The converse theorem is a way to tell when a representation of $GL_n(\mathbb{A})$ is automorphic based on the analytic properties of its L-functions (see Section 2 below). As a vehicle for functoriality, the input to the converse theorem must be checked, and this is done by controlling the L-functions of the automorphic representations to be functorially transferred. There are two ways to control these L-functions: via integral representations and via Eisenstein series, or the Langlands-Shahidi method. For the examples discussed here, these are controlled by the Langlands-Shahidi method, which we discuss in Sections 3-5. The functorial liftings themselves are obtained in Sections 7 and 8.

While Piatetski-Shapiro viewed this method as being simpler, it is in many ways more flexible and still mysterious. For example, as presented in Section 8, by this method you can obtain the third and fourth symmetric power lifting from GL_2 and these are still not attainable by the trace formula. In general, whenever you can control enough twisted L-functions, you can apply the converse theorem and obtain functoriality to GL_n. Of course controlling these L-functions is hard and for most of them we do not have a way to analyze them: they either fall outside the range of the Langlands-Shahidi method or we do not yet know if we can find integral representations for them. So there is still much work to be done.

These notes represent the lectures I gave at the CIMPA-UNESCO-CHINA Research School on *Automorphic forms and L-functions* in August of 2010. Their purpose was to present an introduction to the L-function approach

The author was supported in part by NSF Grants DMS-0654017 and DMS-0968505.

to functoriality. My original lectures were to be on GL_n and functoriality. These comprise Sections 1 and 2 here, which present the theory of automorphic representations and L-functions for GL_n, including the converse theorem, and then Sections 6 and 7 which were devoted to an exposition of functoriality via L-functions and then the example of functoriality for the classical groups. However, Shahidi was not able to attend the conference at the last moment, so I also gave an informal introduction to the Langlands-Shahidi method, based on Shahidi's notes [10]. These appear as Sections 3-5 and 8 here. These sections can be viewed as a "gentle introduction" to [10] and I have tried to indicate where the results here can be found in Shahidi's contribution. However, having the opportunity to give a single self-contained introduction to our approach to functoriality by the method of L-functions, I took the opportunity to integrate both series of lecture into a single contribution. I hope the reader finds this useful.

A word on references. I have surveyed the material in Sections 1,2,6, and 7 many times. Rather than burden this set of informal notes with pages of references, I refer instead to the sources I used for these talks, which are mainly my previous surveys. One can find more extensive references there. Similarly, for Sections 3,4,5 and 8 I have included in the bibliography the sources that I have used, which were surveys by Kim and Shahidi, Shahidi's new book, and Shahidi's contribution to this volume. I hope the reader does not mind this informality.

Finally I would like to thank Jianya Liu, the faculty at Shandong University, and all the students that attended for giving me the opportunity for presenting these lectures.

I. L-functions for $GL(n)$ and Converse Theorems

1. Modular forms and automorphic representations

1.1. **Classical modular forms and their L-functions.** We begin our tale by recalling the classical results of Hecke and Weil.

Let $\mathfrak{H} = \{z = x + iy \mid y > 0\}$ denote the complex upper half-plane. Then a (classical) modular form of weight k for $\Gamma = SL_2(\mathbb{Z})$ is a function $f : \mathfrak{H} \to \mathbb{C}$ which is holomorphic on $\mathfrak{H}$ and at the cusps of Γ and for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ we have

$$f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z).$$

We denote the space of these by $M_k(\Gamma)$. These arise in the theory of theta series, elliptic modular forms, etc.. Since $f(z+1) = f(z)$, by modularity under $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, this will have a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z}$$

and is called a *cusp form* if

$$a_0 = \int_0^1 f(x+iy)\ dx = 0.$$

Recall that $\Gamma\backslash\mathfrak{H}$ is the Riemann sphere with the "cusp at ∞" removed, so cusp forms are those modular forms that vanish at the cusps. The subspace of cusp forms of weight k is denoted by $S_k(\Gamma)$.

To each cusp form Hecke associated an L-function given by the Dirichlet series

$$L(s,f) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

which is absolutely convergent for $\mathrm{Re}(s) > \frac{k}{2} + 1$. The L-function is related to f through a Mellin transform:

$$\Lambda(s,f) = (2\pi)^{-s}\Gamma(s)L(s,f) = \int_0^{\infty} f(iy)y^s\ d^{\times}y.$$

By transferring the analytic properties of f to $L(s,f)$, Hecke showed:

Theorem 1.1. *If $f \in S_k(\Gamma)$ then $\Lambda(s,f)$ is nice, i.e.,*

(1) *$\Lambda(s,f)$ extends to an entire function of s,*
(2) *$\Lambda(s,f)$ is bounded in vertical strips (BVS),*
(3) *$\Lambda(s,f)$ satisfied the functional equation (FE)*

$$\Lambda(s,f) = (i)^k\Lambda(k-s,f).$$

Note that the functional equation results from the modular transformation law of $f(z)$ under $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, namely $f(Sz) = f\left(\frac{-1}{z}\right) = z^k f(z)$.

Remark 1.1. In general $L(s,f)$ need not have an *Euler product*. Hecke introduced an algebra of operators $\mathcal{H} = \langle T_p \mid p \text{ prime}\rangle$, the original Hecke algebra, such that if f is a simultaneous eigen-function for all T_n, i.e., $T_n f = \lambda_n f$, and if we normalize f such that $a_1 = 1$ then $\lambda_n = a_n$ and we have

$$L(s,f) = \prod_p (1 - a_p p^{-s} + p^{2k-1}p^{-2s})^{-1}.$$

Hecke was able to invert this process, via the inverse Mellin transform, and prove the "Hecke Converse Theorem".

Theorem 1.2. *Let*

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

be a Dirichlet series that converges in some right half-plane. Set $\Lambda(s) = (2\pi)^{-s}\Gamma(s)D(s)$. *If* $\Lambda(s)$ *is* nice *as above, i.e., (1) entire continuation, (2) BVS, and (3) satisfies the functional equation* $\Lambda(s) = (i)^k\Lambda(k-s)$, *then*

$$f(z) = \sum_{n=1}^{\infty} a_n e^{2\pi inz}$$

is a cusp form of weight k *for* $\Gamma = SL_2(\mathbb{Z})$.

Note that by the Fourier expansion $f(z+1) = f(z)$, so $f(z)$ is modular under the translation matrix T. Since $SL_2(\mathbb{Z})$ is generated by T and the inversion S, we only need the transformation law under S. This follows from the functional equation for $\Lambda(s)$ via Mellin inversion

$$f(iy) = \sum_{n=1}^{\infty} a_n e^{-2\pi ny} = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=2} \Lambda(s)y^{-s}\, ds.$$

If f is a cusp form not for all of $SL_2(\mathbb{Z})$ but say for the Hecke congruence group of level N

$$\Gamma = \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \middle| c \equiv 0 \pmod{N} \right\}$$

then the inversion S is no longer in Γ. Weil still defined $L(s,f)$ for $f \in S_k(\Gamma)$ and proved they were nice, using now $S_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$, which normalizes $\Gamma_0(N)$, in place of S. (Now the functional equation relates f and a form g related to f through S_N.) To invert this process, i.e., to establish the "Weil Converse Theorem", Weil showed that besides knowing that $D(s)$ was nice, one had to also control twisted Dirichlet series; one needed that the

$$D_\chi(s) = \sum_{n=1}^{\infty} \frac{\chi(n)a_n}{n^s}$$

were *nice* (essentially) for all Dirichlet characters χ of conductor relatively prime to N. The important thing to note here is that one Dirichlet series no longer suffices for inversion, one must control a family of twisted Dirichlet series as well.

1.2. **Automorphic representations of $GL(n)$.** The modern theory of automorphic representations is a theory of functions on adele groups. To make the connection note that

$$\mathfrak{H} = PGL_2^+(\mathbb{R})/PSO_2(\mathbb{R}).$$

So functions of $\mathfrak{H}$ can be lifted to functions on $GL_2(\mathbb{R})$. So we now have functions on a group.

To make an adelic theory, take k a number field ... but for now we can take $k = \mathbb{Q}$. The the adele ring of $\mathbb{Q}$ is

$$\mathbb{A} = \mathbb{A}_{\mathbb{Q}} = \varinjlim_{S} (\mathbb{R} \times \prod_{p \in S} \mathbb{Q}_p \times \prod_{p \notin S} \mathbb{Z}_p) = {\prod_v}' \mathbb{Q}_v$$

where the limit is over all finite subsets S of primes of $\mathbb{Z}$. Then $\mathbb{Q} \hookrightarrow \mathbb{A}$ as a discrete, co-compact subgroup with $\mathbb{Q} \cap (\mathbb{R} \times \prod_p \mathbb{Z}_p) = \mathbb{Z}$, so that $\mathbb{Z}\backslash\mathbb{R} \simeq \mathbb{Q}\backslash\mathbb{A}/\prod_p \mathbb{Z}_p$.

Similarly, for $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$,

$$GL_n(\mathbb{A}) = \varinjlim_{S} (GL_n(\mathbb{R}) \times \prod_{p \in S} GL_n(\mathbb{Q}_p) \times \prod_{p \notin S} GL_n(\mathbb{Z}_p)) = {\prod_v}' GL_n(\mathbb{Q}_v)$$

and $GL_n(\mathbb{Q}) \hookrightarrow GL_n(\mathbb{A})$ as a canonical discrete subgroup. While the quotient $GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})$ is no longer compact, it is finite volume modulo the center $Z_n(\mathbb{A}) \simeq \mathbb{A}^\times$.

The analogue of the space of classical modular forms is the space of automorphic forms

$$\mathcal{A}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}); \omega)$$

consisting of functions $\varphi : GL_n(\mathbb{A}) \to \mathbb{C}$ such that $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in GL_n(\mathbb{Q})$ plus regularity and growth conditions to match the classical conditions of holomorphy and holomorphy at the cusps. Here ω is a fixed unitary central character, i.e., φ satisfies $\varphi(zg) = \omega(z)\varphi(g)$ for $z \in Z_n(\mathbb{A})$. $GL_n(\mathbb{A})$ acts on the space $\mathcal{A}$ by right translations. The irreducible subquotients of this representation are the *automorphic representations* (π, V_π). In this passage we have traded the tool of one complex variable (i.e., holomorphy) for the tool of non-abelian harmonic analysis.

The notion of a cusp form models the classical idea that translation integrals are zero.

Definition 1.1. *$\varphi \in \mathcal{A}$ is a cusp form if for each pair of nonzero integers n_1, n_2 with $n_1 + n_2 = n$ we have*

$$\int_{M_{n_1,n_2}(\mathbb{Q})\backslash M_{n_1,n_2}(\mathbb{A})} \varphi\left(\begin{pmatrix} I_{n_1} & X \\ & I_{n_2} \end{pmatrix} g\right) \, dX = 0.$$

Note that the subgroups

$$N_{n_1,n_2} = \left\{ \begin{pmatrix} I_{n_1} & X \\ & I_{n_2} \end{pmatrix} \middle| X \in M_{n_1,n_2} \right\}$$

are the unipotent radicals of the two-block maximal parabolic subgroups P_{n_1,n_2} of GL_n.

Let $\mathcal{A}_0 \subset \mathcal{A}$ denote the space of cusp forms. Then a theorem of Gelfand and Piatetski-Shapiro says that in general the space of cusp forms is completely reducible with finite multiplicities:

$$\mathcal{A}_0(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}); \omega) = \bigoplus m(\pi) V_\pi.$$

(In fact, for GL_n, $m(\pi) = 0$ or 1.) The (π, V_π) that occur with non-zero multiplicities are the *cuspidal automorphic representations*. Moreover, just as $GL_n(\mathbb{A}) = \prod_v{}' GL_n(\mathbb{Q}_v)$ we have

$$(\pi, V_\pi) \simeq \otimes_v{}'(\pi_v, V_{\pi_v})$$

where each (π_v, V_{π_v}) is an irreducible, smooth, admissible representation of $GL_n(\mathbb{Q}_v)$ and for almost all v, the representation π_v has a unique $K_v = GL_n(\mathbb{Z}_v)$-fixed vector, i.e., is *unramified*. (N.B. This abstract restricted tensor decomposition of an irreducible representation will be responsible for the Euler product of the L-function. It is the adelic avatar of the Hecke operators.)

Key to Hecke's theory was the notion of a Fourier expansion. Here there is a Fourier expansion for cusp forms for GL_n due to Piatetski-Shapiro and Shalika. It is as follows. Let

$$N = \left\{ n = \begin{pmatrix} 1 & x_{1,2} & \cdots & * \\ & \ddots & \ddots & \vdots \\ 0 & & \ddots & x_{n-1,n} \\ 0 & 0 & & 1 \end{pmatrix} \right\}$$

be the maximal unipotent subgroup of GL_n If $\psi : \mathbb{Q}\backslash\mathbb{A} \to \mathbb{C}$ is a non-trivial additive character of $\mathbb{A}$, invariant by translations by $\mathbb{Q}$, then ψ defines a character of $N(\mathbb{A})$, left invariant by $N(\mathbb{Q})$, via

$$\psi(n) = \psi(x_{1,2} + \cdots + x_{n-1,n}).$$

If $\varphi \in \mathcal{A}_0$ set

$$W_\varphi(g) = \int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \varphi(ng)\psi^{-1}(n)\ dn.$$

Then $W_\varphi(ng) = \psi(n)W_\varphi(g)$ for all $n \in N(\mathbb{A})$. W_φ is called the *Whittaker function* of φ and it is what occurs in the following Fourier expansion of φ.

Theorem 1.3. *If $\varphi \in \mathcal{A}_0$ then*

$$\varphi(g) = \sum_{\gamma \in N_{n-1}(\mathbb{Q})\backslash GL_{n-1}(\mathbb{Q})} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right).$$

This turns out to be the proper notion of a Fourier expansion in this context. (The proof isn't hard once you realize this is the correct notion – it is simply an inductive abelian Fourier expansion beginning along the last column of N.)

1.3. **Uniqueness of Whittaker models.**

The space of functions

$$\mathcal{W}(\pi, \psi) = \left\{ W_\varphi(g) \middle| \varphi \in V_\pi \right\}$$

gives a model of π in the space of functions W on $GL_n(\mathbb{A})$ satisfying $W(ng) = \psi(n)W(g)$, on which the group $GL_n(\mathbb{A})$ acts by right translation. This realization is called a *Whittaker model* for π.

Similarly, if $\pi \simeq \otimes' \pi_v$ then a Whittaker model for a local representation π_v is a realization of π_v in a space of functions

$$\mathcal{W}(\pi_v, \psi_v) = \left\{ W_{\xi_v}(g) \middle| \xi_v \in V_{\pi_v} \right\}$$

where again we require that $W_{\xi_v}(n_v g_v) = \psi_v(n_v) W_{\xi_v}(g_v)$ for $n_v \in N(\mathbb{Q}_v)$, $g_v \in GL_n(\mathbb{Q}_v)$ and where ψ_v is a local component of ψ. It is a theorem of Gelfand-Kazhdan and Shalika that locally an irreducible smooth admissible representation (π_v, V_{π_v}) can have at most one such model (with a continuity assumption at the archimedean place). This result is *the local uniqueness of the Whittaker model*.

From the local statements one can deduce *the uniqueness of the global Whittaker model*. From this it follows that if $\varphi \in V_\pi \subset \mathcal{A}_0$ and under the isomorphism $\pi \simeq \otimes' \pi_v$ we have $\varphi \simeq \otimes \xi_v$ then we have a factorization

$$W_\varphi(g) = \prod_v W_{\xi_v}(g_v).$$

This is a highly non-trivial factorization!

2. L-functions for GL_n and Converse Theorems

Let's recall Hecke: for $f \in S_k(\Gamma)$, a Hecke eigen-form, we have

$$\int_0^\infty f(iy) y^s \, d^\times y = \Lambda(s, f) = (2\pi)^{-s} \Gamma(s) \prod_p (1 - a_p p^{-s} + p^{2k-1} p^{-2s})^{-1}.$$

The analogous integral would be for $\varphi \in \mathcal{A}_0(GL_2(\mathbb{Q}) \backslash GL_2(\mathbb{A}); \omega)$ and would be

$$\int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \varphi \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^s \, d^\times a = \int_{GL_1(\mathbb{Q}) \backslash GL_1(\mathbb{A})} \varphi \begin{pmatrix} a & \\ & 1 \end{pmatrix} |a|^s \, d^\times a.$$

However, we will want to include twists as necessary for a Converse Theorem. Here that would be twists by an adelic Dirichlet character χ, i.e., an automorphic form for GL_1:

$$\int_{GL_1(\mathbb{Q}) \backslash GL_1(\mathbb{A})} \varphi \begin{pmatrix} a & \\ & 1 \end{pmatrix} \chi(a) |a|^s \, d^\times a.$$

2.1. **Global Rankin-Selberg integrals (d'apres Jacquet, Piatetski-Shapiro, and Shalika).** In the higher rank case, the analogous integrals are the Rankin-Selberg integrals for $GL_n \times GL_m$ with $n > m$. The theory which is closest to the classical theory of Hecke is for $GL_n \times GL_{n-1}$. So let (π, V_π) be a cuspidal automorphic representation of $GL_n(\mathbb{A})$ and let $(\pi', V_{\pi'})$ be a cuspidal automorphic representation of $GL_{n-1}(\mathbb{A})$. For cusp forms $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$ we set

$$I(s, \varphi, \varphi') = \int_{GL_{n-1}(\mathbb{Q}) \backslash GL_{n-1}(\mathbb{A})} \varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) |\det(h)|^{s - \frac{1}{2}} \, dh.$$

Note that in fact we have a *family* of such integrals, one for each pair of φ and φ'.

The basic facts about these integrals are the following.

(i) The individual integrals are *nice*. As in the classical case, here we have that the integrals are absolutely convergent for all s and hence entire; they are bounded in vertical strips (BVS); they satisfy a functional equation (FE) of the form $I(s,\varphi,\varphi') = I(1-s,\widetilde{\varphi},\widetilde{\varphi}')$ where $\widetilde{\varphi}(g) = \varphi({}^tg^{-1})$.

(ii) The individual integrals are *Eulerian*. If we insert the Fourier expansion for φ, then the integral unfolds to

$$I(s,\varphi,\varphi') = \int_{N_{n-1}(\mathbb{A})\backslash GL_{n-1}(\mathbb{A})} W_\varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} W'_{\varphi'}(h)|\det(h)|^{s-\frac{1}{2}}\ dh.$$

where $W_\varphi \in \mathcal{W}(\pi,\psi)$ and $W'_{\varphi'} \in \mathcal{W}(\pi',\psi^{-1})$. If we now assume we have decomposable $\varphi \simeq \otimes\xi_v$ and $\varphi' \simeq \otimes\xi'_v$, then the two global Whitaker functions in the integral now factor into a product of local Whittaker functions and then our integral itself factors as

$$\begin{aligned} I(s,\varphi,\varphi') &= \prod_v \int_{N_{n-1}(\mathbb{Q}_v)\backslash GL_{n-1}(\mathbb{Q}_v)} W_{\xi_v} \begin{pmatrix} h_v & \\ & 1 \end{pmatrix} W'_{\xi'_v}(h_v)|\det(h_v)|^{s-\frac{1}{2}}\ dh_v \\ &= \prod_v \Psi_v(s,W_v,W'_v), \end{aligned}$$

which is convergent now only for $\mathrm{Re}(s) >> 1$, with $W_v = W_{\xi_v} \in \mathcal{W}(\pi_v\psi_v)$ and $W'_v = W'_{\xi'_v} \in \mathcal{W}(\pi'_v,\psi_v^{-1})$. As we see, it is the decomposition $\pi \simeq \otimes'\pi_v$ that is responsible for the factorization of the Whittaker functions and hence ultimately for the Eulerian factorization of the global integral into a product of local integrals.

2.2. **Local L-functions (Euler factors).** Now let's look at a local non-archimedean place v of a number field k. We have the completion k_v, its ring of integers $\mathfrak{o}_v$ and maximal ideal $\mathfrak{p}_v = (\varpi_v)$. We set $q_v = |\mathfrak{o}_v/\mathfrak{p}_v| = |\varpi_v|_v^{-1}$.

Let $W_v \in \mathcal{W}(\pi_v,\psi_v)$ and $W'_v \in \mathcal{W}(\pi'_v,\psi_v^{-1})$. Then the basic properties of the local Rankin-Selberg integrals are the following.

(i) Each integral $\Psi_v(s,W,W'_v)$ is a rational function of q_v^{-s} and in fact their span $\langle \Psi_v(s,W_v,W'_v)\rangle$ forms a $\mathbb{C}[q_v^s,q_v^{-s}]$-fractional ideal of $\mathbb{C}(q_v^{-s})$. This fractional ideal has a (normalized) generator of the form $P_v(q_v^{-s})^{-1}$ with $P_v(X) \in \mathbb{C}[X]$ with $P_v(0) = 1$ and $\deg(P_v) \leq nm$.

Definition 2.1. $L(s,\pi_v\times\pi'_v) = P_v(q_v^{-s})^{-1}$.

(ii) For each individual integral we have

$$e(s,W_v,W'_v) = \frac{\Psi_v(s,W_v,W'_v)}{L(s,\pi_v\times\pi'_v)}$$

is entire and for each $s_0 \in \mathbb{C}$ there exist a choice of W_v and W_v' such that $e(s_0, W_v, W_v') \neq 0$. In other words, the local L-function $L(s, \pi_v \times \pi_v')$ exactly captures the poles of the family of local integrals.

(iii) The local functional equation. There exists $\gamma(s, \pi_v \times \pi_v', \psi_v) \in \mathbb{C}(q_v^{-s})$ such that for all choices of W_v and W_v' we have

$$\Psi_v(1-s, \widetilde{W}_v, \widetilde{W}_v') = \omega_{\pi_v'}(-1)^{n-1}\gamma(s, \pi_v \times \pi_v', \psi_v)\Psi_v(s, W_v, W_v')$$

or

$$\frac{\Psi_v(1-s, \widetilde{W}_v, \widetilde{W}_v')}{L(1-s, \widetilde{\pi}_v \times \widetilde{\pi}_v')} = \omega_{\pi_v'}(-1)^{n-1}\varepsilon(s, \pi_v \times \pi_v', \psi_v)\frac{\Psi_v(s, W_v, W_v')}{L(s, \pi_v \times \pi_v')}$$

with the local ε-factor $\varepsilon(s, \pi_v \times \pi_v', \psi_v)$ then being entire without zeroes. More precisely

$$\varepsilon(s, \pi_v \times \pi_v', \psi_v) = \varepsilon(\tfrac{1}{2}, \pi_v \times \pi_v', \psi_v)q_v^{-c(\pi_v \times \pi_v')(s-\frac{1}{2})}$$

where $\varepsilon(\frac{1}{2}, \pi_v \times \pi_v', \psi_v)$ is the local root number and $c(\pi_v \times \pi_v')$ the local conductor (exponent). Here $\widetilde{W}(g) = W\left(\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix} {}^t g^{-1}\right)$. For future reference, let us note that the L-, ε-, and γ-factors are related by

$$\gamma(s, \pi_v \times \pi_v', \psi_v) = \frac{\varepsilon(s, \pi_v \times \pi_v', \psi_v)L(1-s, \widetilde{\pi}_v \times \widetilde{\pi}_v')}{L(s, \pi_v \times \pi_v')}.$$

(iv) If π_v and π_v' are unramified, as well as ψ_v and $k_v/\mathbb{Q}_p$, which is the case at almost all finite places of k, then for the normalized unramified Whittaker functions W_v° and $W_v'^\circ$ we have that the local integral computes the local L-function on the nose and the local ε-factor is 1, i.e.,

$$L(s, \pi_v \times \pi_v') = \Psi_v(s, W_v^\circ, W_v'^\circ)$$
$$\varepsilon(s, \pi_v \times \pi_v', \psi_v) = 1.$$

At an archimedean place $v|\infty$ the local theory is similar yet different. Due to the existence of a local Langlands parametrization for real and complex groups, the form of $L(s, \pi_v \times \pi_v')$ is given as the local archimedean factor of the associated Artin representation. So it will be a product of factors of the form $\Gamma(a_{v,i}s+b_{v,i})$, where now $\Gamma(s)$ is Euler's gamma function, and accounts for the gamma factors in the classical functional equation. Now one must work to show that this "Langlands given" factor behaves well with respect to the local integrals, i.e., that analogues of (ii) and (iii) above hold. This can be quite complex and what we know is mainly due to Jacquet-Shalika and subsequent work of Jacquet.

2.3. **When $n-m>1$.** If we consider (π, V_π) a cuspidal representation of $GL_n(\mathbb{A})$ and $(\pi', V_{\pi'})$ a cuspidal representation of $GL_m(\mathbb{A})$ with $m<n-1$ the theory is quite similar. One begins with a global projection operator $\mathbb{P}': V_\pi \to U_\pi$ where U_π is a representation of the mirabolic subgroup $P_{m+1} \subset GL_{m+1}$, which has the form

$$P_{m+1} = \left\{ \begin{pmatrix} g & u \\ 0 & 1 \end{pmatrix} | g \in GL_m \right\},$$

with the property that

$$\mathbb{P}\varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} = \sum_{\gamma \in N_m(k)\backslash GL_m(k)} W_\varphi \left(\begin{pmatrix} \gamma & \\ & I_{n-m} \end{pmatrix} \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} \right).$$

$\mathbb{P}$ is essentially given by a partial Whittaker transform, as if one was only working out part of the Fourier expansion, twisted by an appropriate power of the determinant. With this operator in hand, the family of global integrals now has the form

$$I(s,\varphi,\varphi') = \int_{GL_m(\mathbb{Q})\backslash GL_m(\mathbb{A})} \mathbb{P}\varphi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) |\det(h)|^{s-\frac{1}{2}} \, dh.$$

which now unfold to

$$I(s,\varphi,\varphi') = \int_{N_m(\mathbb{A})\backslash GL_m(\mathbb{A})} W_\varphi \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} W'_{\varphi'}(h) |\det(h)|^{s-\frac{n-m}{2}} \, dh.$$

These are again Eulerian

$$\begin{aligned} I(s,\varphi,\varphi') &= \prod_v \int_{N_m(k_v)\backslash GL_m(k_v)} W_{\xi_v} \begin{pmatrix} h_v & \\ & I_{n-m} \end{pmatrix} W'_{\xi'_v}(h_v) |\det(h_v)|^{s-\frac{n-m}{2}} \, dh_v \\ &= \prod_v \Psi_v(s, W_v, W'_v) \end{aligned}$$

and from here one establishes the analogues of the results in Section 2.2. The only significant change is in the local functional equation, where the projection operator $\mathbb{P}$ reappears as an unipotent integration in the integral on the left hand side of the functional equation.

2.4. **When $m=n$.** When $m=n$ the construction is slightly different. It is the automorphic analogue of the classical construction of Rankin and Selberg. Now (π, V_π) and $(\pi', V_{\pi'})$ are both cuspidal representations of $GL_n(\mathbb{A})$. There is an extra player now, a (mirabolic) Eisenstein series. This Eisenstein series is of a special type and is associated to a Schwartz function $\Phi \in \mathcal{S}(\mathbb{A}^n)$; let us denote it by $E(g;s,\Phi)$. Then the family of global integrals has the form

$$I(s,\varphi,\varphi',\Phi) = \int_{GL_n(k)\backslash GL_n(\mathrm{A})} \varphi(g)\varphi'(g) E(g;s,\Phi) \, dg$$

for $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$. These integrals converge absolutely for all s and have a functional equation. (The analytic properties, including the functional equation, come from those of the Eisenstein series.) The family is again Eulerian and unfold to a family of local integrals of the form

$$\Psi_v(s, W_v, W_v', \Phi_v) = \int_{N_n(k_v)\backslash GL_n(k_v)} W_v(g) W_v'(g) \Phi_v(e_n g) |\det(g)|^s \, dg$$

where $e_n = (0, \cdots, 0, 1) \in k^n$. Once again, from here one establishes the analogous results as in Section 2.2.

2.5. **Global L-functions.** If we combine the nice properties of the global integrals from Section 2.1, 2.3, and 2.4 with the local analysis of Section 2.2 then we arrive at the global analogue of Hecke's theorem. Again take (π, V_π) a cuspidal automorphic representation of $GL_n(\mathbb{A})$ and $(\pi', V_{\pi'})$ a cuspidal representation of $GL_m(\mathbb{A})$.

We define the global L-factor and ε-factors as Euler products:

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi_v')$$
$$\varepsilon(s, \pi \times \pi') = \prod_v \varepsilon(s, \pi_v \times \pi_v', \psi_v)$$

where the product for the L-factor converges for $\mathrm{Re}(s) >> 0$ and the product for the ε-factor is a finite product. Note that we are now using the automorphic L-function convention where the L-factor is the product over *all* places – this we denoted Λ classically.

Theorem 2.1. *$L(s, \pi \times \pi')$ is nice in the sense that*

(1) *$L(s, \pi \times \pi')$ extends to a meromorphic function of s, entire if $m < n$;*
(2) *$L(s, \pi \times \pi')$ is bounded in vertical strips;*
(3) *we have the global functional equation*

$$L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi') L(1 - s, \widetilde{\pi} \times \widetilde{\pi}').$$

We note that in fact our analysis only gives the BVS in (ii) for the cases of $m = n, n-1$. The more general boundedness in vertical strips comes from the work of Shahidi.

2.6. **Converse Theorems for GL_n.** Converse Theorems for GL_n invert these integral representations much the same way that Hecke and Weil did. However, here we begin with an Euler product rather than a Dirichlet series, since that is what the direct theory, Theorem 2.1, gives us. Conceptually the theorem takes the following form. Given $\pi = \otimes' \pi_v$, an irreducible admissible representation of $GL_n(\mathbb{A})$ on a space V_π, when can we embed

$$V_\pi \hookrightarrow \mathcal{A}(GL_n(k)\backslash GL_n(\mathbb{A}); \omega)?$$

To relate to Hecke and Weil, by our local theory, to each local component π_v of π we can associate a local L- and ε-factor

$$\pi_v \mapsto L(s, \pi_v), \ \varepsilon(s, \pi_v, \psi_v)$$

and so to π itself the Euler products

$$L(s,\pi)=\prod_v L(s,\pi_v),\ \varepsilon(s,\pi,\psi)=\prod_v \varepsilon(s,\pi_v,\psi_v).$$

We think of π as encoding an L-function, as an Euler product of degree n. We must always assume two things:

(1) $L(s,\pi)$ converges in in some right half plane $\mathrm{Re}(s) >> 0$
(2) the central character $\omega=\omega_\pi$ is automorphic, i.e., is an idele class character (which guarantees that the global ε-factor is independent of ψ).

Like Weil, our criterion will involve twisting. So if $(\pi',V_{\pi'})$ is a *cuspidal automorphic* representation of $GL_m(\mathbb{A})$ for $m<n$ we can similarly define

$$L(s,\pi\times\pi')=\prod_v L(s,\pi_v\times\pi'_v)$$
$$\varepsilon(s,\pi\times\pi')=\prod_v \varepsilon(s,\pi_v\times\pi'_v,\psi_v)$$

We will say that the $L(s,\pi\times\pi')$ are *nice* if they behave as they would if π were cuspidal automorphic, i.e.,

(1) $L(s,\pi\times\pi')$ extends to an entire function of s;
(2) $L(s,\pi\times\pi')$ is bounded in vertical strips;
(3) we have the global functional equation

$$L(s,\pi\times\pi')=\varepsilon(s,\pi\times\pi')L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

To state the Theorems, let us set

$$\mathcal{A}_0(m)=\coprod_\omega \mathcal{A}_0(GL_m(k)\backslash GL_m(\mathbb{A});\omega)$$
$$\mathcal{T}_0(m)=\coprod_{1\le d\le m}\mathcal{A}_0(d).$$

Theorem 2.2. *Fix $\pi=\otimes'\pi_v$ an irreducible admissible representation of $GL_n(\mathbb{A})$ as above. If $L(s,\pi\times\pi')$ is nice for all $\pi'\in\mathcal{T}_0(n-1)$, then π is a cuspidal automorphic representation of $GL_n(\mathbb{A})$, i.e., we can embed π into $\mathcal{A}_0(GL_n(k)\backslash GL_n(\mathbb{A});\omega_\pi)$.*

Conceptually, the proof of this is very similar to that of Hecke and Weil. We can also reduce the twists with more work. This requires a delicate local construction and has no classical analogue.

Theorem 2.3. *Fix $\pi=\otimes'\pi_v$ an irreducible admissible representation of $GL_n(\mathbb{A})$ as above, with $n\ge 3$. If $L(s,\pi\times\pi')$ is nice for all $\pi'\in\mathcal{T}_0(n-2)$, then π is a cuspidal automorphic representation of $GL_n(\mathbb{A})$, i.e., we can embed π into $\mathcal{A}_0(GL_n(k)\backslash GL_n(\mathbb{A});\omega_\pi)$.*

There is a very useful variant of these results. Let S be a finite set of finite places of k and let η be a *fixed* idele class character of $\mathbb{A}^\times$. Let

$$\mathcal{T}_0^S(m) = \{\pi' \in \mathcal{T}_0(m) \mid \pi'_v \text{ is unramified for all } v \in S\}.$$

Theorem 2.4. *Fix $\pi = \otimes' \pi_v$ an irreducible admissible representation of $GL_n(\mathbb{A})$ as above and fix an idele class character η. If $L(s, \pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}_0^S(n-2) \otimes \eta$, then π is quasi-automorphic, i.e., there exists an automorphic representation π_1 of $GL_n(\mathbb{A})$ such that $\pi_{1,v} \simeq \pi_v$ for all $v \notin S$.*

Note that we can no longer claim that π is cuspidal, and indeed it may not be.

We can summarize the ideas of this section as follows.

Moral Theorem. *Any suitably nice L-function of degree n must be automorphic (or modular), i.e., associated to an automorphic representation of GL_n.*

In this Moral Theorem, *suitably nice* means that the L-function must be given by a convergent Euler product and that it and suitable twists must be entire, BVS, and satisfy an appropriate functional equation, all the standard properties that L-functions arising from arithmetic or geometry are conjectured to satisfy.

2.7. **On integral representations.** The theory of integral representation for the Rankin-Selberg L-functions, particularly the $n = m$ case, is the paradigm for the study of automorphic L-functions by integral representations. The general outline of the method as practiced are the following steps

(i) Write down a family of global integrals which have nice analytic properties, including continuation, BVS and functional equation.

In general the integrals are variants on the $n = m$ integrals here, where the analytic properties follow from those of an Eisenstein series. In this sense, the Hecke type integrals for $m < n$, and through them the Converse Theorem, are special.

(ii) Show that the global integrals are Eulerian, i.e., factor into a product of local integrals.

As is the case here, this unfolding and then factorization often relies on a local and global uniqueness principle, here the uniqueness of the Whittaker models. It is only in rare instances, such as the "new way" of Rallis and Piatetski-Shapiro, that this is not the case.

(iii) Perform the local unramified calculation.

At almost all places, the data that goes into the local integral is unramified and the integral can be computed explicitly. This is a non-trivial calculation, usually involving an explicit formula for the unramified functions in the integral, here the formula of Shintani for the unramified Whittaker function for GL_n. This is then followed by a calculation in invariant theory which expresses the result in terms of a Langlands L-function coming from an appropriate representation of the L-group. This "identifies" the L-function represented by the integrals in terms of Langlands' paradigm.

Often the process stops here. But for many purposes, such as functoriality, this is not enough. There remain three steps.

(iv) Analyze and compute the local L-function at the finite ramified places, including continuation and functional equations.
(v) Analyze the local L-function at the archimedean places, again including the local functional equation.
(vi) Combine steps (i)–(v) to complete the analysis of the global L-function, including continuation, BVS, and global functional equation.

It is only for the Rankin-Selberg L-functions for GL_n that we have completed these last three steps. (iv) was carried out by Jacquet, Piatetski-Shapiro, and Shalika for Rankin-Selberg convolutions. (v) was initiated by Jacquet and Shalika, but the final papers are by Jacquet alone. These involve passing to the Casselman-Wallach completions and introduces another layer of technical difficulty. (vi) is almost completed for GL_n. There remains the question of BVS, which currently can be done for $m = n$ and $m = n-1$ within the method. (The difficulty comes from the passage to the Casselman-Wallach completion in the archimedean theory.) For BVS in the other cases, we must rely on the Langlands-Shahidi method, which follows.

II. L-functions via Eisenstein series

In many ways, this part serves as a reader's guide to the lectures of Shahidi [10]. I will try to indicate where the results I mention are to be found in his contribution.

3. The Origins: Langlands

3.1. **A classical example.** The Eisenstein series for $\Gamma = SL_2(\mathbb{Z})$ is defined as follows.

Let $B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right\} \subset SL_2(\mathbb{R})$ be the Borel subgroup of $SL_2(\mathbb{R})$. For $z \in \mathfrak{H}$ set

$$\begin{aligned} E(z,s) &= \sum_{(B\cap\Gamma)\backslash\Gamma} \operatorname{Im}(\gamma z)^{\frac{s}{2}+\frac{1}{2}} \qquad \text{for } \operatorname{Re}(s) > 1 \\ &= \sum_{\substack{(c,d)\in\mathbb{Z}^2 \\ (c,d)=1}} \frac{y^{\frac{s+1}{2}}}{|cz+d|^{s+1}} \\ &= \frac{1}{\zeta(s+1)} \sum_{(m,n)\neq(0,0)} \frac{y^{\frac{s+1}{2}}}{|mz+d|^{s+1}}. \end{aligned}$$

The Fourier expansion of $E(z,s)$ has been computed classically. We find, if we set $Z(s) = \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$, then

$$E(z,s) = 2\left(y^{\frac{s+1}{2}} + \frac{Z(s)}{Z(s+1)}y^{\frac{1-s}{2}}\right) + 4\sum_{n=1}^{\infty}\frac{|n|^s\sigma_{-s}(n)}{Z(s+1)}\sqrt{y}K_{\frac{s}{2}}(2\pi|n|y)e^{2\pi inx}$$

where $\sigma_{-s}(n) = \sum_{d|n} d^{-s}$.

Classically, one can use the *known* meromorphic continuation and functional equation of the Riemann zeta function $\zeta(s)$, and its completion $Z(s)$, to obtain the meromorphic continuation and functional equation of the Eisenstein series:

$$E(z,s) = \frac{Z(s)}{Z(s+1)}E(z,-s).$$

The idea of Langlands was to reverse this process: from the analytic properties of an Eisenstein series, deduce analytic properties of the L-functions appearing in its Fourier expansion.

3.2. **Induced representations.** In terms of automorphic representations, Eisenstein series correspond to induced representations.

Let k be a number field. (You can take $k = \mathbb{Q}$ if you want.) Let G be a connected reductive linear algebraic group over k, which we can assume to be split for simplicity, so we can view G as a subgroup of a suitably large GL.

Example. As an example to carry along consider the symplectic group $G = Sp_{2n}$:

$$Sp_{2n} = Isom\left(\begin{pmatrix} & J_n \\ -J_n & \end{pmatrix}\right) \subset GL_{2n} \quad \text{where} \quad J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

G will contain a Borel subgroup B, which we can take to be

$$B = G \cap \left\{\begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ & & * \end{pmatrix}\right\}.$$

Then $B = TU$ where T is a maximal torus, which will be the diagonal elements in B, and U is the unipotent radical of B,

$$U = G \cap \left\{\begin{pmatrix} 1 & \cdots & * \\ & \ddots & \vdots \\ & & 1 \end{pmatrix}\right\}.$$

We will need the following structural objects:

$$X(T) = \mathrm{Hom}_k(T, \mathbb{G}_m): \text{ the } k\text{-rational characters of } T;$$

$$\Phi(T) = \text{the characters of } T \text{ acting on } \mathrm{Lie}(G)\text{: the } \textit{roots} \text{ of } T \text{ in } G;$$

$$\Phi^+(T) = \text{ the characters of } T \text{ acting on } \mathrm{Lie}(U)\text{: the } \textit{positive roots};$$

$$\Delta = \text{ a basis of } \Phi^+(T)\text{: the } \textit{simple roots};$$

$$W = N(T)/Z(T): \text{ the } \textit{Weyl group.}$$

Note that $\Delta \subset \Phi^+(T) \subset \Phi(T) \subset X(T)$. Also $N(T)$ denotes the normalizer of T in G and $Z(T)$ the centralizer.

The Weyl group acts as a reflection group on $\Phi(T)$. It will contain a "longest element" $w_\ell \in W$ with the property that if we let $\Phi^-(T)$ denote the characters of T acting on the $\mathrm{Lie}(U^-)$, U^- being the opposite or lower triangular unipotent subgroup, then $w_\ell(\Phi^+) = \Phi^-$. In GL_n, the longest element of W is represented by $w_\ell = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$.

Let P be a parabolic subgroup of G containing B. We will only need to consider maximal parabolic subgroups and will also restrict to those that are "self-associate". Just as $B = TU$, P will have a decomposition $P = MN$ with $M \supset T$ the reductive Levi component of P and $N \subset U$ the unipotent radical of P.

Example. In the example of Sp_{2n}, P will be a block upper triangular subgroup of the form

$$P = \left\{ \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix} \right\}$$

with

$$M = \left\{ \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & A^* \end{pmatrix} \in Sp_{2n} \middle| A \in GL_r;\ B \in Sp_{2t} \right\} \simeq GL_r \times Sp_{2t}$$

where $2r + 2t = 2n$, and

$$N = \left\{ \begin{pmatrix} I_r & * & * \\ & I_{2t} & * \\ & & I_r \end{pmatrix} \in Sp_{2n} \right\}.$$

Let

$$\delta_P(mn) = |\det(\mathrm{Ad}(m)|\mathrm{Lie}(N))|$$

be the modulus character of P. Then $\delta_P : M \to \mathbb{R}^\times$ and $\delta_P \in X(M) \otimes \mathbb{R}$. If (σ, V_σ) is a (reasonable) representation of $M(k_v)$ in the local situation (resp. an automorphic representation of $M(\mathbb{A})$ in the global situation) we can construct a representation of $G = G(k_v)$ (resp. $G = G(\mathbb{A})$) by

$$I(s, \sigma) = \mathrm{Ind}_P^G(\sigma \delta_P^{s/c_P})$$

where $s \in \mathbb{C}$ and $c_P \in \mathbb{Z}$ is a normalizing denominator depending on P. This is the representation of G by right translation on

$$V(s,\sigma) = \{f : G \to \mathbb{C} \,|\, f(mng) = \delta_P(m)^{\frac{s}{c_P}+\frac{1}{2}}\sigma(m)f(g),$$
$$\text{plus regularity conditions}\}.$$

If $\sigma \simeq \otimes'\sigma_v$ is an automorphic representation of $M(\mathbb{A})$ then we have

$$I(s,\sigma) \simeq \otimes' I(s,\sigma_v)$$

as a representation of $G(\mathbb{A})$, but $I(s,\sigma)$ is not yet automorphic; it is only left invariant under $P(k)$, not all of $G(k)$.

3.3. **Eisenstein series.** We will use the theory of Eisenstein series to embed $I(s,\sigma)$ into the space of automorphic forms. We assume that σ is a unitary cuspidal automorphic form for $M(\mathbb{A})$.

Definition 3.1. *If $f_s \in I(s,\sigma)$ we define the Eisenstein series associated to f_s by*

$$E(g,s,f_s) = \sum_{\gamma \in P(k)\backslash G(k)} f_s(\gamma g).$$

The series which defines the Eisenstein series converges for $\mathrm{Re}(s) >> 0$. It intertwines $I(s,\sigma)$ with the space of automorphic forms on $G(\mathbb{A})$, to give the automorphic model of the induced representation. The basic properties of the Eisenstein series, due to Selberg, Harish-Chandra, and Langlands, are as follows.

(i) $E(g,s,f_s)$ converges for $\mathrm{Re}(s) >> 0$ and has a meromorphic continuation to all of $\mathbb{C}$;

(ii) $E(g,s,f_s)$ satisfies a functional equation of the form

$$E(g,s,f_s) = E(g,-s,M(s,\sigma)f_s)$$

where

$$M(s,\sigma) : V(s,\sigma) \to V(-s,w_0(\sigma))$$

is an intertwining operator defined as follows. First set

$$w_0 = w_\ell w_\ell^M \in W \quad \text{and} \quad w_0(\sigma)(g) = \sigma(w_0^{-1} g w_0).$$

Part of P being "self-associate" is that $w_0(M) = M$. Then

$$M(s,\sigma)f_s(g) = \int_{N(\mathbb{A})} f_s(w_0^{-1}ng)\, dn$$

for $\mathrm{Re}(s) >> 0$.

(iii) The constant term of the Eisenstein series along P is

$$\int_{N(k)\backslash N(\mathbb{A})} E(ng,s,f_s) = f_s + M(s,\sigma)f_s$$

(iv) The analytic properties, and particularly the poles, of $E(g,s,f_s)$ and $M(s,\sigma)$ are the same. There are only a finite number of simple poles for $\mathrm{Re}(s)\geq 0$; there are no poles on $\mathrm{Re}(s)=0$.

3.4. **The L-functions in the constant term.** Langlands computed the Euler products that appear in the constant term of the Eisenstein series. "Recall", from what we said about GL_n, that if $\sigma\simeq\otimes'\sigma_v$ is a cuspidal representation of $M(\mathbb{A})$, then for almost all places v, σ_v will be unramified. This will persist for $I(s,\sigma)$ acting on $V(s,\sigma)$, i.e, if σ_v is unramified, so will be $I(s,\sigma_v)$ and in $V(s,\sigma_v)$ there will be a unique (normalized) K_v-fixed vector $f^\circ_{s,v}$, where K_v is a (reasonable) maximal compact subgroup of $G(k_v)$. So if $f_s\in V(s,\sigma)$ is decomposable, then we can write

$$f_s\simeq\otimes f_{s,v}=\left(\bigotimes_{v\in S}f_{s,v}\right)\otimes\left({\bigotimes_{v\notin S}}' f^\circ_{s,v}\right).$$

When we apply the global intertwining operator it will factor into a product of local intertwining operators, given by similar integral expressions, which is written as

$$M(s,\sigma)f_s=\otimes A(s,\sigma_v,w_0)f_{s,v}.$$

Now for $v\notin S$, $A(s,\sigma_v,w_0)f^\circ_{s,v}$ will again be unramified and hence be a multiple of the unramified vector $\widetilde{f}^\circ_{-s,v}\in V(-s,w_0(\sigma_v))$,i.e.,

$$A(s,\sigma_v,w_0)f^\circ_{s,v}=c_v(s)\widetilde{f}^\circ_{-s,v}$$

with $c_v(s)$ a scalar function of s (which turns out to be a rational function of q_v^s). Using the method of Gindikin-Karpelevich, which reduces to a succession of SL_2 calculations, Langlands computed $c_v(s)$. Without explaining the notation involved, the result of Langlands' calculation was

$$c_v(s)=\prod_{\substack{\beta\in\Phi^+\\ w_0(\beta)\in\Phi^-}}\frac{1-\chi_v\circ\beta^\vee(\varpi_v)q_v^{s\langle\widetilde{\alpha},\beta\rangle+1}}{1-\chi_v\circ\beta^\vee(\varpi_v)q_v^{s\langle\widetilde{\alpha},\beta\rangle}}$$

which looks like a ratio of products of Euler factors! Here $\widetilde{\alpha}\in\Phi$ is a root associated to P, $\beta^\vee$ is the co-root associated to β (see below), and χ_v is a character associated to the unramified representation σ_v.

How are we to interpret this? Langlands had at his disposal the formalism of Artin L-functions and the Satake parametrization of unramified representations. To make sense of these Euler products, Langlands introduced the the L-group and what we now call Langlands L-functions. (For more details on these, see Part III.)

(i) The L-group. Over an algebraically closed field, the algebraic group G determines and is determined by its *root datum* which is the quadruple $(X(T),\Phi(T),X^\vee(T),\Phi^\vee(T))$ where $X^\vee(T)=\mathrm{Hom}_k(\mathbb{G}_m,T)$ is the group of rational co-characters of T and $\Phi^\vee$ the set of co-roots. If we simply dualize this root datum we obtain $(X^\vee(T),\Phi^\vee(T),X(T),\Phi(T))$ which in turn is the root datum for the *Langlands dual group* $\widehat{G}$, which we can take over $\mathbb{C}$, i.e., set $\widehat{G} = \widehat{G}(\mathbb{C})$. The root datum for G inherits an action of the local Galois group $\mathrm{Gal}(\overline{k}_v/k_v)$ and we can transfer this to the dual root system and hence $\widehat{G}$. The L-group of G is then ${}^LG = \widehat{G} \rtimes \mathrm{Gal}(\overline{k}_v/k_v)$.

(ii) Satake parameters. Langlands interpreted Satake's parametrization of the unramified representation σ_v as an assignment $\sigma_v \mapsto t_v \in \widehat{T} \subset \widehat{M} \subset {}^LM$.

(iii) Local Langlands L-functions. If $r : {}^LM \to GL_n(\mathbb{C})$ is a finite dimensional complex representation of ${}^LM$ we define a local Euler factor attached to the unramified representation σ_v and the representation r by

$$L(s, \sigma_v, r) = \det(1 - r(t_v)q_v^{-s})^{-1}.$$

Coming back to the constant term of the Eisenstein series, we consider the representation r of ${}^LM$ acting by the adjoint representation (conjugation) on $\mathrm{Lie}({}^LN)$. This decomposes as

$$r = \bigoplus_{j=1}^{m} r_j \qquad \text{with } r_j \text{ irreducible.}$$

The irreducible piece r_j is actually the sum of the the co-root spaces in $\mathrm{Lie}({}^LN)$ spanned by the co-roots $\beta^\vee$ for $\beta \in \Phi^+$ with $\langle \widetilde{\alpha}, \beta \rangle = j$, where $\widetilde{\alpha}$, β and $\beta^\vee$ are as in Langlands' formula for $c_v(s)$. Then Langlands' formula for $c_v(s)$ takes the form

$$c_v(s) = \prod_{j=1}^{m} \frac{L(js, \sigma_v, \widetilde{r}_j)}{L(js+1, \sigma_v, \widetilde{r}_j)}.$$

Thus we arrive at the following result.

Theorem 3.1 ([10] Lemma 4.1)**.**

$$M(s,\sigma)f_s = \left(\bigotimes_{v \in S} A(s, \sigma_v, w_0) f_{s,v} \right) \otimes \left(\prod_{v \notin S} \prod_{j=1}^{m} \frac{L(js, \sigma_v, \widetilde{r}_j)}{L(js+1, \sigma_v, \widetilde{r}_j)} \bigotimes_{v \notin S} \widetilde{f}^\circ_{-s,v} \right)$$

for $\mathrm{Re}(s) >> 0$*.*

We set

$$L^S(s, \sigma, r) = \prod_{v \notin S} L(s, \sigma_v, r).$$

This is the partial L-function attached to the automorphic representation σ, the representation r, and the set of places S. Langlands proved that this converges absolutely in some right half-plane. So the product

$$\prod_{j=1}^{m} \frac{L^S(js, \sigma, \widetilde{r}_j)}{L^S(js+1, \sigma, \widetilde{r}_j)}$$

is the (partial) Euler product in the constant term of the Eisenstein series. From here Langlands went on to formulate the local and global Langlands Conjectures and Functoriality, which we will come back to in Part III.

Example. In our example of $G = Sp_{2n}$ with maximal parabolic $P = MN$ with $M \simeq GL_r \times Sp_{2t}$, the ${}^LG = SO_{2n+1}(\mathbb{C})$ and ${}^LM = GL_r(\mathbb{C}) \times SO_{2t+1}(\mathbb{C})$. In $\mathrm{Lie}({}^LN) \subset \mathrm{Lie}(SO_{2t+1}(\mathbb{C}))$ the representation r decomposes as follows:

$$\begin{pmatrix} 0 & r_1 & r_2 \\ & 0 & {}^t r_1 \\ & & 0 \end{pmatrix}$$

and we have $m = 2$ and

$$r_1 = St_{GL_r} \otimes St_{SO_{2t+1}}$$
$$r_2 = \Lambda^2(St_{GL_r})$$

where St_G denotes the standard or defining representation of G. If we decompose $\sigma_v = \sigma_{1,v} \otimes \tau_v$ according to the decomposition of M, then $L(s, \sigma_v, r_1) = L(s, \sigma_{1,v} \times \tau_v)$ gives the Rankin-Selberg convolution L-function of $GL_r \times Sp_{2t}$ and $L(s, \sigma_v, r_2) = L(s, \sigma_{1,v}, \Lambda^2)$ gives the exterior square L-function of GL_r.

Our goal is now to use the meromorphic continuation and functional equation of $E(g, s, f_s)$ to obtain the meromorphic continuation and functional equation of the $L(s, \sigma, \widetilde{r}_j)$. However there are three immediate problems.

(1) The problem of the ratios. Every L-function occurs as part of a ratio

$$\frac{L(js, \sigma, \widetilde{r}_j)}{L(js+1, \sigma, \widetilde{r}_j)}.$$

(2) The problem of the products. Each ratio occurs as part of a product

$$\prod_{j=1}^{m} \frac{L(js, \sigma, \widetilde{r}_j)}{L(js+1, \sigma, \widetilde{r}_j)}.$$

(3) The problem of the missing local factors. How can we define and analyze the local factors $L(s, \sigma_v, \widetilde{r}_j)$ for places $v \notin S$ where σ_v is ramified?

4. The Method: Langlands-Shahidi

4.1. **Resolution of the problem of the ratios.** The resolution of the problem of the ratios is central to the Langlands-Shahidi method and involves Whittaker models.

Fix a non-trivial additive character $\psi = \otimes\psi_v$ of $k\backslash\mathbb{A}$. We can use it to define a non-degenerate character χ of the unipotent radical U of the Borel subgroup B of G through the composition of the following sequence of maps

$$U \longrightarrow U^{ab} = U/[U,U] \xrightarrow{\sim} \bigoplus\nolimits_{\alpha \in \Delta} k_\alpha \xrightarrow{tr} k \xrightarrow{\psi} \mathbb{C}.$$

The choice of an isomorphism between U^{ab} and $\oplus k_\alpha$ is called a *splitting.* The trace map is simply $(x_\alpha) \mapsto \sum x_\alpha$. Since globally we assume ψ is trivial on k, we will have χ is trivial on $U(k)$ as well.

Definition 4.1. *A representation* (π, V_π) *of* G *(which we can take to be the local* $G(k_v)$ *or global* $G(\mathbb{A})$*) is called* χ-generic *if there exists a non-zero functional* $\lambda_\chi : V_\pi \to \mathbb{C}$ *such that for all* $u \in U$ *we have*

$$\lambda_\chi(\pi(u)v) = \chi(u)\lambda_\chi(v).$$

Such a functional is called a χ-Whittaker functional. *For* $v \in V_\pi$ *the function* $W_v(g) = \lambda_\chi(\pi(g)v)$ *is the associated* χ-Whittaker function.

Globally, this abstract notation of a generic representation is too weak. What is needed is something seemingly stronger, the notion of *globally* χ*-generic.*

Definition 4.2. *A cuspidal automorphic representation* (π, V_π) *of* $G(\mathbb{A})$ *is called* globally χ-generic *if the* explicit χ*-Whittaker functional*

$$\varphi \mapsto \int_{U(k)\backslash U(\mathbb{A})} \varphi(u)\chi^{-1}(u)\ du$$

is non-vanishing on V_π.

Let us now return to our Eisenstein series. Suppose now that (σ, V_σ) is a globally χ-generic representation of $M(\mathbb{A})$ with respect to the restriction of χ to $U_M = M \cap U$. Denote the globally χ-generic Whittaker functional on σ by λ_χ^M. Then $I(s, \sigma)$ is also χ-generic with explicit Whittaker functional λ_χ on $V(s, \sigma)$ given by

$$\lambda_\chi(f_s) = \int_{N(k)\backslash N(\mathbb{A})} \lambda_\chi^M(f_s(w_0^{-1}n))\chi^{-1}(n)\ dn.$$

If we then form the Eisenstein series $E(g, s, f_s)$ we can compute its χ-Fourier coefficient using the same methods that Langlands used for the constant term.

By definition, the Fourier coefficient is

$$E_\chi(g, s, f_s) = \int_{U(k)\backslash U(\mathbb{A})} E(ug, s, f_s)\chi^{-1}(u)\ du.$$

If f_s is decomposable, $f_s = \otimes f_{s,v}$ we get

$$E_\chi(g, s, f_s) = \otimes\lambda_{\chi_v}(f_{s,v})(g) = \prod_v W_{f_{s,v}}(g)$$

and at the unramified places $v \notin S$, if $f^\circ_{s,v}$ is the unramified vector then $W_{f^\circ_{s,v}} = W^\circ_v$ the associated unramified Whittaker function. The Whittaker functions are not "multiplicative" in the same sense that the constant terms were and the method of Gindikin-Karpelevich no longer applies. However one can evaluate the unramified Whitaker functions by using the Casselman-Shalika formula to obtain the following result.

Theorem 4.1 ([10] Theorem 6.2)**.**

$$E_\chi(e,s,f_s)=\left(\prod_{v\in S}W_{f_{s,v}}(e)\right)\cdot\prod_{j=1}^{m}\frac{1}{L^S(js+1,\sigma,\widetilde{r}_j)}.$$

Compare this with the classical situation where $Z(s+1)$ occurred in the denominator of the Fourier coefficient. This resolves the problem of the ratios.

As an application, we have the following corollary. Since $E(g,s,f_s)$ has no poles on $\mathrm{Re}(s)=0$, neither does $E_\chi(g,s,f_s)$. This then leads to non-vanishing on the 1–line!

Corollary 4.1.1 ([10] Corollary 6.3)**.** *For* $\mathrm{Re}(s)=0$

$$\prod_{j=1}^{m}L^S(js+1,\sigma,\widetilde{r}_j)\neq 0.$$

Remark 4.1. If $G\neq GL_n$, not all cuspidal representations are globally generic. This is a restriction on the representations under consideration. However, it is conjectured that every tempered L-packet should contain a globally generic member and that every globally generic representation, or even a representation that is consistently locally generic, should be tempered. So this should be no restriction on the level of the L-functions themselves.

4.2. **Resolution of the problem of the products.** This is resolved by a nice inductive procedure which is embodied in the following proposition.

Proposition 4.1 ([10] Proposition 7.1)**.** *Given* $1<j\leq m$*, there exists a (split) group* G_j *over* k *and a maximal* k*-parabolic subgroup* $P_j=M_jN_j\subset G_j$ *and a globally generic cuspidal automorphic representation* σ_j *of* $M_j(\mathbb{A})$ *such that if the adjoint action* r' *of* ${}^LM_j$ *on* $\mathrm{Lie}({}^LN_j)$ *decomposes as*

$$r'=\bigoplus_{k=1}^{m'}r'_k$$

then we have (i) $m'<m$ *and (ii)* $L^S(s,\sigma,\widetilde{r}_j)=L^S(s,\sigma_j,\widetilde{r}'_1)$.

Example. If we go back to our example in the symplectic group, in $G=\mathrm{Sp}_{2n}\supset P=MN$ with $M\simeq GL_r\times Sp_{2t}$ then we saw that ${}^LM\simeq GL_r(\mathbb{C})\times SO_{2t+1}(\mathbb{C})$ and that $m=2$ with $r_1=St_{GL_r}\otimes St_{SO_{2t+1}}$ and $r_2=\Lambda^2(St_{GL_r})$. If we take $G'=SO_{2r}\supset P'=M'N'$ with P' the Siegel parabolic with $M'\simeq GL_r$, then ${}^LM'\simeq GL_r(\mathbb{C})$ and in fact

$${}^LM'=\left\{\begin{pmatrix}A&\\&A^*\end{pmatrix}\middle|A\in GL_r\right\}\quad\text{and}\quad\mathrm{Lie}({}^LN')=\left\{\begin{pmatrix}0&S\\&0\end{pmatrix}\middle|{}^tS=-S\right\}$$

so that $m'=1$ and $r'_1=\Lambda^2(St_{GL_r})$.

If we apply this proposition inductively we can understand *all*

- $m = 1$ cases;
- $m = 2$ cases, since $L^S(s, \sigma, \widetilde{r}_2) = L^S(s, \sigma', \widetilde{r}'_1)$, which we would now understand;
- $m = 3$ cases, etc ...

Note: The $G \supset P = MN$ fall into a finite number of infinite families plus a finite number of exceptional cases. So we could prove this proposition by "checking the list".

As an application, we have the following result.

Proposition 4.2. *Each $L^S(s, \sigma, \widetilde{r}_j)$ has a meromorphic continuation to $\mathbb{C}$.*

To see this, since the Eisenstein series $E(g, s, f_s)$ has a meromorphic continuation iff its constant term does, we see that $M(s, \sigma)f_s$ must have a meromorphic continuation. But by the explicit computation, this implies that

$$\prod_{j=1}^{m} \frac{L^S(js, \sigma, \widetilde{r}_j)}{L^S(js + 1, \sigma, \widetilde{r}_j)}$$

has a meromorphic continuation. By the above proposition, we can inductively strip off factors and can conclude that

$$\frac{L^S(js, \sigma, \widetilde{r}_j)}{L^S(js + 1, \sigma, \widetilde{r}_j)}$$

has a meromorphic continuation for each j. But now we utilize the shift in the numerator and denominator to set up the shift equation

$$L^S(s, \sigma, \tilde{r}_j) = \frac{L^S(js, \sigma, \widetilde{r}_j)}{L^S(js + 1, \sigma, \widetilde{r}_j)} L^S(s + 1, \sigma, \widetilde{r}_j)$$

and inductively shift from the original half-plane of convergence to all of $\mathbb{C}$.

4.3. **Resolution of the problem of the missing local factors.** There are two main tools here.

(i) A local/global principle. Since we hope to use the Eisenstein, which are global in nature, we will need some type of a local/global principal. Here it is.

Proposition 4.3 ([10] Prpopsition 7.3)**.** *Let F be a local field, σ' a χ_F-generic supercuspidal representation of $M'(F)$. Then there exists a number field k and a group M/k and a globally χ-generic cuspidal representation σ of $M(\mathbb{A})$ such that*

(a) *for some place v_0 of k we have $F = k_{v_0}$, $M(k_{v_0}) = M'(F)$, $\chi_{v_0} = \chi_F$, and $\sigma_{v_0} = \sigma'$.*
(b) *for all $v \neq v_0$, σ_v is unramified (or v is archimedean).*

So, not only can we embed any local generic supercuspidal representation into a globally generic cuspidal representation, but at all the other places we completely understand the local theory!

(ii) Shahidi's local coefficient. One of the tools we will exploit is the functional equation of the Eisenstein series

$$E(g,s,f_s) = E(g,-s,M(s,\sigma)f_s)$$

which involves the global intertwining operator. We will also use the global Fourier coefficient $E_\chi(g,s,f_s)$. Shahidi's local coefficient combines these.

Begin with $I(s,\sigma_v)$ acting on $V(s,\sigma_v)$. We have local intertwining operators and local Whittaker functionals. If we combine them, we get the following diagram.

$$\begin{tikzcd} V(s,\sigma_v) \arrow[r, "A(s,\sigma_v,w_0)"] \arrow[d, "\lambda_{\chi_v}"'] & V(-s,w_0(\sigma_v)) \arrow[d, "\lambda_{\chi_v}"] \\ \mathbb{C} & \mathbb{C} \arrow[l, "C_{\chi_v}(s,\sigma_v)"] \end{tikzcd}$$

By uniqueness of the Whittaker functionals or models, the functionals λ_{χ_v} and $\lambda_{\chi_v} \circ A(s,\sigma_v,w_0)$ differ by a complex scalar. This scalar is *Shahidi's local coefficient* and denoted by $C_{\chi_v}(s,\sigma_v)$, i.e., for each $f_{s,v} \in V(s,\sigma_v)$ we have

$$\lambda_{\chi_v}(f_{s,v}) = C_{\chi_v}(s,\sigma_v) \cdot \lambda_{\chi_v}(A(s,\sigma_v,w_0)(f_{s,v})).$$

Remark 4.2. The analytic properties of $C_{\chi_v}(s,\sigma_v)$ will be related to those of the local intertwining operator $A(s,\sigma_v,w_0)$ just as those of $E(g,s,f_s)$ are related to $M(s,\sigma)$.

How should we think of these local coefficients? Look at the places where we understand the local L-function, i.e., $v \notin S$.

Proposition 4.4 ([10] Proposition 7.4)**.** *Let $v|\infty$ or $v<\infty$ and σ_v unramified. Then*

$$C_{\chi_v}(s,\sigma_v) = \prod_{j=1}^{m} \gamma(js,\sigma_v,\widetilde{r}_j,\psi_v^{-1})$$

where

$$\gamma(s,\sigma_v,\widetilde{r}_j,\psi_v^{-1}) = \frac{\varepsilon(s,\sigma_v,\widetilde{r}_j,\psi_v^{-1})L(1-s,\sigma_v,r_j)}{L(s,\sigma_v,\widetilde{r}_j)}.$$

Note. This expression for the localγ-factor as the same as for GL_n given in Part I.

Remark 4.3. Since in these cases, we have the either the local Langlands conjecture at the archimedean places or Langlands' interpretation of the Satake parameters in terms of the L-group, the L-functions that appear can all be interpreted as Artin L-functions.

As an application of Shahidi's local coefficients, we obtain the *crude functional equation* for the partial global L-functions. We begin with the functional equation of the Eisenstein series

$$E(g,s,f_s) = E(g,-s,M(s,\sigma)f_s),$$

take the χ-Fourier coefficient

$$E_\chi(g,s,f_s) = E_\chi(g,-s,M(s,\sigma)f_s),$$

then use the calculation of the local Fourier coefficient at the unramified places $v \notin S$ and the definition of the local coefficient at $v \in S$ we obtain a crude functional equation.

Theorem 4.2 ([10] Theorem 6.4)**.**

$$\prod_{j=1}^{m} L^S(js,\sigma,r_j) = \prod_{v\in S} C_{\chi_v^{-1}}(s,\widetilde{\sigma}_v) \prod_{j=1}^{m} L^S(1-js,\sigma,\widetilde{r}_j).$$

5. The Results: Shahidi

5.1. **The local γ-factor.** We have seen that at the unramified places and the archimedean places, there is a relation between Shahidi's local coefficient and the local γ-factor. Shahidi now uses *all* our tools plus the crude functional equation to establish a consistent theory of local γ-factors for all places.

Theorem 5.1 ([10] Theorem 7.5)**.** *Let G be a split reductive algebraic group over a local field F of characteristic* 0*. Let $P = MN$ be a (self associate) maximal parabolic subgroup of G σ an irreducible admissible χ-generic representation of $M = M(F)$. Then there exist m complex functions*

$$\gamma(s,\sigma,r_j,\psi) \quad 1 \le j \le m$$

such that

(1) *if F is archimedean or F is non-archimedean and σ is unramified, then*
$$\gamma(s,\sigma,r_j,\psi) = \frac{\varepsilon(s,\sigma,r_j,\psi)L(1-s,\sigma,\widetilde{r}_j)}{L(s,\sigma,r_j)};$$

(2)
$$C_\chi(s,\sigma) = \prod_{j=1}^{m} \gamma(js,\sigma,\widetilde{r}_j,\psi^{-1});$$

(3) *$\gamma(s,\sigma,r_j,\psi)$ is "multiplicative" with respect to induction;*

(4) *if σ_v is the local component of a globally generic representation σ of $M(\mathbb{A})$ then*
$$L^S(s,\sigma,r_j) = \prod_{v\in S} \gamma(s,\sigma_v,r_j,\psi_v) L^S(1-s,\sigma,\widetilde{r}_j).$$

Moreover, (1), (3), and (4) characterize the γ-factors uniquely.

Example. We will illustrate what is meant by "multiplicativity" in part (3) in our example. So we return, once again, to $G = Sp_{2n}$ with $n = r+t$. We have our maximal parabolic subgroup $P = MN$ with $M \simeq GL_r \times Sp_{2t}$. Our representation σ of $M(F)$ will then decompose as $\sigma \simeq \sigma_1 \otimes \tau$ with σ_1 an irreducible admissible representation of $GL_r(F)$ and τ a similar representation of $Sp_{2t}(F)$. Now the Levi subgroup M is itself reductive and within M we take a parabolic subgroup $P' = M'N'$ with

$$M' \simeq (GL_{r_1} \times GL_{r_2}) \times (GL_a \times Sp_{2b}) \subset GL_r \times Sp_{2t} \simeq M$$

with $r_1 + r_2 = r$ and $2a + 2b = 2t$. Let

$$\sigma' = (\sigma'_1 \otimes \sigma'_2) \otimes (\sigma'' \otimes \tau')$$

be a representation of $M'(F)$ with the tensor decomposition of the representation corresponding to the decomposition of M'. Suppose that in the decomposition $\sigma = \sigma_1 \otimes \tau$ we have

$$\sigma_1 \subset \mathrm{Ind}(\sigma'_1 \otimes \sigma'_2) \quad \text{and} \quad \tau \subset \mathrm{Ind}(\sigma'' \otimes \tau')$$

so that $\sigma \subset \mathrm{Ind}(\sigma')$.

We also have

$$\begin{aligned} {}^L M' &\simeq (GL_{r_1}(\mathbb{C}) \times GL_{r_2}(\mathbb{C}) \times (GL_a(\mathbb{C}) \times SO_{2b+1}(\mathbb{C})) \\ &\subset GL_r(\mathbb{C}) \times SO_{2t+1}(\mathbb{C}) \simeq {}^L M. \end{aligned}$$

Recall that the action r of ${}^L M$ on $\mathrm{Lie}({}^L N)$ decomposes as $r = r_1 \oplus r_2$ with $r_1 = St_{GL_4} \otimes St_{SO_{2t+1}}$ and $r_2 = \Lambda^2(St_{GL_r})$. Both of these will further decompose when we restrict to ${}^L M' \subset {}^L M$. This decomposition then gives the following decomposition of the resulting γ-factors, which is what is referred to as "multiplicativity".

$$\begin{aligned} \gamma(s, \sigma_1 \otimes \tau, r_1, \psi) &= \gamma(s, \sigma_1 \times \tau, \psi) \\ &= \prod_{k=1}^{2} \left[\gamma(s, \sigma'_k \times \sigma'', \psi) \gamma(s, \sigma'_k \times \widetilde{\sigma}'', \psi) \right] \prod_{k=1}^{2} \gamma(s, \sigma'_k \times \tau', \psi) \end{aligned}$$

and

$$\begin{aligned} \gamma(s, \sigma_1 \times \tau, r_2, \psi) &= \gamma(s, \sigma_1, \Lambda^2, \psi) \\ &= \gamma(s, \sigma'_1 \times \sigma'_2, \psi) \prod_{k=1}^{2} \gamma(s, \sigma'_k, \Lambda^2, \psi). \end{aligned}$$

5.2. **The definition of local L- and ε-factors at the ramified places.** F is again a local field. Recall that if $v|\infty$ or $v < \infty$ and σ is unramified,then

$$\gamma(s, \sigma, r_j, \psi) = \frac{\varepsilon(s, \sigma, r_j, \psi) L(1 - s, \sigma, \widetilde{r}_j)}{L(s, \sigma, r_j)}.$$

For the other places, we take this relation as the paradigm for defining the L- and ε-factors.

(a) Tempered representations For tempered representations we can make the following definition.

Definition 5.1. *If σ is χ-generic and* tempered, *set*

$$L(s, \sigma, r_j) = P_{\sigma,j}(q^{-s})^{-1},$$

where $P_{\sigma,j}(X) \in \mathbb{C}[X]$ with $P(0) = 1$, where $P_{\sigma,j}(q^{-s})$ is the normalized polynomial in q^{-s} in the numerator of $\gamma(s, \sigma, r_j, \psi)$. Then we define $\varepsilon(s, \sigma, r_j, \psi)$ by

$$\gamma(s, \sigma, r_j, \psi) = \frac{\varepsilon(s, \sigma, r_j, \psi) L(1 - s, \sigma, \widetilde{r}_j)}{L(s, \sigma, r_j)}.$$

This determines L- and ε-uniquely for σ tempered.

From the behavior of the local intertwining operator $A(s, \sigma, w_0)$ for σ tempered, we can draw the following conclusion.

Theorem 5.2 ([10] Theorem 7.6)**.** *Suppose σ is χ-generic and tempered. Then $L(s, \sigma, r_j)$ is holomorphic for* $\mathrm{Re}(s) > 0$.

Corollary 5.2.1. *$L(s, \sigma, r_j)$ is multiplicative for σ tempered.*

(b) General representations If σ is χ-generic but not necessarily tempered we appeal to the (representation theoretic) Langlands classification. This allows us to write σ uniquely as a sub-representation

$$\sigma \subset \mathrm{Ind}_{M'N'}^{M}(\sigma'_\nu)$$

with σ' a tempered representation of $M' = M'(F)$ and σ'_ν a quasi-tempered (or deformed) representation with ν in the negative Weyl chamber. Then $L(s, \sigma, r_j)$ is defined in terms of σ'_ν by formal multiplicativity, as for γ and for L- in the tempered case, and the $\varepsilon(s, \sigma, r_j, \psi)$ is derived from $L(s, \sigma, r_j)$ and $\gamma(s, \sigma, r_j, \psi)$ so the standard relation holds.

This now defines $L(s, \sigma, r_j)$ and $\varepsilon(s, \sigma, r_j)$ for *all* irreducible admissible χ-generic representations of $M = M(F)$.

5.3. **Global L-functions.** We now return to k to be a number field and $\sigma \simeq \otimes' \sigma_v$ a globally χ-generic cuspidal representation of $M(\mathbb{A})$. Let r_j be one of the representations of ${}^L M$ occurring in the action of ${}^L M$ on $\mathrm{Lie}({}^L N)$.

Definition 5.2.

$$L(s, \sigma, r_j) = \prod_v L(s, \sigma_v, r_j)$$
$$\varepsilon(s, \sigma, r_j) = \prod_v \varepsilon(s, \sigma_v, r_j, \psi_v)$$

Then from property (4) of the γ-factor (the local/global principal for γ) and the definition of L- and ε for $v \in S$ we obtain the true global functional equation.

Theorem 5.3 ([10] Theorem 7.7)**.** *For each $j = 1, \dots, m$*

$$L(s, \sigma, r_j) = \varepsilon(s, \sigma, r_j) L(1 - s, \sigma, \widetilde{r}_j).$$

5.4. **Holomorphy and BVS.** In general, the $L(s,\sigma,r_j)$ can have poles coming from the analytic properties of the Eisenstein series. If an Eisenstein series has a pole in $\mathrm{Re}(s)\geq 0$, its residual representation lies in the residual L^2-spectrum. If you "twist" the Eisenstein series, the residue can't be in L^2, i.e., there can be no pole. Transferring this to the L-functions gives the following result.

Theorem 5.4 ([10] Theorem 8.2)**.** *Let ξ be the k-rational character of M given by*

$$\xi(m)=\det(\mathrm{Ad}(m)|\mathrm{Lie}(N)).$$

Let S be a non-empty set of finite places of k. For every globally χ-generic cuspidal representation σ of $M(\mathbb{A})$ there exists non-negative integers f_v for $v\in S$ such that for every idele class character $\eta=\otimes\eta_v$ with $\mathrm{cond}(\eta_v)\geq f_v$ for all $v\in S$, if we set

$$\sigma_\eta=\sigma\otimes(\eta\circ\xi)$$

then $L(s,\sigma_\eta,r_j)$ is entire for all $1\leq j\leq m$.

The above remarks on the Eisenstein series will rule out the poles in $\mathrm{Re}(s)\geq\frac{1}{2}$. Then use the global functional equation.

Also, although this requires other analytic techniques, we have the following result.

Theorem 5.5 ([10] Theorem 8.3)**.** *With the assumptions of the previous theorem, so η_v sufficiently highly ramified at $v\in S$, we have $L(s,\sigma_\eta,r_j)$ is bounded in vertical strips (BVS) for all $1\leq j\leq m$.*

5.5. **Summary.** Let G be a (quasi)split connected reductive algebraic group over a number field k. Let $P=MN$ be a maximal parabolic subgroup defined over k. Let σ be a globally χ-generic cuspidal representation of $M(\mathbb{A})$. Let r be the representation of ${}^LM$ on $\mathrm{Lie}({}^LN)$, decomposed as

$$r=\bigoplus_{j=1}^{m}r_j.$$

Then each global L-function $L(s,\sigma,r_j)$ is nice in the sense that

(i) $L(s,\sigma,r_j)$ converges for $\mathrm{Re}(s)>>0$ and has a meromorphic continuation to all $\mathbb{C}$;
(ii) for every sufficiently ramified idele class character η, $L(s,\sigma_\eta,r_j)$ is entire;
(iii) for every sufficiently ramified idele class character η, $L(s,\sigma_\eta,r_j)$ is BVS;
(iv) $L(s,\sigma,r_j)=\varepsilon(s,\sigma,r_j)L(1-s,\sigma,\widetilde{r}_j)$.

Concluding remark. While this method covers a large family of Langlands L-functions $L(s,\sigma,\rho)$, it has two limitations.

(a) It requires σ to be globally generic.

(b) The representations σ are restricted to representations of a group M which must occur as the Levi subgroup of a maximal parabolic $P=MN\subset$

G for some larger group G. Moreover, the representations ρ of ${}^LM$ must be one of the r_j occurring in the action of ${}^LM$ on $\mathrm{Lie}({}^LN)$.

However, other than these restrictions that are forced by the method, the analytic results on the $L(s,\sigma,r_j)$ are quite complete. As we will see, they suffice, when combined with the Converse Theorem of Part I, to establish many cases of Langlands' conjecture by the method of L-functions.

III. Functoriality

6. Langlands Conjectures and Functoriality

As we indicated in Section 3, the Langlands Conjectures and Functoriality have their origin in the theory of Eisenstein series. Here I want to present another way to think about these ideas and what they entail. Recall that we ended Part I with a Moral Theorem, which we repeat.

Moral Theorem. *Any suitably nice L-function of degree n must be automorphic (or modular), i.e., associated to an automorphic representation of GL_n.*

What is an interesting source of degree n L-functions?

6.1. **Global Class Field Theory.** The problem of a global class field theory is to

- understand the global Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ or more generally $\mathrm{Gal}(\overline{k}/k)$.

Of course one way to understand the structure of a complicated group is through its linear representations. So our question becomes

- understand the representations

$$\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{C}) \quad (\text{ or } GL_n(\widehat{\overline{\mathbb{Q}_\ell}})).$$

Artin attached to every such complex representation ρ a degree n L-function. If $\rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL(V)$ with V an n-dimensional complex vector space, then

$$L(s,\rho) = \prod_v L_v(s,\rho) = \prod_v L(s,\rho_v)$$

where

$$L(s,\rho_v) = \begin{cases} \det(1 - p^{-s}\rho(Frob_p)|V^{I_p})^{-1} & \text{if } v = p \\ \Gamma(s,\rho) & \text{if } v = \infty \end{cases}$$

with $\Gamma(s,\rho)$ the appropriate product of Γ-functions as defined by Artin. He also defined an ε-factor $\varepsilon(s,\rho)$ So then our problem can possibly be reformulated as

- for each ρ as above, understand its L-function $L(s,\rho)$.

We know the following facts about the Artin L-functions.

(i) $L(s,\rho)$ converges for $\mathrm{Re}(s) >> 0$;
(ii) $L(s,\rho)$ has a meromorphic continuation to $\mathbb{C}$;

(iii) $L(s,\rho)$ is bounded in vertical strips;
(iv) we have the functional equation $L(s,\rho) = \varepsilon(s,\rho)L(1-s,\widetilde{\rho})$.

We also have the

Artin Conjecture: *$L(s,\rho)$ is entire if ρ is irreducible and not the trivial representation.*

So, morally, this should imply that $L(s,\rho)$ is automorphic. (The twists in our Moral Theorem are here given by the tensor product of Galois representations.) It is this automorphy that is the substance of the Langlands Conjectures.

6.2. **The Langlands Conjectures for GL_n.** Based on our Moral Theorem, we are led to a first naive formulation of the Langlands conjecture.

Naive Global Langlands Conjecture. *There exist bijections*

$$\left\{ \begin{matrix} \rho : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{C}) \\ \textit{irreducible} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \textit{cuspidal automorphic} \\ \textit{representations } \pi \textit{ of } GL_n(\mathbb{A}) \end{matrix} \right\}$$

such that

$$L(s,\rho) = L(s,\pi)$$
$$L(s,\rho_1 \otimes \rho_2) = L(s,\pi_1 \times \pi_2)$$

with similar equalities for ε-factors.

One advantage of the adelic theory of automorphic representations is a natural formulation of a local version of the conjecture.

Naive Local Langlands Conjecture. *There exist bijections*

$$\left\{ \begin{matrix} \rho_v : \mathrm{Gal}(\overline{\mathbb{Q}_v}/\mathbb{Q}_v) \to GL_n(\mathbb{C}) \\ \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \textit{irreducible admissible} \\ \textit{representations } \pi_v \textit{ of } GL_n(\mathbb{Q}_v) \end{matrix} \right\}$$

such that

$$L(s,\rho_v) = L(s,\pi_v)$$
$$L(s,\rho_{1,v} \otimes \rho_{2,v}) = L(s,\pi_{1,v} \times \pi_{2,v})$$

with similar equalities for ε-factors.

Note that these local and global Langlands Conjectures are compatible. This local/global compatibility principal will be important in our later formulations.

Why did we call this naive? It is not even true for $n = 1$. On the left hand side, we would have only finite order Galois characters, while on the right hand side we have all Hecke characters (or all characters of $\mathbb{Q}_v^\times$ in the local case). Weil realized this and remedied it by introducing what we now refer to as the Weil group $W_{\mathbb{Q}}$ or $W_{\mathbb{Q}_v}$. Locally at a finite place we have

$$I_p \hookrightarrow W_{\mathbb{Q}_p} \to \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$$

where the Weil group has dense image in the Galois group; at the infinite place we have a short exact sequence

$$1 \to \mathbb{Q}^\times \to W_{\mathbb{R}} \to \mathrm{Gal}(\mathbb{C}/\mathbb{R}) \to 1.$$

The global Weil group $W_{\mathbb{Q}}$ has only a cohomological definition, but there is a compatibility between the local and global Weil groups. If we replace $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by $W_{\mathbb{Q}}$ and $\mathrm{Gal}(\overline{\mathbb{Q}}_v/\mathbb{Q}_v)$ by $W_{\mathbb{Q}_v}$, then the Langlands conjectures for $n = 1$ become the theorems of global and local Class Field Theory.

Unfortunately, the replacement of the Galois group by the Weil group did not completely solve our problem. Deligne pointed out that even this modification of the local Langlands conjecture was not true for $GL_2(\mathbb{Q}_p)$ for finite primes p. There were still too many representations on the right hand side. Deligne observed that if one used ℓ-adic coefficients rather than complex coefficients in the representations, one had an extra structure, as shown by Grothendieck. ℓ-adic Galois representations come with a monodromy operator, an extra nilpotent endomorphism that is normalized by the Galois representation. So Deligne imposed this structure in the complex case by introducing what we now call the (local) Weil-Deligne group $W'_{\mathbb{Q}_p}$. It has the structure of a semi-direct product $W'_{\mathbb{Q}_p} = W_{\mathbb{Q}_p} \ltimes \mathbb{G}_a$, where in any representation, the generator of the additive group acts by a nilpotent endomorphism N. So a representation of the Weil-Deligne group is a pair (ρ, N) with ρ a representation of the Weil group on a complex vector space V and N a nilpotent endomorphism of V, with certain compatibilities. (The archimedean Weil-Deligne group is just the Weil group; there is no need to change the theory at these places.) Deligne also extended the definition of the L- and ε-factors to these representations. This resolved the problem of the missing representations for $GL_2(\mathbb{Q}_p)$ and gives us a good formulation of a local Langlands conjecture.

Local Langlands Conjecture. *There exist bijections*

$$\left\{ \begin{matrix} \rho_v : W'_{\mathbb{Q}_v} \to GL_n(\mathbb{C}) \\ \textit{Frobenius semi-simple} \end{matrix} \right\} \leftrightarrow \left\{ \begin{matrix} \textit{irreducible admissible} \\ \textit{representations } \pi_v \textit{ of } GL_n(\mathbb{Q}_v) \end{matrix} \right\}$$

such that

$$L(s, \rho_v) = L(s, \pi_v)$$
$$L(s, \rho_{1,v} \otimes \rho_{2,v}) = L(s, \pi_{1,v} \times \pi_{2,v})$$

with similar equalities for ε-factors.

This is indeed the correct formulation, for this is now a Theorem, due to Langlands for $\mathbb{Q}_v = \mathbb{R}$ and by Harris-Taylor, followed by Henniart, for $\mathbb{Q}_v = \mathbb{Q}_p$. In fact, their proofs work for any local field of characteristic 0. It is also known for local fields of characteristic p by Laumon-Rapoport-Stuhler.

Unfortunately, there is no known construction of a global Weil-Deligne group. (This is often formulated as the conjectural Langlands group.) Instead we rely on a formulation of a local/global compatibility at least for representations of the Weil group. If we begin with a global representation $\rho : W_{\mathbb{Q}} \to GL_n(\mathbb{C})$ then, from the local/global compatibility of the Weil groups, ρ induces local representations ρ_v of $W_{\mathbb{Q}_v}$ for each v. Then we have the following diagram

$$\begin{tikzcd}
\rho \arrow[r, dashed] \arrow[d] & \pi = \otimes' \pi_v \\
\{\rho_v\} \arrow[r, "LLC"'] & \{\pi_v\} \arrow[u]
\end{tikzcd}$$

where the top arrow is defined by the composition of the other three.

Global Langlands Conjecture. *With the above formalism, $\pi = \otimes'\pi_v$ is automorphic.*

There are two ways of thinking about the information flow in these conjectures and theorems. For non-abelian Class Field Theory, we think of the information flow as going from the automorphic side (where we can control the analytic properties of L-functions) to the Galois side. However, for the formulation of Functoriality, we view the flow of information as going from the Galois side to the automorphic side, thus giving us *an arithmetic parametrization* of admissible or automorphic representations.

6.3. **Langlands Conjectures for other groups.** Another powerful aspect of the adelic theory of automorphic representations is that it can be formulated for any reductive algebraic group H over $\mathbb{Q}$ or k, as we have seen in Part II. Thinking of the Langlands Conjectures as arithmetic parametrizations of automorphic representations of $H(\mathbb{A})$ or admissible representations of $H(\mathbb{Q}_v)$, what do we replace the $GL_n(\mathbb{C})$ by in our Galois representations? This is where the formulation of the dual group $\widehat{H}$ or the L-group ${}^LH$ that we saw in Part II comes into play.

Local Langlands Conjecture for H. *There is a finite-to-one surjective map*

$$\left\{ \begin{matrix} \textit{irreducible admissible} \\ \textit{representations } \pi_v \textit{ of } H(\mathbb{Q}_v) \end{matrix} \right\} \to \left\{ \begin{matrix} \textit{admissible homomorphisms} \\ \phi_v : W'_{\mathbb{Q}_v} \to {}^LH \end{matrix} \right\}$$

which satisfies .... a list of representation theoretic desiderata.

Part of the desiderata should be an equality of L-functions. But Langlands used this formulation to define the L-functions, namely if $r : {}^LH \to GL_n(\mathbb{C})$ is an L-homomorphism of the type we saw in Part II (although now not restricted to come from the theory of induced representations as there) then Langlands *defined*

$$L(s, \pi_v, r) = L(s, r \circ \phi_v)$$

where now the L-function on the right is a generalized Artin L-function. So we should think of the right hand side of this local Langlands conjecture as giving special families of Galois representations that parametrize representations of H and lead to a formulations of L-functions for H as Artin L-functions. The fibres of the parametrization are called *L-packets*, because under this formalism, the representations in the fibres will have the same L-functions for any representation r of the dual group ${}^L H$.

What to we know in the direction of this conjecture? We know the conjecture in the following instances. (Lets talk over a number field k and its completions.)

(i) $k_v = \mathbb{R}$, $\mathbb{C}$ and any H, by Langlands,
(ii) k_v non-archimedean and π_v unramified, by Satake.
(iii) k_v non-archimedean and $H = GL_n$, the characteristic 0 case by Harris-Taylor, Henniart.
(iv) k_v non-archimedean and $H = GSp_4$ or Sp_4, by Gan and Takeda.
(v) k_v non-archimedean, $H = SO_{2n+1}$, and π_v generic, by Jiang and Soudry.
(vi) Possible other miscellaneous cases that I am unaware of.

As for GL_n, we formulate a version of the **Global Langlands Conjecture for** H through a local/global compatibility.

6.4. **Functoriality.** Thinking of the local or global Langlands conjectures in terms of arithmetic parametrizations of admissible or automorphic representations, these Langlands conjectures lead to a formulation of transferring admissible/automorphic representations from H to G. We will concentrate on the case where $G = GL_N$, as this is most compatible with out methods, but any quasi-split target group will do.

There is a new ingredient necessary for this transfer. This is an **L-homomorphism**, which is a complex analytic map

$$u : {}^L H \to {}^L G = {}^L GL_N = GL_N(\mathbb{C}) \times \mathrm{Gal}(\bar{k}/k)$$

which is compatible with the projections to the Galois factors. (In the local formulation, we use $\mathrm{Gal}(\bar{k}_v/k_v)$.) In our example from Part II, where $H = Sp_{2n}$, then $\widehat{H} = SO_{2n+1}$ and we can take for u the map coming from the natural embedding

$$u : SO_{2n+1}(\mathbb{C}) \hookrightarrow GL_{2n+1}(\mathbb{C}).$$

There are similar embeddings for the other split and even quasi-split classical groups.

A formulation of local functoriality from H to GL_N relies on assuming the local Langlands conjecture for H and recalling that we know the local Langlands conjecture for GL_N by Harris-Taylor/Henniart.

Local Functoriality. *Assume we know the local Langlands conjecture for $H(k_v)$. Let π_v be an irreducible admissible representation of $H(k_v)$ with* *associated Langlands parameter ϕ_v. Then we simply follow the diagram*

$$\begin{tikzcd}
 & & {}^L H \arrow[rr, "u"] & & {}^L Gl_N & & \\
\pi_v \arrow[r, mapsto] & \phi_v & & & & \Phi_v \arrow[r, mapsto, "HT/H"] & \Pi_v . \\
 & & & W'_{k_v} \arrow[uul] \arrow[uur] & & &
\end{tikzcd}$$

to obtain an irreducible admissible representation Π_v of $GL_n(k_v)$.

This formalism comes complete with an equality of L-and ε-factors, for example

$$L(s, \pi_v, u) = L(s, u \circ \phi_v) = L(s, \Phi_v) = L(s, \Pi_v).$$

Note that we know how to carry this out whenever we know the local Langlands parametrization for π_v, i.e., when k_v is archimedean or when k_v is non-archimedean and π_v is unramified. So in fact, if we are given a global automorphic representation $\pi = \otimes' \pi_v$ of $H(\mathbb{A})$, we can transfer the local representation π_v for all v outside a finite set S of finite places.

We can formulate a global version of this using the local/global principal and the local functoriality diagram. It relies on knowing the local Langlands conjecture for H at all places.

Global Functoriality Conjecture. *Assume the local Langlands conjecture for H at all places of k. If $\pi = \otimes' \pi_v$ is a cuspidal automorphic representation of $H(\mathbb{A})$ then the representation $\Pi = \otimes' \Pi_v$ of $GL_N(\mathbb{A})$ obtained by following the local functoriality diagrams*

$$\begin{tikzcd}
 & & & {}^L H \arrow[rr, "u"] & & {}^L GL_N & & & \\
\pi = \otimes' \pi_v & \pi_v \arrow[r, mapsto] & \phi_v & & & & \Phi_v \arrow[r, mapsto] & \Pi_v & \Pi = \otimes' \Pi_v \\
 & & & & W'_{k_v} \arrow[uul] \arrow[uur] & & & &
\end{tikzcd}$$

is automorphic.

This global formalism also comes complete with an equality of L- and ε-factors

$$L(s, \pi, u) = \prod_v L(s, \pi_v, u) = \prod_v L(s, \Pi_v) = L(s, \Pi)$$

along with similar equalities for twisted versions.

7. The Converse Theorem and Functoriality

In the global functoriality diagram, one begins with a cuspidal automorphic representation $\pi = \otimes' \pi_v$ of $H(\mathbb{A})$ and produce an irreducible admissible representation $\Pi = \otimes' \Pi_v$ of $GL_N(\mathbb{A})$. Such a representation of GL_N could be the input to the Converse Theorem for GL_N. Since the diagram comes with an identity of twisted L-functions, one could verify the conditions of the Converse Theorem for Π by controlling the analytic properties of the twisted L-functions for H. But this is exactly what came out of the process in Part II. So we are now in the position of combining the L-function techniques of Part I, which come from the theory of integral representations for GL_N, and the L-functions techniques of Part II, which come from the relation with Eisenstein series, to attack the question of functoriality via L-function techniques.

7.1. **Functoriality for the classical groups.** The first instances where we could do this were the families of (quasi)split classical groups. For now, we will content ourselves with the split cases. As we have seen in our example that we carried through in Part II, for Sp_{2n}, we can control the twisted L-functions $L(s, \pi \times \tau)$ for π a globally generic cuspidal representation of $Sp_{2n}(\mathbb{A})$ and τ a cuspidal representation of $GL_m(\mathbb{A})$. The twisted L-functions for the other split classical groups occur in a similar fashion. So we will be able to consider functoriality for globally generic cuspidal representations in the following split situations.

H	${}^L H$	$u : {}^L H \to {}^L GL_N$	${}^L GL_N$	GL_N
SO_{2n+1}	$Sp_{2n}(\mathbb{C})$	$\hookrightarrow$	$\mathrm{GL}_{2n}(\mathbb{C})$	GL_{2n}
SO_{2n}	$SO_{2n}(\mathbb{C})$	$\hookrightarrow$	$\mathrm{GL}_{2n}(\mathbb{C})$	GL_{2n}
Sp_{2n}	$SO_{2n+1}(\mathbb{C})$	$\hookrightarrow$	$\mathrm{GL}_{2n+1}(\mathbb{C})$	GL_{2n+1}

Take an H from this table and let $\pi = \otimes' \pi_v$ be a globally generic cuspidal representation of $H(\mathbb{A})$. For almost all places v of k we know the local Langlands conjecture for $H(k_v)$, namely for the archimedean places and the finite places for which π_v is unramified. So for these places we have our local functoriality diagram

$$\begin{tikzcd}
 & {}^L H \arrow[rr, "u"] & & {}^L Gl_N & \\
\pi_v \arrow[r, mapsto, "LLC"] & \phi_v & & \Phi_v \arrow[r, mapsto, "LLC"] & \Pi_v. \\
 & & W'_{k_v} \arrow[uul] \arrow[uur] & &
\end{tikzcd}$$

which preserves L-functions. However, there remains a finite set of S of non-archimedean places for which we do not know the local Langlands parameters for π_v. In spite of this we can still prove the following result.

Theorem 7.1. *Let H be a (quasi)split classical group, $\pi = \otimes'\pi_v$ a globally generic cuspidal representation of $H(\mathbb{A})$, and $u : {}^L H \to {}^L GL_N$ the natural L-homomorphism. Then there exists an automorphic representation $\Pi = \otimes'\Pi_v$ of $GL_n(\mathbb{A})$ such that for all $v \notin S$, Π_v is the local Langlands lift of π_v as in the above diagram.*

The proof has three main steps, but three lingering questions about their implementation.

Step 1. Construct a candidate lift $\Pi = \otimes'\Pi_v$ as an irreducible admissible representation of $GL_N(\mathbb{A})$ with the property that for all cuspidal automorphic representations τ of $GL_m(\mathbb{A})$ in an appropriate twisting set we have

$$L(s, \pi \times \tau) = L(s, \Pi \times \tau)$$
$$\varepsilon(s, \pi \times \tau) = \varepsilon(s, \Pi \times \tau)$$

For the places $v \notin S$, we can obtain this locally from the local functoriality diagram.
Question 1. What to do about Π_v for $v \in S$?

Step 2. Control the analytic properties of $L(s, \Pi \times \tau)$ through those of $L(s, \pi \times \tau)$. We do this using the Langlands-Shahidi method, as in the Summary in Section 5, where we are guaranteed that these twisted L-functions are meromorphic, BVS and satisfy an appropriate functional equation. But the Converse Theorem requires that all L-functions be entire, which is more that is guaranteed.
Question 2. How to uniformly control the poles of the L-functions for all the twists simultaneously?

Step 3. Apply the Converse Theorem to $\Pi = \otimes'\Pi_v$.
Question 3. Which one, i.e., what is the appropriate twisting set?

7.2. **Resolution of the difficulties.** We will take our difficulties in order, since the resolution of Questions 1 & 2 will force the answer to Question 3.

Question 1. What to do about the $v \in S$ where we do not know the local Langlands parametrization for π_v. We finesse the lack of the local Langlands parametrization by a purely local result called the "stability of L and ε under highly ramified twists". The proof of this is quite involved and is established in the context of the "stability of γ under highly ramified twists". The statement of stability is the following.

Theorem 7.2. *Suppose that η_v is a suitably highly ramified character of $k_v^\times$. Then for irreducible admissible representations π_v of $H(k_v)$ or Π_v of $GL_N(k_v)$ we have*

$$L(s, \pi_v \times \eta_v) \equiv 1 \equiv L(s, \Pi_v \times \eta_v)$$

and both $\varepsilon(s, \pi_v \times \eta_v, \psi_v)$ and $\varepsilon(s, \Pi_v \times \eta_v, \psi_v)$ stabilize, with the stable form only depending on the central character of the local representations.

The degree of ramification depends on π_v or Π_v respectively. Once you have stability, one can replace π_v and Π_v by a full induced representation with the same central character and then compute these stable form of the ε-factors (from the stable form of the γ-factors) by using multiplicativity of the γ-factors. When one does this, one obtains the following corollary.

Corollary 7.2.1. *Assume H is split. Then for any irreducible admissible generic representation π_v of $H(k_v)$ and any irreducible admissible representation Π_v of $GL_N(k_v)$ with trivial central character, for sufficiently highly ramified character η_v, depending on both representations π_v of $H(k_v)$ and Π_v of $GL_N(k_v)$, we have*

$$L(s, \pi_v \times \eta_v) = L(s, \Pi_v \times \eta_v)$$
$$\varepsilon(s, \pi_v \times \eta_v, \psi_v) = \varepsilon(s, \Pi_v \times \eta_v, \psi_v).$$

So if we have the freedom to twist, we can take as our "local lift" any Π_v with trivial central character. We can extend this to certain twists by $GL_m(k_v)$, again by multiplicativity of L- and ε- for certain representations.

Corollary 7.2.2. *Assume H is split. Let π_v be an irreducible admissible generic representation of $H(k_v)$ and Π_v irreducible admissible representation of $GL_N(k_v)$ with trivial central character. Let η_v be a sufficiently highly ramified character of $k_v^\times$ as in the above corollary. Then for every unramified representation τ_v° of $GL_m(k_v)$ if we set $\tau_v = \tau_v^\circ \otimes \eta_v$ then*

$$L(s, \pi_v \times \tau_v) = L(s, \Pi_v \times \tau_v)$$
$$\varepsilon(s, \pi_v \times \tau_v, \psi_v) = \varepsilon(s, \Pi_v \times \tau_v, \psi_v).$$

If we combine this last corollary at the places $v \in S$ with the local functoriality diagram for $v \notin S$ we arrive at the end of Step 1. Let $\pi = \otimes' \pi_v$ be a generic cuspidal automorphic representation of $H(\mathbb{A})$. For $v \notin S$, let Π_v be the local lift of π_v obtained through the local functoriality diagram. For

$v \in S$ take Π_v any irreducible admissible representation of $GL_n(k_v)$ with trivial central character. Consider the candidate lift $\Pi = \otimes \Pi_v$ of $GL_N(\mathbb{A})$.

Theorem 7.3. *Let $\eta = \otimes \eta_v$ be a fixed idele class character such that at all $v \in S$, η_v is sufficiently highly ramified so that the previous corollaries hold for π_v and Π_v. Then for every $\tau \in \mathcal{T}_0^S(N-2) \otimes \eta$ we have*

$$L(s, \pi \times \tau) = L(s, \Pi \times \tau)$$
$$\varepsilon(s, \pi \times \tau) = \varepsilon(s, \Pi \times \tau).$$

For the definition of $\mathcal{T}_0^S(N-2)$ see the paragraph before Theorem 2.4. This completes Step 1.

Question 2. The answer to our second question, controlling the poles of $L(s, \Pi \times \tau)$ and hence $L(s, \Pi \times \tau)$ we can now find in the Summary in Section 5 and the previous theorem. In Section 5 we saw that if we twisted by a sufficiently ramified character η then the resulting L-function had no poles. When one unravels the twisting there in terms of the Rankin-Selberg L-functions we are considering here, then we have the following result.

Theorem 7.4. *Let H be a split classical group and $\pi = \otimes' \pi$ a globally generic cuspidal representation of $H(\mathbb{A})$. Let S be a non-empty finite set of finite places. Let $\eta = \otimes \eta_v$ be a idele class character which is sufficiently highly ramified at the places $v \in S$ so that Theorems 5.4 and 5.5 are satisfied. Then for every $\tau \in \mathcal{T}_0^S(N-2) \otimes \eta$ we have that $L(s, \pi \times \tau)$ is nice, that is*

(i) *$L(s, \pi \times \tau)$ is an entire function of s;*
(ii) *$L(s, \pi \times \tau$ is bounded in vertical strips;*
(iii) *$L(s, \pi \times \tau) = \varepsilon(s, \pi \times \tau) L(1-s, \widetilde{\pi} \times \widetilde{\tau})$.*

So a similar twisting by a ramified character gives us the the twisted L-functions on the classical group are nice as desired in the Converse Theorem. This completes Step 2.

Question 3. We are now in a position to apply a Converse Theorem. We just need to combine Theorems 7.3 and 7.4. So we take our globally generic cuspidal representation $\pi = \otimes \pi_v$ of $H(\mathbb{A})$. We construct a candidate lift $\Pi = \otimes' \Pi_v$ as in Step 1. We take S to be a finite set of finite places such that (i) S contains all finite places where we do not have the local Langlands parametrization for π_v and (ii) S is not empty. We fix an idele class character $\eta = \otimes \eta_v$ such that the local components at $v \in S$ are sufficiently ramified that both Theorems 7.3 and 7.4 hold. Then combining these with the flexible Converse Theorem given in Theorem 2.4, we obtain Theorem 7.1, the functorial lifting from classical groups to GL_N for globally generic cuspidal representations. This completes Step 3.

7.3. **The image of functoriality.** We have two remaining issues.

(1) When is the functorial lift Π cuspidal?
(2) Can we characterize the image of functoriality?

These questions are answered by yet another set of techniques. This is the method of *automorphic descent* of Ginzburg, Rallis, and Soudry. It was

- motivated by integral representations of twisted L-functions for classical groups
- implemented in terms of residues of Eisenstein series.

So it represents a nice synthesis of ideas from the two methods for studying L-functions. The descent allows you to begin with a self-dual representation of GL_N, local or global, and descend back to a representation of the classical group H (or a metaplectic group).

If we combine our approach to functoriality via the Converse Theorem with the Ginzburg-Rallis-Soudry theory of descent we obtain the following more complete result.

Theorem 7.5. *Let H be a split classical group, π a globally generic cuspidal representation of $H(\mathbb{A})$. Then there is a representation $R = R_H$ of ${}^LGL_n$ such that the functorial lift of π to an automorphic representation Π of $GL_N(\mathbb{A})$ as above has trivial central character and is of the form*

$$\Pi = \mathrm{Ind}(\Pi_1 \otimes \cdots \otimes \Pi_d) = \Pi_1 \boxplus \cdots \boxplus \Pi_d$$

where each Π_i is a unitary self-dual representation of $GL_{N_i}(\mathbb{A})$ such that the L-function $L(s, \Pi_i, R)$ has a pole at $s = 1$ and $\Pi_i \not\simeq \Pi_j$ for $i \neq j$. Moreover any such Π is the functorial lift of some globally generic cuspidal representation π of $H(\mathbb{A})$.

To finish, we only need to identify $R = R_H$. For $H = Sp_{2n}$ or SO_{2n}, so that ${}^LH$ is an orthogonal group, $R = \mathrm{Sym}^2$, while for $H = SO_{2n+1}$, where ${}^LH = Sp_{2n}$, we have $R = \Lambda^2$.

For the case of the quasisplit classical groups, the result is similar, but there is also a central character condition.

8. Symmetric powers and applications

There are a number of other functorialities that one can obtain by this method that have striking arithmetic applications. As the method is the same as before, here I will describe the L-homomorphisms and how one controls the twisted L-functions needed for the Converse Theorem within the Langlands-Shahidi method.

8.1. **The tensor product lifting from $GL_2 \times GL_3$ to GL_6.** The L-group of GL_n is $GL_n(\mathbb{C})$. The tensor product isomorphism $\mathbb{C}^2 \otimes \mathbb{C}^3 \simeq \mathbb{C}^6$ induces a map

$$GL_2(\mathbb{C}) \times GL_3(\mathbb{C}) \xrightarrow{\quad\otimes\quad} GL_6(\mathbb{C})$$

and there should be an associated map of automorphic representations given by the following functoriality diagram.

Let $\pi_1 = \otimes' \pi_{1,v}$ be a cuspidal automorphic representation of $GL_2(\mathbb{A})$ and $\pi_2 = \otimes' \pi_{2,v}$ a cuspidal automorphic representation of $GL_3(\mathbb{A})$. Write $\pi_1 \times \pi_2$

for the corresponding representation of $GL_2(\mathbb{A}) \times GL_3(\mathbb{A})$. Then for each place v of k, since we know local Langlands for GL_n, we have

$$\begin{tikzcd}
 & & GL_2(\mathbb{C}) \times GL_3(\mathbb{C}) \arrow[rr, "\otimes"] & & GL_6(\mathbb{C}) & \\
\pi_{1,v} \times \pi_{2,v} \arrow[r, mapsto] & \phi_{1,v} \times \phi_{2,v} & & & \Phi_v \arrow[r, mapsto] & \Pi_v \\
 & & & W'_{k_v} \arrow[uul] \arrow[uur] & &
\end{tikzcd}$$

which then gives us an irreducible admissible representation $\Pi = \otimes \Pi_v$ of $GL_6(\mathbb{A})$. Let us denote $\Pi_v = \pi_{1,v} \boxtimes \pi_{2,v}$ and $\Pi = \pi_1 \boxtimes \pi_2$.

Theorem 8.1 ([10] Theorem 9.1)**.** $\pi_1 \boxtimes \pi_2$ *is an automorphic representation of* $GL_6(\mathbb{A})$

To apply the Converse Theorem for GL_6 we must now control the triple product L-functions $L(s, \pi_1 \times \pi_2 \times \tau)$ for τ a cuspidal representation of $GL_m(\mathbb{A})$ with $m = 1, 2, 3, 4$. Where do these occur in the Langlands-Shahidi list?

GL_1 **twists.** We take $G = GL_5$. Its Dynkin diagram is

$$\bullet \text{ --- } \bullet \text{ --- } \bullet \text{ --- } \bullet$$

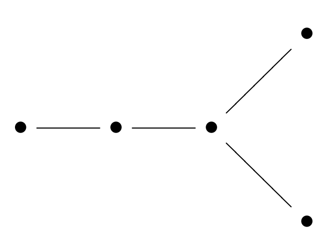
and if we remove the second vertex

$$\bullet \text{ --- } \times \text{ --- } \bullet \text{ --- } \bullet$$

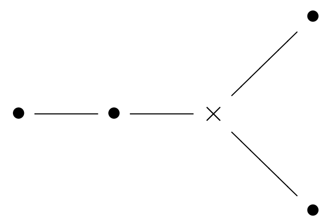
we obtain a maximal parabolic subgroup $P = MN$ whose Levi has derived group $M \simeq GL_2 \times GL_3$. If we take the representation of M to be $\sigma \simeq \pi_1 \otimes (\pi_2 \otimes \tau)$, then this is an $m = 1$ situation and we have, in the notation of Part II,

$$L(s, \sigma, r_1) = L(s, \pi_1 \times \pi_2 \times \tau).$$

GL_2 **twists.** We take $G = D_5^{sc} = Spin_{10}$. Its Dynkin diagram is



and if we remove the third vertex

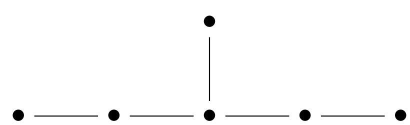


we obtain a maximal parabolic subgroup $P = MN$ whose Levi subgroup has derived group $M_{der} \simeq SL_3 \times SL_2 \times SL_2$. This is the same as the derived group of $GL_3 \times GL_2 \times GL_2$ and there is a k-rational morphism from M to $GL_3 \times GL_2 \times GL_2$ which is the identity on the derived groups. There is then a way to transfer the representation $\pi_2 \otimes \pi_1 \otimes \tau$ of $GL_3 \times GL_2 \times GL_2$ to a representation σ of M such that

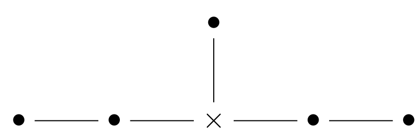
$$L(s, \sigma, r_1) = L(s, \pi_1 \times \pi_2 \times \tau).$$

We will denote this (and similar) process by writing $\sigma \sim \pi_2 \otimes \pi_1 \otimes \tau$. This is an $m = 2$ situation.

GL_3 **twists.** We take G to be the exceptional group E_6^{sc}. Its Dynkin diagram is



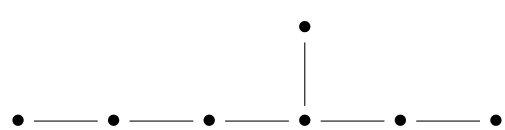
and if we remove the third vertex



we obtain a maximal parabolic subgroup $P = MN$ whose Levi subgroup has derived group $M_{der} \simeq SL_3 \times SL_2 \times SL_3$. If we take the representation of M so that $\sigma \sim \pi_2 \otimes \pi_1 \otimes \tau$, then this is an $m = 3$ situation with

$$L(s, \sigma, r_1) = L(s, \pi_1 \times \pi_2 \times \tau).$$

GL_4 **twists.** We take G to be the exceptional group E_7^{sc}. Its Dynkin diagram is



and if we remove the fourth vertex

$$\begin{tikzcd}[column sep=small]
& & & \bullet \arrow[d, dash] & & \\
\bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet \arrow[r, dash] & \times \arrow[r, dash] & \bullet \arrow[r, dash] & \bullet
\end{tikzcd}$$

we obtain a maximal parabolic subgroup $P = MN$ whose Levi subgroup has derived group $M_{der} \simeq SL_4 \times SL_2 \times SL_3$. If we take the representation of M so that $\sigma \sim \tau \otimes \pi_1 \otimes \pi_2$, then this is an $m = 4$ situation with

$$L(s, \sigma, r_1) = L(s, \pi_1 \times \pi_2 \times \tau).$$

Even in this relatively simple case of a tensor product functoriality, we need Eisenstein series on exceptional groups! As usual, the Converse Theorem gives an automorphic representation Π' with $\Pi'_v \simeq \pi_{1,v} \boxtimes \pi_{2,v}$ for almost all places v. To fill in the correct lift at the $v \in S$ we need other techniques: base change and the theory of types.

8.2. **Symmetric power functorialities.** $GL_2(\mathbb{C})$, the L-group of GL_2, has an irreducible $n+1$ dimensional representation on the space of symmetric n-tensors $\mathrm{Sym}^n(\mathbb{C}^2)$. This gives a map

$$GL_2(\mathbb{C}) \xrightarrow{\mathrm{Sym}^n} GL_{n+1}(\mathbb{C})$$

and there should be an associated lifting of automorphic representations from $GL_2(\mathbb{A})$ to $GL_{n+1}(\mathbb{A})$. The relevant functoriality diagram is

$$\begin{tikzcd}
& & GL_2(\mathbb{C}) \arrow[rr, "\mathrm{Sym}^n"] & & GL_{n+1}(\mathbb{C}) & & \\
\pi = \otimes' \pi_v & \pi_v \arrow[r, mapsto] & \phi_v & & \Phi_v \arrow[r, mapsto] & \Pi_v & \Pi = \otimes' \Pi_v . \\
& & & W'_{k_v} \arrow[uul] \arrow[uur] & & &
\end{tikzcd}$$

Let us then write $\Pi_v = \mathrm{Sym}^n(\pi_v)$, a well defined irreducible admissible representation of $GL_{n+1}(k_v)$, and $\Pi = \mathrm{Sym}^n(\pi) = \prod \mathrm{Sym}^n(\pi_v)$ an irreducible admissible representation of $GL_{n+1}(\mathbb{A})$. The question of Sym^n functoriality for GL_2 is then whether $\mathrm{Sym}^n(\pi)$ is automorphic.

As a consequence of the $GL_2 \times GL_3 \to GL_6$ tensor product functoriality, we have that $\mathrm{Sym}^3(\pi)$ is indeed automorphic.

Theorem 8.2 ([10] Theorem 9.2)**.** *If π is a cuspidal automorphic representation of $GL_2(\mathbb{A})$ then $\mathrm{Sym}^3(\pi)$ is an automorphic representation of $GL_4(\mathbb{A})$. It is cuspidal unless π is of dihedral or tetrahedral type.*

This result is finessed from the tensor product functoriality as follows. Begin with π, our cuspidal automorphic representation of $GL_2(\mathbb{A})$. Gelbart and Jacquet had proved that $\mathrm{Sym}^2(\pi)$ is an automorphic representation of $GL_3(\mathbb{A})$ via the Converse Theorem for GL_3 in 1978. Then by the tensor product functoriality for $GL_2 \times GL_3$ above we have that $\pi \boxtimes \mathrm{Sym}^2(\pi)$ is an automorphic representation of $GL_6(\mathbb{A})$. Moreover, we have a decomposition

$$\pi \boxtimes (\mathrm{Sym}^2(\pi) \otimes \omega_\pi^{-1}) = (\mathrm{Sym}^3(\pi) \otimes \omega_\pi^{-1}) \boxplus \pi$$

from which we deduce that $\mathrm{Sym}^3(\pi)$ is automorphic.

8.3. **The exterior square lifting for GL_4.** From the six dimensional irreducible representation of $GL_4(\mathbb{C})$ on the space of antisymmetric two tensors $\Lambda^2(\mathbb{C}^4)$ we obtain an L-homomorphism

$$GL_4(\mathbb{C}) \xrightarrow{\Lambda^2} GL_6(\mathbb{C})$$

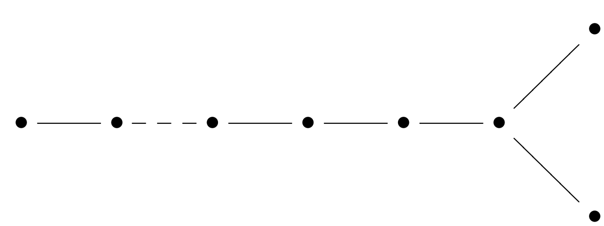
and from it local and global lifts. If $\pi = \otimes \pi_v$ is a cuspidal automorphic representation of $GL_4(\mathbb{C})$ then $\Lambda^2(\pi_v)$ is an irreducible admissible representation of $GL_6(k_v)$ and $\Lambda^2(\pi) = \otimes' \Lambda^2(\pi_v)$ is an irreducible admissible representation of $GL_6(\mathbb{A})$. Henry Kim established the corresponding functoriality in the following form.

Theorem 8.3 ([10] Theorem 9.3)**.** *Let π be a cuspidal automorphic representation of $GL_4(\mathbb{A})$. Then there exists an automorphic representation Π' of $GL_6(\mathbb{A})$ such that for almost all places v, $\Pi'_v \simeq \Lambda^2(\pi_v)$*

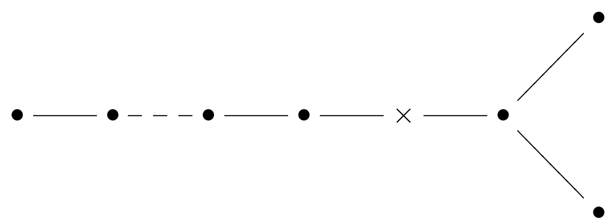
Again, we use the Converse Theorem for GL_6 and thereby must control the twisted L-functions

$$L(s, \pi \otimes \tau, \Lambda^2 \otimes St_m)$$

for τ cuspidal automorphic representations of $GL_m(\mathbb{A})$ for $1 \leq m \leq 4$, where St_m is the standard representation of $GL_m(\mathbb{C})$. These are controlled as in Part II, by the Langlands-Shahidi method. In that notation, we take $G = D_{k+4}^{sc} = Spin_{2k+8}$ for $k = 0, 1, 2, 3$. The Dynkin diagram for G is



and if we remove the appropriate vertex



we obtain a maximal parabolic subgroup $P = MN$ whose Levi subgroup has derived group $M_{der} \simeq SL_{k+1} \times SL_4$. If we take the representation of M so that $\sigma \sim \tau \otimes \pi$, then this is an $m = 2$ situation with

$$L(s, \sigma, r_1) = L(s, \pi \otimes \tau, \Lambda^2 \otimes St_{k+1}).$$

This then gives the theorem.

From this result we can deduce another symmetric power functoriality for GL_2.

Corollary 8.3.1 ([10] Corollary 9.4)**.** *If π is a cuspidal automorphic representation of $GL_2(\mathbb{A})$ then $\mathrm{Sym}^4(\pi)$ is an automorphic representation of $GL_5(\mathbb{A})$.*

To derive this, we begin with our cuspidal representation π of $GL_2(\mathbb{A})$. We know that $\mathrm{Sym}^3(\pi)$ is an automorphic representation of $GL_4(\mathbb{A})$, so we can now form $\Lambda^2(\mathrm{Sym}^3(\pi))$ which is an automorphic representation of $GL_6(\mathbb{A})$. This is not irreducible, but decomposes as

$$\Lambda^2(\mathrm{Sym}^3(\pi)) = (\mathrm{Sym}^4(\pi) \otimes \omega_\pi) \boxplus \omega_\pi^3$$

from which we deduce the automorphy of $\mathrm{Sym}^4(\pi)$.

8.4. **Applications.** The new symmetric power functorialities give us improved bounds towards the Ramanujan and Selberg conjectures for GL_2 over an arbitrary number field.

Let $\pi = \otimes' \pi_v$ be a unitary cuspidal representation of $GL_2(\mathbb{A})$. If v is a finite place where π_v is unramified and $\phi_v : W'_{k_v} \to GL_2(\mathbb{C})$ the arithmetic Langlands parameter for π_v, then ϕ_v is also unramified and

$$\phi_v(Frob_v) = \begin{pmatrix} \alpha_v & \\ & \beta_v \end{pmatrix} = t_v \in GL_2(\mathbb{C}).$$

These are the Satake parameters for π_v. The Ramanujan conjecture states that

$$q_v^{-\epsilon} < |\alpha_v|, |\beta_v| < q_v^{\epsilon}$$

for all $\epsilon > 0$

Theorem 8.4 ([10] Theorem 9.6)**.** *For π a unitary cuspidal representation of $GL_2(\mathbb{A})$, the Satake parameters for π_v satisfy*

$$q_v^{-1/9} < |\alpha_v|, |\beta_v| < q_v^{1/9}.$$

The first idea is to use the automorphy of the symmetric powers. Suppose that $\mathrm{Sym}^4(\pi)$ is cuspidal. (The non-cuspidal case is in fact easier.) Then the Satake parameters for $\mathrm{Sym}^4(\pi)$ are given by

$$\mathrm{Sym}^4(t_v) = \mathrm{Sym}^4\left(\begin{pmatrix} \alpha_v & \\ & \beta_v \end{pmatrix}\right) = \begin{pmatrix} \alpha_v^4 & & & & \\ & \alpha_v^3\beta_v & & & \\ & & \ddots & & \\ & & & \alpha_v\beta_v^3 & \\ & & & & \beta_v^4 \end{pmatrix}.$$

Now there is a general first non-trivial bound towards Ramanujan valid for any GL_n due to Jacquet and Shalika. It says that the Satake parameters for any unitary cuspidal representation for $GL_n(\mathbb{A})$ satisfy

$$q_v^{-1/2} \leq |\alpha_{v,i}| \leq q_v^{1/2}.$$

Applying this to the Satake parameters for $\mathrm{Sym}^4(\pi)$ gives

$$q_v^{-1/2} \leq |\alpha_v^4|, \beta_v^4| \leq q_v^{1/2}$$

or

$$q_v^{-1/8} \leq |\alpha_v|, \beta_v| \leq q_v^{1/8}.$$

To achieve the "1/9 bound" one has to invoke other techniques. What Kim and Shahidi prove is that $L(s, \pi_v, \mathrm{Sym}^9)$ is holomorphic for $\mathrm{Re}(s) \geq 1$ and this gives the stated 1/9 bound

There is an archimedean analogue giving a 1/9 bound towards the general Selberg conjecture for GL_2.

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