ICM 2002 · Vol. II · 119–128

Converse Theorems, Functoriality, and Applications to Number Theory

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Abstract

There has been a recent coming together of the Converse Theorem for GL_n and the Langlands-Shahidi method of controlling the analytic properties of automorphic L-functions which has allowed us to establish a number of new cases of functoriality, or the lifting of automorphic forms. In this article we would like to present the current state of the Converse Theorem and outline the method one uses to apply the Converse Theorem to obtain liftings. We will then turn to an exposition of the new liftings and some of their applications.

2000 Mathematics Subject Classification: 11F70 (22E55).
Keywords and Phrases: Automorphic forms, L-functions, Converse Theorems, Functoriality.

1. Introduction

Converse Theorems traditionally have provided a way to characterize Dirichlet series associated to modular forms in terms of their analytic properties. Most familiar are the Converse Theorems of Hecke and Weil. Hecke first proved that L-functions associated to modular forms enjoyed "nice" analytic properties and then proved "Conversely" that these analytic properties in fact characterized modular L-functions. Weil extended this Converse Theorem to L-functions of modular forms with level.

In their modern formulation, Converse Theorems are stated in terms of automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ instead of modular forms. Jacquet, Piatetski-Shapiro, and Shalika have proved that the L-functions associated to automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ have nice analytic properties via integral representations similar to those of Hecke. The relevant "nice" properties are: analytic continuation, boundedness in vertical strips, and functional equation. Converse Theorems in this

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context invert these integral representations. They give a criterion for an irreducible admissible representation Π of $\mathrm{GL}_n(\mathbb{A})$ to be automorphic and cuspidal in terms of the analytic properties of Rankin-Selberg convolution L-functions $L(s,\Pi\times\pi')$ of Π twisted by cuspidal representations π' of $\mathrm{GL}_m(\mathbb{A})$ of smaller rank groups.

To use Converse Theorems for applications, proving that certain objects are automorphic, one must be able to show that certain L-functions are "nice". However, essentially the only way to show that an L-function is nice is to have it associated to an automorphic form. Hence the most natural applications of Converse Theorems are to functoriality, or the lifting of automorphic forms, to GL_n. More explicit number theoretic applications then come as consequences of these liftings.

Recently there have been several applications of Converse Theorems to establishing functorialities. These have been possible thanks to the recent advances in the Langlands-Shahidi method of analysing the analytic properties of general automorphic L-functions, due to Shahidi and his collaborators [21]. By combining our Converse Theorems with their control of the analytic properties of L-functions many new examples of functorial liftings to GL_n have been established. These are described in Section 4 below. As one number theoretic consequence of these liftings Kim and Shahidi have been able to establish the best general estimates over a number field towards the Ramamujan-Selberg conjectures for GL_2, which in turn have already had other applications.

2. Converse Theorems for GL_n

Let k be a global field, $\mathbb{A}$ its adele ring, and ψ a fixed non-trivial (continuous) additive character of $\mathbb{A}$ which is trivial on k. We will take $n\geq 3$ to be an integer.

To state these Converse Theorems, we begin with an irreducible admissible representation Π of $\mathrm{GL}_n(\mathbb{A})$. It has a decomposition $\Pi=\otimes'\Pi_v$, where Π_v is an irreducible admissible representation of $\mathrm{GL}_n(k_v)$. By the local theory of Jacquet, Piatetski-Shapiro, and Shalika [9, 11] to each Π_v is associated a local L-function $L(s,\Pi_v)$ and a local ε-factor $\varepsilon(s,\Pi_v,\psi_v)$. Hence formally we can form

$$L(s,\Pi)=\prod L(s,\Pi_v) \qquad \text{and} \qquad \varepsilon(s,\Pi,\psi)=\prod \varepsilon(s,\Pi_v,\psi_v).$$

We will always assume the following two things about Π:

1. $L(s,\Pi)$ converges in some half plane $Re(s)>>0$,
2. the central character ω_Π of Π is automorphic, that is, invariant under $k^\times$.

Under these assumptions, $\varepsilon(s,\Pi,\psi)=\varepsilon(s,\Pi)$ is independent of our choice of ψ [4].

As in Weil's case, our Converse Theorems will involve twists but now by cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A})$ for certain m. For convenience, let us set $\mathcal{A}(m)$ to be the set of automorphic representations of $\mathrm{GL}_m(\mathbb{A})$, $\mathcal{A}_0(m)$ the set of (irreducible) cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A})$, and $\mathcal{T}(m)=\bigcup_{d=1}^m \mathcal{A}_0(d)$. If S is a finite set of places, we will let $\mathcal{T}^S(m)$ denote the subset of representations $\pi\in\mathcal{T}$ with local components π_v unramified at all places $v\in S$ and let $\mathcal{T}_S(m)$ denote those π which are unramified for all $v\notin S$.

Let $\pi' = \otimes' \pi'_v$ be a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$ with $m < n$. Then again we can formally define

$$L(s, \Pi \times \pi') = \prod L(s, \Pi_v \times \pi'_v) \qquad \text{and} \qquad \varepsilon(s, \Pi \times \pi') = \prod \varepsilon(s, \Pi_v \times \pi'_v, \psi_v)$$

since the local factors make sense whether Π is automorphic or not. A consequence of (1) and (2) above and the cuspidality of π' is that both $L(s, \Pi \times \pi')$ and $L(s, \widetilde{\Pi} \times \widetilde{\pi}')$ converge absolutely for $Re(s) >> 0$, where $\widetilde{\Pi}$ and $\widetilde{\pi}'$ are the contragredient representations, and that $\varepsilon(s, \Pi \times \pi')$ is independent of the choice of ψ.

We say that $L(s, \Pi \times \pi')$ is *nice* if it satisfies the same analytic properties it would if Π were cuspidal, i.e.,

1. $L(s, \Pi \times \pi')$ and $L(s, \widetilde{\Pi} \times \widetilde{\pi}')$ have continuations to *entire* functions of s,
2. these entire continuations are *bounded in vertical strips* of finite width,
3. they satisfy the standard *functional equation*

$$L(s, \Pi \times \pi') = \varepsilon(s, \Pi \times \pi') L(1 - s, \widetilde{\Pi} \times \widetilde{\pi}').$$

The basic converse theorem for GL_n is the following.

Theorem 1. [6] *Let Π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ as above. Let S be a finite set of finite places. Suppose that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}^S(n-2)$. Then Π is quasi-automorphic in the sense that there is an automorphic representation Π' such that $\Pi_v \simeq \Pi'_v$ for all $v \notin S$. If S is empty, then in fact Π is a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$.*

It is this version of the Converse Theorem that has been used in conjunction with the Langlands–Shahidi method of controlling analytic properties of L-functions in the new examples of functoriality explained below.

Theorem 2. [4] *Let Π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ as above. Let S be a non-empty finite set of places, containing S_∞, such that the class number of the ring $\mathfrak{o}_S$ of S-integers is one. Suppose that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}_S(n-1)$. Then Π is quasi-automorphic in the sense that there is an automorphic representation Π' such that $\Pi_v \simeq \Pi'_v$ for all $v \in S$ and all $v \notin S$ such that both Π_v and Π'_v are unramified.*

This version of the Converse Theorem was specifically designed to investigate functoriality in the cases where one controls the L-functions by means of integral representations where it is expected to be more difficult to control twists.

The proof of Theorem 1 with S empty and $n-2$ replaced by $n-1$ essentially follows the lead of Hecke, Weil, and Jacquet–Langlands. It is based on the integral representations of L-functions, Fourier expansions, Mellin inversion, and finally a use of the weak form of Langlands spectral theory. For Theorems 1 and 2 where we have restricted our twists either by ramification or rank we must impose certain local conditions to compensate for our limited twists. For Theorem 1 are a finite number of local conditions and for Theorem 2 an infinite number of local conditions. We must then work around these by using results on generation of congruence subgroups and either weak approximation (Theorem 1) or strong approximation (Theorem 2).

As for our expectations of what form the Converse Theorem may take in the future, we refer the reader to the last section of [6].

3. Functoriality via the Converse Theorem

In order to apply these these theorems, one must be able to control the analytic properties of the L-function. However the only way we have of controlling global L-functions is to associate them to automorphic forms or representations. A minute's thought will then convince one that the primary application of these results will be to the lifting of automorphic representations from some group H to GL_n.

Suppose that H is a reductive group over k. For simplicity of exposition we will assume throughout that H is split and deal only with the connected component of its L-group, which we will (by abuse of notation) denote by ${}^L\mathrm{H}$ [1]. Let $\pi = \otimes' \pi_v$ be a cuspidal automorphic representation of H and ρ a complex representation of ${}^L\mathrm{H}$. To this situation Langlands has associated an L-function $L(s,\pi,\rho)$ [1]. Let us assume that ρ maps ${}^L\mathrm{H}$ to $\mathrm{GL}_n(\mathbb{C})$. Then by Langlands' general Principle of Functoriality to π should be associated an automorphic representation Π of $\mathrm{GL}_n(\mathbb{A})$ satisfying $L(s,\Pi) = L(s,\pi,\rho)$, $\varepsilon(s,\Pi) = \varepsilon(s,\pi,\rho)$, with similar equalities locally and for the twisted versions [1]. Using the Converse Theorem to establish such liftings involves three steps: construction of a candidate lift, verification that the twisted L-functions are "nice", and application of the appropriate Converse Theorem.

1. *Construction of a candidate lift*: We construct a candidate lift $\Pi = \otimes' \Pi_v$ on $\mathrm{GL}_n(\mathbb{A})$ place by place. We can see what Π_v should be at almost all places. Since we have the arithmetic Langlands (or Hecke-Frobenius) parameterization of representations π_v of $\mathrm{H}(k_v)$ for all archimedean places and those non-archimedean places where the representations are unramified [1], we can use these to associate to π_v and the map $\rho_v : {}^L\mathrm{H}_v \to {}^L\mathrm{H} \to \mathrm{GL}_n(\mathbb{C})$ a representation Π_v of $\mathrm{GL}_n(k_v)$. This correspondence preserves local L– and ε–factors

$$L(s,\Pi_v) = L(s,\pi_v,\rho_v) \qquad \text{and} \qquad \varepsilon(s,\Pi_v,\psi_v) = \varepsilon(s,\pi_v,\rho_v,\psi_v)$$

along with the twisted versions. If H happens to be GL_m or a related group then we in principle know how to associate the representation Π_v at all places now that the local Langlands conjecture has been solved for GL_m. For other situations, we may not know what Π_v should be at the ramified places. We will return to this difficulty momentarily and show how one can work around this with the use of a highly ramified twist. But for now, let us assume we can finesse this local problem and arrive at a global representation $\Pi = \otimes' \Pi_v$ such that

$$L(s,\Pi) = \prod L(s,\Pi_v) = \prod L(s,\pi_v,\rho_v) = L(s,\pi,\rho)$$

and similarly $\varepsilon(s,\Pi) = \varepsilon(s,\pi,\rho)$ with similar equalities for the twisted versions. Π should then be the Langlands lifting of π to $\mathrm{GL}_n(\mathbb{A})$ associated to ρ.

2. *Analytic properties of global L-functions*: For simplicity of exposition, let us now assume that ρ is simply a standard embedding of ${}^L\mathrm{H}$ into $\mathrm{GL}_n(\mathbb{C})$, such as will be the case if we consider H to be a split classical group, so that $L(s,\pi,\rho) = L(s,\pi)$ is the standard L-function of π. We have our candidate Π for the lift of π to GL_n from above. To be able to assert that the Π which we constructed place by place is automorphic, we will apply a Converse Theorem. To do so we must control the twisted L-functions $L(s,\Pi\times\pi') = L(s,\pi\times\pi')$ for $\pi' \in \mathcal{T}$ with an appropriate

twisting set $\mathcal{T}$ from Theorem 1 or 2. In the examples presented below, we have used Theorem 1 above and the analytic control of $L(s, \pi \times \pi')$ achieved by the so-called Langlands-Shahidi method of analyzing the L-functions through the Fourier coefficients of Eisenstein series [21]. Currently this requires us to take k to be a number field. The *functional equation* $L(s, \pi \times \pi') = \varepsilon(s, \pi \times \pi')L(1-s, \tilde{\pi} \times \tilde{\pi}')$ has been proved in wide generality by Shahidi [18]. The *boundedness in vertical strips* has been proved in close to the same generality by Gelbart and Shahidi [7]. As for the entire continuation of $L(s, \pi \times \pi')$, a moments thought will tell you that one should not always expect a cuspidal representation of $\mathrm{H}(\mathbb{A})$ to necessarily lift to a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$. Hence it is unreasonable to expect all $L(s, \pi \times \pi')$ to be entire. We had previously understood how to work around this difficulty from the point of view of integral representations by again using a highly ramified twist. Kim realized that one could also control the entirety of these twisted L-functions in the context of the Langlands–Shahidi method by using a highly ramified twist. We will return to this below. Thus in a fairly general context one has that $L(s, \pi \times \pi')$ is *entire* for π' in a suitably modified twisting set $\mathcal{T}'$.

3. *Application of the Converse Theorem*: Once we have that $L(s, \pi \times \pi')$ is nice for a suitable twisting set $\mathcal{T}'$ then from the equalities

$$L(s, \Pi \times \pi') = L(s, \pi \times \pi') \qquad \text{and} \qquad \varepsilon(s, \Pi \times \pi') = \varepsilon(s, \pi \times \pi')$$

we see that the $L(s, \Pi \times \pi')$ are nice and then we can apply our Converse Theorems to conclude that Π is either cuspidal automorphic or at least that there is an automorphic Π' such that $\Pi_v = \Pi'_v$ at almost all places. This then effects the (possibly weak) automorphic lift of π to Π or Π'.

4. *Highly ramified twists*: As we have indicated above, there are both local and global problems that can be finessed by an appropriate use of a highly ramified twist. This is based on the following simple observation.

Observation. *Let Π be as in Theorem 1 or 2. Suppose that η is a fixed character of $k^\times \backslash \mathbb{A}^\times$. Suppose that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}' = \mathcal{T} \otimes \eta$, where $\mathcal{T}$ is either of the twisting sets of Theorem 1 or 2. Then Π is quasi-automorphic as in those theorems.*

The only thing to observe is that if $\pi' \in \mathcal{T}$ then $L(s, \Pi \times (\pi' \otimes \eta)) = L(s, (\Pi \otimes \eta) \times \pi')$ so that applying the Converse Theorem for Π with twisting set $\mathcal{T} \otimes \eta$ is equivalent to applying the Converse Theorem for $\Pi \otimes \eta$ with the twisting set $\mathcal{T}$. So, by either Theorem 1 or 2, whichever is appropriate, $\Pi \otimes \eta$ is quasi-automorphic and hence Π is as well.

If we now begin with π automorphic on $\mathrm{H}(\mathbb{A})$, we will take T to be the set of finite places where π_v is ramified. For applying Theorem 1 we want $S = T$ and for Theorem 2 we would want $S \cap T = \emptyset$. We will now take η to be highly ramified at all places $v \in T$, so that at $v \in T$ our twisting representations are all locally of the form (unramified principal series)$\otimes$(highly ramified character).

In order to finesse the lack of knowledge of an appropriate local lift, we need to know the following two local facts about the local theory of L-functions for H.

Multiplicativity of γ–factors. *If $\pi'_v = Ind(\pi'_{1,v} \otimes \pi'_{2,v})$, with $\pi'_{i,v}$ and irreducible admissible representation of $\mathrm{GL}_{r_i}(k_v)$, then we have $\gamma(s, \pi_v \times \pi'_v, \psi_v) = \gamma(s, \pi_v \times \pi'_{1,v}, \psi_v)\gamma(s, \pi_v \times \pi'_{2,v}, \psi_v)$.*

Stability of γ–factors. *If $\pi_{1,v}$ and $\pi_{2,v}$ are two irreducible admissible representations of* $\mathrm{H}(k_v)$ *with the same central character, then for every sufficiently highly ramified character η_v of $GL_1(k_v)$ we have $\gamma(s, \pi_{1,v} \times \eta_v, \psi_v) = \gamma(s, \pi_{2,v} \times \eta_v, \psi_v)$.*

Both of these facts are known for GL_n, the multiplicativity being found in [9] and the stability in [10]. Multiplicativity in a fairly wide generality useful for applications has been established by Shahidi [19]. Stability is in a more primitive state at the moment, but Shahidi has begun to establish the necessary results in a general context in [20].

To utilize these local results, what one now does is the following. At the places where π_v is ramified, choose Π_v to be arbitrary, except that it should have the same central character as π_v. This is both to guarantee that the central character of Π is the same as that of π and hence automorphic and to guarantee that the stable forms of the γ–factors for π_v and Π_v agree. Now form $\Pi = \otimes' \Pi_v$. Choose our character η so that at the places $v \in T$ we have that the L– and γ–factors for both $\pi_v \otimes \eta_v$ and $\Pi_v \otimes \eta_v$ are in their stable form and agree. We then twist by $\mathcal{T}' = \mathcal{T} \otimes \eta$ for this *fixed* character η. If $\pi' \in \mathcal{T}'$, then for $v \in T$, π'_v is of the form $\pi'_v = Ind(|\ |^{s_1} \otimes \cdots \otimes |\ |^{s_m}) \otimes \eta_v$. So at the places $v \in T$, applying both multilplcativity and stability, we have

$$\begin{aligned}\gamma(s, \pi_v \times \pi'_v, \psi_v) &= \prod \gamma(s + s_i, \pi_v \otimes \eta_v, \psi_v) \\ &= \prod \gamma(s + s_i, \Pi_v \otimes \eta_v, \psi_v) = \gamma(s, \Pi_v \times \pi'_v, \psi_v)\end{aligned}$$

from which one deduces a similarly equality for the L- and ε–factors. From this it will then follow that globally we will have $L(s, \pi \times \pi') = L(s, \Pi \times \pi')$ for all $\pi' \in \mathcal{T}'$ with similar equalities for the ε–factors. This then completes Step 1.

To complete our use of the highly ramified twist, we must return to the question of whether $L(s, \pi \times \pi')$ can be made entire. In analysing L-functions via the Langlands–Shahidi method, the poles of the L-function are controlled by those of an Eisenstein series. In general, the inducing data for the Eisenstein series must satisfy a type of self–contragredience for there to be poles. The important observation of Kim is that one can use a highly ramified twist to destroy this self-contragredience at one place, which suffices, and hence eliminate poles. The precise condition will depend on the individual construction. A more detailed explanation of this can be found in Shahidi's article [21]. This completes Step 2 above.

4. New Examples of Functoriality

Now take k to be a number field. There has been much progress recently in utilizing the method described above to establish global liftings from split groups H over k to an appropriate GL_n. Among them are the following.

1. *Classical groups.* Take H to be a split classical group over k, more specifically, the split form of either SO_{2n+1}, Sp_{2n}, or SO_{2n}. The the L-groups ${}^L\mathrm{H}$ are then $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{SO}_{2n+1}(\mathbb{C})$, or $\mathrm{SO}_{2n}(\mathbb{C})$ and there are natural embeddings into the general linear group $\mathrm{GL}_{2n}(\mathbb{C})$, $\mathrm{GL}_{2n+1}(\mathbb{C})$, or $\mathrm{GL}_{2n}(\mathbb{C})$ respectively. Associated to each there should be a lifting of admissible or automorphic representations from

H($\mathbb{A}$) to the appropriate $\mathrm{GL}_N(\mathbb{A})$. The first lifting that resulted from the combination of the Converse Theorem and the Langlands-Shahidi method of controlling automorphic L-functions was the weak lift for generic cuspidal representations from SO_{2n+1} to GL_{2n} over a number field k obtained with Kim and Shahidi [2]. We can now extend this to the following result.

Theorem. [2, 3] *Let* H *be a split classical group over k as above and π a globally generic cuspidal representation of* H($\mathbb{A}$). *Then there exists an automorphic representation Π of $\mathrm{GL}_N(\mathbb{A})$ for the appropriate N such that Π_v is the local Langlands lift of π_v for all archimedean places v and almost all non-archimedean places v where π_v is unramified.*

In these examples the local Langlands correspondence is not understood at the places v where π_v is ramified and so we must use the technique of multiplicativity and stability of the local γ–factors as outlined in Section 3. Multiplicativity has been established in generality by Shahidi [19] and in our first paper [2] we relied on the stability of γ–factors for SO_{2n+1} from [5]. Recently Shahidi has established an expression for his local coefficients as Mellin transforms of Bessel functions in some generality, and in particular in the cases at hand one can combine this with the results of [5] to obtain the necessary stability in the other cases, leading to the extension of the lifting to the other split classical groups [3].

2. *Tensor products.* Let $H = \mathrm{GL}_m \times \mathrm{GL}_n$. Then ${}^L\mathrm{H} = \mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$. Then there is a natural simple tensor product map from $\mathrm{GL}_m(\mathbb{C}) \times \mathrm{GL}_n(\mathbb{C})$ to $\mathrm{GL}_{mn}(\mathbb{C})$. The associated functoriality from $\mathrm{GL}_n \times \mathrm{GL}_m$ to to GL_{mn} is the *tensor product lifting.* Now the associated local lifting is now understood in principle since the local Langlands conjecture for GL_n has been solved. The question of global functoriality has been recently solved in the cases of $\mathrm{GL}_2 \times \mathrm{GL}_2$ to GL_4 by Ramakrishnan [17] and $\mathrm{GL}_2 \times \mathrm{GL}_3$ to GL_6 by Kim and Shahidi [15, 16].

Theorem. [17, 15] *Let π_1 be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ and π_2 a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ (respectively $\mathrm{GL}_3(\mathbb{A})$). Then there is an automorphic representation Π of $\mathrm{GL}_4(\mathbb{A})$ (respectively $\mathrm{GL}_6(\mathbb{A})$) such that Π_v is the local tensor product lift of $\pi_{1,v} \times \pi_{2,v}$ at all places v.*

In both cases the authors are able to characterize when the lift is cuspidal.

In the case of Ramakrishnan [17] $\pi = \pi_1 \times \pi_2$ with each π_i cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ and Π is to be an automorphic representation of $\mathrm{GL}_4(\mathbb{A})$. To apply the Converse Theorem Ramakrishnan needs to control the analytic properties of $L(s, \Pi \times \pi')$ for π' cuspidal representations of $\mathrm{GL}_1(\mathbb{A})$ and $\mathrm{GL}_2(\mathbb{A})$, that is, the Rankin triple product L-functions $L(s, \Pi \times \pi') = L(s, \pi_1 \times \pi_2 \times \pi')$. This he was able to do using a combination of results on the integral representation for this L-function due to Garrett, Rallis and Piatetski-Shapiro, and Ikeda and the work of Shahidi on the Langlands-Shahidi method.

In the case of Kim and Shahidi [15, 16] π_2 is a cuspidal representation of $\mathrm{GL}_3(\mathbb{A})$. Since the lifted representation Π is to be an automorphic representation of $\mathrm{GL}_6(\mathbb{A})$, to apply the Converse Theorem they must control the analytic properties of $L(s, \Pi \times \pi') = L(s, \pi_1 \times \pi_2 \times \pi')$ where now π' must run over appropriate cuspidal representations of $\mathrm{GL}_m(\mathbb{A})$ with $m = 1, 2, 3, 4$. The control of these triple products is an application of the Langlands-Shahidi method of analysing L-functions and involves coefficients of Eisenstein series on GL_5, Spin_{10}, and simply connected E_6 and E_7 [15, 21]. We should note that even though the complete local lifting theory is understood, they still use a highly ramified twist to control the global properties of the L-functions involved. They then show that their lifting is correct at all local places by using a base change argument.

3. *Symmetric powers.* Now take $H = \mathrm{GL}_2$, so ${}^L\mathrm{H} = \mathrm{GL}_2(\mathbb{C})$. For each $n \geq 1$ there is the natural symmetric n-th power map $sym^n : \mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_{n+1}(\mathbb{C})$. The associated functoriality is the *symmetric power lifting* from representations of GL_2 to representations of GL_{n+1}. Once again the local symmetric powers liftings are understood in principle thanks to the solution of the local Langlands conjecture for GL_n. The global symmetric square lifting, so GL_2 to GL_3, is an old theorem of Gelbart and Jacquet. Recently, Kim and Shahidi have shown the existence of the global symmetric cube lifting from GL_2 to GL_4 [15] and then Kim followed with the global symmetric fourth power lifting from GL_2 to GL_5 [14].

Theorem. [15, 14] *Let π be a cuspidal automorphic representation of* $\mathrm{GL}_2(\mathbb{A})$*. Then there exists an automorphic representation* Π *of* $\mathrm{GL}_4(\mathbb{A})$ *(resp.* $\mathrm{GL}_5(\mathbb{A})$*) such that Π_v is the local symmetric cube (resp. symmetric fourth power) lifting of π_v.*

In either case, Kim and Shahidi have been able to give a very interesting characterization of when the image is in fact cuspidal [15, 16].

The original symmetric square lifting of Gelbart and Jacquet indeed used the converse theorem for GL_3. For Kim and Shahidi, the symmetric cube was deduced from the functorial $\mathrm{GL}_2 \times \mathrm{GL}_3$ tensor product lift above [15, 16] and did not require a new use of the Converse Theorem. For the symmetric fourth power lift, Kim first used the Converse Theorem to establish the *exterior square* lift from GL_4 to GL_6 by the method outlined above and then combined this with the symmetric cube lift to deduce the symmetric fourth power lift [14].

5. Applications

These new examples of functoriality have already had many applications. We will discuss the primary applications in parallel with our presentation of the examples. k remains a number field.

1. Classical groups: The applications so far of the lifting from classical groups to GL_n have been "internal" to the theory of automorphic forms. In the case of the lifting from SO_{2n+1} to GL_{2n}, once the weak lift is established, then the theory of Ginzburg, Rallis, and Soudry [8] allows one to show that this weak lift is indeed a strong lift in the sense that the local components Π_v at those $v \in S$ are completely determined and to completely characterize the image locally and globally. This will be true for the liftings from the other classical groups as well. Once one knows that these lifts are rigid, then one can begin to define and analyse the local lift for ramified representations by setting the lift of π_v to be the Π_v determined by the global lift. This is the content of the papers of Jiang and Soudry [12, 13] for the case of $\mathrm{H} = SO_{2n+1}$. In essence they show that this local lift satisfies the relations on L-functions that one expects from functoriality and then deduce the *local Langlands conjecture for* SO_{2n+1} from from that for GL_{2n}. We refer to their papers for more detail and precise statements.

2. Tensor product lifts: Ramakrishnan's original motivation for establishing the tensor product lifting from $\mathrm{GL}_2 \times \mathrm{GL}_2$ to GL_4 was to prove the multiplicity one conjecture for SL_2 of Langlands and Labesse.

Theorem. [17] *In the spectral decomposition*

$$L^2_{cusp}(\mathrm{SL}_2(k)\backslash \mathrm{SL}_2(\mathbb{A})) = \bigoplus m_\pi \pi$$

into irreducible cuspidal representations, the multiplicities m_π are at most one.

This was previously known to be true for GL_n and false for SL_n for $n \geq 3$. For further applications, for example to the Tate conjecture, see [17].

The primary application of the tensor product lifting from $\mathrm{GL}_2 \times \mathrm{GL}_3$ to GL_6 of Kim and Shahidi was in the establishment of the the symmetric cube lifting and through this the symmetric fourth power lifting, so the applications of the symmetric power liftings outlined below are applications of this lifting as well.

3. Symmetric powers: It was early observed that the existence of the symmetric power liftings of GL_2 to GL_{n+1} for all n would imply the Ramanujan-Petersson and Selberg conjectures for modular forms. Every time a symmetric power lift is obtained we obtain better bounds towards Ramanujan. The result which follows from the symmetric third and fourth power lifts of Kim and Shahidi is the following.

Theorem. [16] *Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ such that the symmetric cube lift of π is again cuspidal. Let $diag(\alpha_v, \beta_v)$ be the Satake parameter for an unramified local component. Then $|\alpha_v|, |\beta_v| < q_v^{1/9}$. If in addition the fourth symmetric power lift is not cuspidal, the full Ramanujan conjecture is valid.*

The corresponding statement at infinite places, i.e., the analogue of the Selberg conjecture on the eigenvalues of Mass forms, is also valid [14]. Estimates towards Ramanujan are a staple of improving any analytic number theoretic estimates obtained through spectral methods. Both the 1/9 non-archimedean and 1/9 archimedean estimate towards Ramanujan above were applied in obtaining the precise form of the exponent in our recent result with Sarnak breaking the convexity bound for twisted Hilbert modular L-series in the conductor aspect, which in turn was the key ingredient in our work on Hilbert's eleventh problem for ternary quadratic forms. Similar in spirit are the applications by Kim and Shahidi to the hyperbolic circle problem and to estimates on sums of shifted Fourier coefficients [15].

In addition Kim and Shahidi were able to obtain results towards the Sato-Tate conjecture.

Theorem. [16] *Let π be a cuspidal representation of $\mathrm{GL}_2(\mathbb{A})$ with trivial central character. Let $diag(\alpha_v, \beta_v)$ be the Satake parameter for an unramified local component and let $a_v = \alpha_v + \beta_v$. Assuming π satisfies the Ramanujan conjecture, there are sets $T^{\pm}$ of positive lower density for which $a_v > 2\cos(2\pi/11) - \epsilon$ for all $v \in T^+$ and $a_v < -2\cos(2\pi/11) + \epsilon$ for all $v \in T^-$. [Note: $2\cos(2\pi/11) = 1.68...$]*

Kim and Shahidi have other conditional applications of their liftings such as the conditional existence of Siegel modular cusp forms of weight 3 (assuming Arthur's multiplicity formula for Sp_4). We refer the reader to [15] for details on these applications and others.

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