LANGLANDS CONJECTURES FOR GL_n

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One of the principle goals of modern number theory is to understand the Galois group $\mathcal{G}_k = Gal(\overline{k}/k)$ of a local or global field k, such as $\mathbb{Q}$ for example. One way to try to understand the group $\mathcal{G}_k$ is by understanding its finite dimensional representation theory. In the case of a number field, to every finite dimensional representation $\rho : \mathcal{G}_k \to \mathrm{GL}_n(\mathbb{C})$ Artin attached a complex analytic invariant, its L-function $L(s, \rho)$. One approach to understanding ρ is through this invariant. For one dimensional ρ this idea was fundamental for the analytic approach to abelian class field theory and the understanding of $\mathcal{G}_k^{ab}$. To obtain a more complete understanding of $\mathcal{G}_k$ we would hope for a more complete understanding of the $L(s, \rho)$ for higher dimensional representations.

There is another class of objects which possess similar analytic invariants. These are the automorphic representations π of $\mathrm{GL}_n(\mathbb{A})$, where $\mathbb{A}$ is the adele ring of k. The analytic properties of the L-functions $L(s, \pi)$ attached to automorphic representations are well understood [13].

The Langlands conjectures predict the existence of a correspondence between the n-dimensional representations of $\mathcal{G}_k$ and the automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ which preserves these analytic invariants. There is a concomitant correspondence between n-dimensional representations of $\mathcal{G}_k$ for a local field k and the admissible representations of $\mathrm{GL}_n(k)$, the local Langlands conjecture. There are two ways to view such correspondences. If one views the passage of information from the automorphic side to the Galois side, as we have done above, this is a local or global non-abelian class field theory. If one views the passage if information from the Galois side to the automorphic side this is an arithmetic parameterization of admissible or automorphic representations.

Over the past ten years there has been significant progress made in the understanding of these Langlands conjectures. It began in the early nineties with the proof of the local Langlands conjecture for local fields k of characteristic p by Laumon, Rapoport, and Stuhler [44]. In the late nineties it was followed by a proof of the local Langlands conjecture for non-archimedean fields of characteristic zero by Harris and Taylor [27], followed quickly by a simplified proof due to Henniart [30]. Around the same time, following the program of Drinfeld from his proof of the Langlands conjecture for GL_2 over a global function field [23], L. Lafforgue established the global Langlands conjecture for GL_n in the function field case [35].

In this survey we would like to present an overview of these results, emphasizing their common features. There are already several excellent surveys on the individual works, namely those of Carayol for the local conjectures [9, 10] and Laumon for the global conjectures

[42, 43] and we refer the reader to these for more in depth coverage. The first two sections of this paper discuss Galois representations, automorphic representations, and their L-functions. We next discuss the local Langlands conjectures in both the representation theoretic version, proved by Langlands in the archimedean case around 1973 [38], and the L-function version, which was the version established by Laumon, Rapoport, Stuhler, Harris, Taylor, and Henniart in the non-archimedean case. Finally we discuss the version of the global Langlands conjecture established by Drinfeld and Lafforgue in characteristic p.

Although there has been little general progress on the global Langlands conjecture for number fields, there have been spectacular special cases established recently. Most notable among these is the proof by Wiles of the modularity of certain 2-dimensional ℓ-adic representations of $\mathcal{G}_{\mathbb{Q}}$ associated to elliptic curves over $\mathbb{Q}$, which he established on his way to the proof of Fermat's last theorem [59], and related results. Unfortunately, we will not discuss these results here.

1. Galois Representations and their L-functions

If G is a topological group and F is a topological field then let $\mathrm{Rep}_n(G;F)$ denote the set of equivalence classes of continuous representations $\rho : G \to \mathrm{GL}_n(F)$. Let $\mathrm{Rep}^0_n(G;F)$ be the subset of irreducible representations. For the most part we will be interested in complex representations and so we will use $\mathrm{Rep}_n(G)$ for $\mathrm{Rep}_n(G;\mathbb{C})$ and similarly for Rep^0_n. At times we will be interested in $F = \overline{\mathbb{Q}}_\ell$ and when we do, we will use the coefficient field in the notation.

If k is either a local or global field we will let $\overline{k}$ be a separable algebraic closure of k. Let $\mathcal{G}_k = Gal(\overline{k}/k)$ be the (absolute) Galois group and W_k the (absolute) Weil group [51].

Let k be a non-archimedean local field. Let p be the characteristic and q the order of its residue field κ. Let $I \subset \mathcal{G}_k$ be the inertia subgroup. If we let Φ denote a choice of geometric Frobenius element of $\mathcal{G}_k$ then W_k can be taken as the subgroup of $\mathcal{G}_k$ algebraically generated by Φ and I but topologized such that I has the induced topology from $\mathcal{G}_k$, I is open, and multiplication by Φ is a homeomorphism. This can also be given the structure of a scheme over $\mathbb{Q}$ [51]. Then we have a continuous homomorphism $\mathcal{G}_k \to W_k$ with dense image. Thus we have a natural inclusion $\mathrm{Rep}_n(\mathcal{G}_k) \to \mathrm{Rep}_n(W_k)$. The image, that is, the those representations that factor through continuous representations of $\mathcal{G}_k$, are the representations of W_k of Galois type. We also have a natural character $\omega^s \in \mathrm{Rep}_1(W_k)$ defined by $\omega^s(I) = 1$ and $\omega^s(\Phi) = q^{-s}$. This is also denoted by $\omega^s(w) = \|w\|^{-s}$ and gives a homomorphism $\nu : W_k \to \mathbb{Z}$ defined by $\|w\| = q^{-\nu(w)}$. Then every irreducible representation ρ of W_k is of the form $\rho = \rho^\circ \otimes \omega^s$ where ρ° is of Galois type [49, 19].

The representations that arise most naturally in arithmetic algebraic geometry, for example those associated with the ℓ-adic cohomology of algebraic varieties, are not complex representations but rather representations in $\mathrm{Rep}_n(\mathcal{G}_k; \overline{\mathbb{Q}}_\ell)$, with $\ell \neq p$, or $\mathrm{Rep}_n(W_k; \overline{\mathbb{Q}}_\ell)$. The representation theory for ℓ-adic representations is richer than for complex representations due to the difference in topologies in the two fields. Recognizing this, Deligne introduced what is now known as the Weil-Deligne group W_k' of the local field so that its representation theory is essentially algebraic, so in essence it doesn't distinguish between $\mathbb{C}$ and $\overline{\mathbb{Q}}_\ell$, and whose category of representations is the same that of the continuous ℓ-adic representations of $\mathcal{G}_k$ or W_k [19]. Following Tate [51], let us define W_k' to be the group scheme over $\mathbb{Q}$ which is the semidirect product of the Weil group W_k with the additive group $\mathbb{G}_a$, i.e., $W_k' = W_k \ltimes \mathbb{G}_a$, where W_k acts on $\mathbb{G}_a$ by $wxw^{-1} = \|w\|x$. If F is any field of characteristic 0, such as $\overline{\mathbb{Q}}_\ell$ or $\mathbb{C}$, the F-points of W_k' is just $W_k \times F$ with composition $(w_1, x_1)(w_w, x_2) = (w_1w_2, x_1 + \|w_1\|x_2)$. But what is really important is the representation theory of W_k'. An n-dimensional representation of W_k' over F is a pair $\rho' = (\rho, N)$ consisting of (i) an n-dimensional F-vector space V with a group homomorphism $\rho : W_k \to \mathrm{GL}(V)$ whose kernel contains an open subgroup of I, that is, which is continuous for the *discrete* topology on $\mathrm{GL}(V)$, and (ii) a nilpotent endomorphism N of V such that $\rho(w)N\rho(w)^{-1} = \|w\|N$ [19, 49, 51].

If $\rho' = (\rho, N)$ is a representation of W_k', there is a unique unipotent automorphism u of V which commutes with both N and $\rho(W_k)$ and such that $e^{aN}\rho(w)u^{-\nu(w)}$ is semisimple for all $a \in F$ and all $w \in W_k - I$ [19, 51]. The Φ semisimplification of ρ' is then $\rho'_{ss} = (\rho u^{-\nu}, N)$. ρ' is called Φ-semisimple (or Frobenius semisimple) if $\rho' = \rho'_{ss}$, for in this case u is the identity and all the Frobeniuses act semisimply. This is equivalent to the representation ρ being semisimple in the ordinary sense.

We will let $\mathrm{Rep}_n(W_k'; F)$ denote the equivalence classes of n-dimensional Φ-semisimple F-representations of the Weil-Deligne group W_k'. When $F = \mathbb{C}$ we will simply write $\mathrm{Rep}_n(W_k')$ for $\mathrm{Rep}_n(W_k'; \mathbb{C})$.

The importance of the Weil-Deligne group is in that it lets us capture, in an algebraic way, the continuous ℓ-adic representations of $\mathcal{G}_k$ or W_k [19, 49, 51]: for every semisimple ℓ-adic representation $\rho_\ell \in \mathrm{Rep}_n(W_k; \overline{\mathbb{Q}}_\ell)$ there is an open subgroup of the inertia group I on which ρ_ℓ is trivial and hence ρ_ℓ gives rise to an (ordinary) Φ-semisimple $\overline{\mathbb{Q}}_\ell$-representation ρ' of W_k'.

Note that by condition (ii) in the definition of a representation of W_k' the topology on F plays no role, so that if we have a fixed isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ we may identify $\mathrm{Rep}_n(W_k'; \overline{\mathbb{Q}}_\ell) \simeq \mathrm{Rep}_n(W_k'; \mathbb{C}) = \mathrm{Rep}_n(W_k')$. Furthermore, note that in an irreducible representation of W_k' we must have that $N = 0$, since the kernel of N would be an invariant subspace, and so $\mathrm{Rep}_n^0(W_k') = \mathrm{Rep}_n^0(W_k)$.

If $\rho' = (\rho, N) \in \mathrm{Rep}_n(W_k'; F)$ is an representation of W_k' on the vector space V, let $V_N^I = (Ker\, N)^{\rho(I)}$ be the invariants of the inertia subgroup I on the kernel of N. The we can define the local L-factor by setting

$$Z(t, V) = \det(1 - t\rho(\Phi)|V_N^I)^{-1} \in F(t)$$

to be the inverse of the characteristic polynomial of Φ acting on V_N^I and if we have an embedding $F \hookrightarrow \mathbb{C}$, so if $F = \mathbb{C}$ or we use the isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$, then we view F as a subfield of $\mathbb{C}$ and set

$$L(s, \rho') = Z(q^{-s}, \rho').$$

The definition of the local constants $\varepsilon(s, \rho', \psi)$, with ψ an additive character of k, is more delicate and we refer the reader to Deligne [19], Rohrlich [49] or Tate [51] for their precise definition. Of course, in the case $N = 0$ these are the usual local Artin-Weil L-functions and ε-factors.

If the local field k is archimedean, so $k = \mathbb{R}$ or $\mathbb{C}$, then we are interested only in complex representations of $\mathcal{G}_k$ or W_k. When $k = \mathbb{C}$ the Weil group is simply $\mathbb{C}^\times$ while if $k = \mathbb{R}$ then $W_\mathbb{R} \simeq \mathbb{C}^\times \coprod j\mathbb{C}^\times$ where $j^2 = -1$ and $jcj^{-1} = \overline{c}$ for $c \in \mathbb{C}^\times$. In either case we have

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_k \longrightarrow \mathcal{G}_k \longrightarrow 1.$$

There is no archimedean Weil-Deligne group, so for consistency we will set $W'_k = W_k$ in these cases. The L-and ε-factors are then defined in terms of the classical Γ-function and a local functional equation [51].

When k is a global field we will at least be interested in the representations of the global Galois group $\mathcal{G}_k$, the global Weil group W_k, or possibly the conjectural Langlands group $\mathcal{L}_k$ [46].

When $F = \overline{\mathbb{Q}}_\ell$ we will let $\mathrm{Rep}_n(G; \overline{\mathbb{Q}}_\ell)$ denote the set of global ℓ-adic representations in the following sense. They should be continuous, algebraic (in the sense that they take values in $\mathrm{GL}(E_\lambda)$ for a finite dimensional extension $E_\lambda/\mathbb{Q}_\ell$), and almost everywhere unramified (in the sense that there is a finite set of places $S(\rho)$ of k such that for all $v \notin S(\rho)$ the representation ρ is unramified at v).

For any global representation ρ of $\mathcal{G}_k$ or W_k we have a local representation ρ_v for each completion v of k obtained by composing ρ with the natural maps $\mathcal{G}_{k_v} \to \mathcal{G}_k$ or $W_{k_v} \to W_k$. The conjectural Langlands group $\mathcal{L}_k$ should have similar local-global compatibility with the local Weil-Deligne groups.

To any n-dimensional complex or ℓ-adic representation of either the Galois group or the Weil group we have attached a global complex analytic invariant, the global L-function $L(\rho, s)$ defined by the Euler product

$$L(s, \rho) = \prod_v L(s, \rho_v) \qquad \varepsilon(s, \rho) = \prod_v \varepsilon(s, \rho_v, \psi_v)$$

where $\psi = \prod_v \psi_v$ is an additive character of k.

These global analytic invariants are conjectured to be nice in the sense that

(1) $L(s, \rho)$ should have a meromorphic continuation with at most a finite number of poles, entire if ρ is irreducible but not trivial;
(2) these continuations should be bounded in vertical strips;
(3) they satisfy the functional equation $L(s, \rho) = \varepsilon(s, \rho) L(1 - s, \tilde{\rho})$.

For $G = \mathcal{G}_k$ or W_k and $F = \mathbb{C}$ these are the classical Artin-Weil L-functions and by Brauer's Theorem are known to converge in a right half plane, have meromorphic continuation to $\mathbb{C}$, and satisfy a functional equation.

If k is a global function field, with constant field of order q, and ρ is an ℓ-adic representation of $\mathcal{G}_k$ as above, then Grothendieck has shown that the L-function is in fact a rational function of q^{-s} and satisfies a functional equation and Deligne later showed that the ε-factor of the functional equation had a local factorization and was given as above [51].

2. Automorphic representations and their L-functions

On the automorphic side, if k is a local field, we let $\mathcal{A}_n(k)$ denote the set of equivalence classes of irreducible admissible complex representations of $\mathrm{GL}_n(k)$. When k is non-archimedean local, we let $\mathcal{A}_n^0(k)$ denote the subset of equivalence classes of supercuspidal representations of $\mathrm{GL}_n(k)$. By the theory of Godement–Jacquet [24], or the theory of Jacquet–Piatetski-Shapiro–Shalika outlined in [13], there a complex analytic invariant attached to every $\pi \in \mathcal{A}_n(k)$, namely its L-function $L(s,\pi)$ and a local ε-factor $\varepsilon(s,\pi,\psi)$ depending on a choice of additive character. If in addition we have an irreducible admissible representation π' of $\mathrm{GL}_m(k)$ then we have the local Rankin-Selberg convolution L-functions $L(s,\pi\times\pi')$ and ε-factor $\varepsilon(s,\pi\times\pi',\psi)$.

If k is a global field we let $\mathbb{A}$ denote its ring of adeles. Let $\mathcal{A}_n(k)$ denote the set of irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ and $\mathcal{A}_n^0(k)$ the subset of cuspidal automorphic representations. If $\pi = \otimes'\pi_v$ is an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ and $\pi' = \otimes'\pi'_v$ an automorphic representation of $\mathrm{GL}_m(\mathbb{A})$ then we have its associated L-function and ε-factor defined by Euler products

$$L(s,\pi\times\pi') = \prod_v L(s,\pi_v\times\pi') \qquad \varepsilon(s,\pi\times\pi') = \prod_v \varepsilon(s,\pi_v\times\pi'_v,\psi_v).$$

As we have seen [13], these invariants are known to be nice, that is if π and π' are unitary cuspidal representations, then

(1) $L(s,\pi\times\pi')$ has an analytic continuation to all of $\mathbb{C}$ with at most simple poles at $s=0,1$ iff $\pi'=\tilde{\pi}$;
(2) these continuations are bounded in vertical strips;
(3) they satisfy the functional equation $L(s,\pi\times\pi') = \varepsilon(s,\pi\times\pi')L(1-s,\tilde{\pi}\times\tilde{\pi}')$.

When considering representations that occur in ℓ-adic cohomologies it is most natural to use $\overline{\mathbb{Q}}_\ell$-valued automorphic forms and representations, which we denote by $\mathcal{A}_n(k;\overline{\mathbb{Q}}_\ell)$. For example, we will need to consider the space of $\overline{\mathbb{Q}}_\ell$-valued cuspidal representations whose central character is of finite order, which we will denote by $\mathcal{A}_n^0(k;\overline{\mathbb{Q}}_\ell)_f$. These are the representations of $\mathrm{GL}_n(\mathbb{A})$, or the associated Hecke algebra $\mathcal{H}$ of locally constant $\overline{\mathbb{Q}}_\ell$-valued functions of compact support on $\mathrm{GL}_n(\mathbb{A})$, in the space of certain $\overline{\mathbb{Q}}_\ell$-valued cusp forms on $\mathrm{GL}_n(\mathbb{A})$. For the convenience of the reader, we will review the definition from [42] for the case of function fields over finite fields. The $\overline{\mathbb{Q}}_\ell$-valued cusp form on $\mathrm{GL}_n(\mathbb{A})$ of interest is a function $\varphi : \mathrm{GL}_n(\mathbb{A}) \to \overline{\mathbb{Q}}_\ell$ such that

(i) $\varphi(\gamma g) = \varphi(g)$ for all $\gamma \in \mathrm{GL}_n(k)$;

(ii) There is a compact open subgroup $K_\varphi \subset K = \mathrm{GL}_n(\mathcal{O}_k)$ such that $\varphi(gk) = \varphi(g)$ for all $k \in K_\varphi$;
(iii) There is an $a \in \mathbb{A}^\times$ with $\deg(a) \neq 0$ such that $\varphi(ag) = \varphi(g)$;
(iv) φ is cuspidal in the usual sense that the integral $\int_U \varphi(ug)\, du \equiv 0$ for each unipotent radical U of a maximal parabolic subgroup of $\mathrm{GL}_n(\mathbb{A})$.

Note that the condition (iii) implies that the central character of φ is of finite order. The theory can essentially be identified with the complex theory through the isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$ and the natural L-functions can be identified with the usual complex analytic ones or they can be left as ℓ-adic valued rational functions as in the appendix of [35] .

3. The Local Langlands Conjecture

In its most basic form, the local Langlands conjecture is a non-abelian generalization of (abelian) local class field theory. The conjecture as first formulated by Langlands was in terms of the Weil group. An early formulation, possibly the first, can be found in [36]. Langlands never restricted himself to GL_n but always formulated in terms of reductive algebraic groups in general. Deligne first pointed the necessity of passing to what is now known as the Weil-Deligne group to be able to include the special representations of $GL_2(k)$ for a local field k [18]. The current formulation of the conjecture which is closest to Langlands original is to be found in Borel's article in Corvallis [4].

Local Langlands Conjecture I: *Let k be a local field. Then there are a series of natural bijections*

$$\mathrm{Rep}_n(W_k') \leftrightarrow \mathcal{A}_n(k) \qquad \rho = \rho_\pi \leftrightarrow \pi = \pi_\rho$$

satisfying a set of representation theoretic desiderata, including:

(i) *For $n = 1$ it should be given by the local class field theory isomorphism.*
(ii) *The central character of π_ρ corresponds to the determinant* $\det(\rho)$ *under the $n = 1$ correspondence;*
(iii) *Compatibility with twisting, i.e., if χ is a character of $k^\times$ then $\rho_{\pi\otimes\chi} = \rho \otimes \chi$;*
(iv) *π_ρ is square integrable iff $\rho(W_k')$ does not lie in a proper Levi subgroup of* $\mathrm{GL}_n(\mathbb{C})$*;*
(v) *π_ρ is tempered iff $\rho(W_k)$ is bounded.*
(vi) *If H is a reductive connected k-group and $H(k) \to \mathrm{GL}_n(k)$ is a k morphism with commutative kernel and co-kernel, then there is a required compatibility between these bijections for* $\mathrm{GL}_n(k)$ *and similar maps for $H(k)$.*

For more details on the the compatibility condition (vi), see the article of Borel in Corvallis [4] or the accompanying article [14]. This is related to Langlands' general functoriality conjecture. Langlands himself never separated this version of his conjecture from his general principle of functoriality [40]. Note that there is no mention of L-functions in this formulation.

For non-archimedean local fields, it was in the book by Jacquet and Langlands [31] and then in the work of Deligne [18] that a version of the local Langlands conjecture was phrased not in terms of representation theoretic properties, but rather in terms of the complex analytic invariants, or L-functions, of the two sets in question. Deligne gave the complete formulation for GL_2. It was in this paper that he utilized for the first time the Weil-Deligne group, which he had introduced in [19] in the context of ℓ-adic representations, in order to have a correct formulation on the case of GL_2 over a non-archimedean local field.

Local Langlands Conjecture II: *Let k be a non-archimedean local field. For each $n \geq 1$ there exists a bijective map $\mathcal{A}_n(k) \to \mathrm{Rep}_n(W_k')$ denoted $\pi \mapsto \rho_\pi$ with the following properties.*

(i) *For $n = 1$ the bijection is given by local class field theory, normalized so that the uniformizer of k corresponds to the geometric Frobenius.*

(ii) *For any $\pi \in \mathcal{A}_n(k)$ and $\pi' \in \mathcal{A}_{n'}(k)$ we have*

$$L(s, \rho_\pi \otimes \rho_{\pi'}) = L(s, \pi \times \pi') \qquad \varepsilon(s, \rho_\pi \otimes \rho_{\pi'}, \psi) = \varepsilon(s, \pi \times \pi', \psi).$$

(iii) *For any $\pi \in \mathcal{A}_n(k)$ the determinant of ρ_π corresponds to the central character of π under local class field theory.*

(iv) *For any π in $\mathcal{A}_n(k)$ we have $\rho_{\tilde{\pi}} = \widetilde{\rho_\pi}$.*

(v) *for any $\pi \in \mathcal{A}_n(k)$ and any character χ of $k^\times$ of finite order we have $\rho_{\pi\otimes\chi} = \rho_\pi \otimes \chi$.*

There are two ways to think about what these conjectures offer. If one views the primary passage of information to be from $\mathcal{A}_n(k)$ to $\mathrm{Rep}_n(W_k')$, then this can be thought of as Langlands formulation of a *non-abelian local class field theory.* If one views the primary passage of information from $\mathrm{Rep}_n(W_k')$ to $\mathcal{A}_n(k)$ then this gives an arithmetic parameterization of irreducible admissible representations of $\mathrm{GL}_n(k)$. This is the *arithmetic Langlands classification* of $\mathcal{A}_n(k)$.

3.1. k **local archimedean, i.e.,** $k = \mathbb{R}$ **or** $\mathbb{C}$. In this case $\mathcal{G}_k$ is well understood; it is either $\mathbb{Z}/2\mathbb{Z}$ or trivial. So the passage of information in this case is in the opposite direction. This was done in great generality by Langlands about 1973 [38], and not only for GL_n but for general real reductive groups. For archimedean local fields there is no Weil-Deligne group. The representation theoretic version is what is now known as the Langlands classification or the Langlands parameters for representations of real groups. In fact, Langlands did this in conjunction with the *arithmetic* parameterization in terms of $\mathrm{Rep}_n(W_k)$ for GL_n (or admissible homomorphisms $W_k \to^L G$ for general G). The deep and interesting part is the classification of representations in term of the information obtained from these maps, particularly their relation with the construction of the discrete series.

Theorem 3.1. *Let k be $\mathbb{R}$ or $\mathbb{C}$. Then there are a are a series of natural bijections*

$$\mathrm{Rep}_n(W_k) \leftrightarrow \mathcal{A}_n(k)$$

satisfying the properties (i)–(vi) of version I of the local Langlands conjecture.

For the precise relation with the usual Langlands classification for real algebraic groups, see [38]. The statement proved is of course that originally given by Langlands and this may well have motivated the precise conditions in the conjecture. Note again that the conditions are representation theoretic and the L-functions and ε-factors play no role.

3.2. k **local non-archimedean.** Recently, this second version of the local Langlands conjecture has been established for non-archimedean local fields, first by Laumon, Rapoport, and Stuhler in the positive characteristic case in 1993 [44] and then in the characteristic 0 case by by Harris and Taylor [27] in 1999 and by Henniart [30] in 2000. In both cases, the correspondence is established at the level of a correspondence between irreducible Galois representations and supercuspidal representations. Much of the original representation theoretic desiderata of the original conjecture has been replaced by an equality of twisted L-functions, i.e., of the associated families of complex analytic invariants.

Let $\mathcal{A}_n^0(k)_f$ denote the set of isomorphism classes of irreducible admissible representations of $\mathrm{GL}_n(k)$ having central character of finite order. Then the theorem of Laumon, Rapoport, and Stuhler is the following [44].

Theorem 3.2. *Let k be a local field of characteristic $p > 0$. For each $n \geq 1$ there exists a bijective map $\mathcal{A}_n^0(k)_f \to \mathrm{Rep}_n^0(\mathcal{G}_k)$ denoted $\pi \mapsto \rho_\pi$ satisfying the conditions (i)–(v) of version II of the local Langlands conjecture.*

When the local field k is of characteristic 0 the local Langlands conjecture established by Harris and Taylor [27] and Henniart [28] has precisely the same statement.

Theorem 3.3. *Let k be a local field of characteristic 0. For each $n \geq 1$ there exists a bijective map $\mathcal{A}_n^0(k)_f \to \mathrm{Rep}_n^0(\mathcal{G}_k)$ denoted $\pi \mapsto \rho_\pi$ satisfying conditions (i)–(v) of version II of the local Langlands conjecture.*

The proofs involve the use of $\overline{\mathbb{Q}}_\ell$-representations on both the Galois and automorphic side, and is translated into the statements above in terms of complex analytic L-functions through the isomorphism ι of $\overline{\mathbb{Q}}_\ell$ with $\mathbb{C}$.

3.2.1. **Reductions and Constructions**. In any of the non-archimedean local cases, the proof passes through a chain of identical reductions which reduces one to proving the existence of a single map having the desired properties. The three proofs then differ in the constructions used to prove the existence of at least one correspondence.

There are essentially three steps in the reduction. Assume that we have any correspondence $\mathcal{A}_n^0(k)_f \to \mathrm{Rep}_n^0(\mathcal{G}_k)$, still denoted $\pi \mapsto \rho_\pi$, which satisfies (i)–(v) of the theorem.

1. Injectivity: Poles of L-functions. For ρ and ρ' in $\mathrm{Rep}_n^0(\mathcal{G}_k)$ we have that $L(s, \rho \otimes \rho')$ has a pole at $s = 0$ iff $\rho' \cong \widetilde{\rho}$. Similarly, if π and π' are both in $\mathcal{A}_n^0(k)$ then $L(s, \pi \times \pi')$ has a pole at $s = 0$ iff $\pi' \simeq \widetilde{\pi}$. Thus we see that any such correspondence satisfying (ii) is automatically injective.

2. Bijectivity: Numerical Local Langlands. For $\rho \in \mathrm{Rep}_n^0(\mathcal{G}_k)$ let $a(\rho)$ denote the exponent of the Artin conductor of ρ [51]. This is determined by the ε–factor $\varepsilon(s, \rho, \psi)$. Let $\mu : k^\times \to \mathbb{C}^\times$ be identified with a character of the Galois group via local class field theory. If we let $\mathrm{Rep}_n^0(\mathcal{G}_k)_{m,\mu}$ denote the set of $\rho \in \mathrm{Rep}_n^0(\mathcal{G}_k)$ with $a(\rho) = m$ and $\det(\rho) = \mu$ then this set is finite. Similarly, we let $\mathcal{A}_n^0(k)_{m,\mu}$ denote the set of $\pi \in \mathcal{A}_n^0(k)$ with $f(\pi) = m$ and central character $\omega_\pi = \mu$, where now $f(\pi)$ is the exponent of the conductor of Jacquet, Piatetski-Shapiro, and Shalika [13], and this set is also finite. The statement of the numerical local Langlands conjecture, which had been established by Henniart in 1988 [30], is that for fixed $m \in \mathbb{Z}_+$ and multiplicative character μ of finite order we have $|\mathrm{Rep}_n^0(\mathcal{G}_k)_{m,\mu}| = |\mathcal{A}_n^0(k)_{m,\mu}|$. Since (ii) and (iii) guarantee that our correspondence preserves the character μ and the conductors, then once we know that the correspondence is injective, the numerical local Langlands conjecture gives that the correspondence is surjective and hence bijective.

3. Uniqueness: The Local Converse Theorem. The uniqueness of a correspondence satisfying (i)–(v) is a consequence of the *local converse theorem for* $\mathrm{GL}(n)$. This result was first stated by Jacquet, Piatetski-Shapiro, and Shalika [32] but the first published proof was by Henniart [29] with precisely this application in mind. The statement is the following. Suppose that π and π' are both elements of $\mathcal{A}_n^0(k)$ and that the twisted ε–factors agree, that is

$$\varepsilon(s, \pi \times \tau, \psi) = \varepsilon(s, \pi' \times \tau, \psi),$$

for all $\tau \in \mathcal{A}_m^0(k)$ with $1 \leq m \leq n-1$. Then $\pi \cong \pi'$. (Note that the corresponding twisted L-functions are all identically 1 [13].) From this, by induction on n, one sees that any such (now bijective) correspondence satisfying (i)–(v) must be unique.

These three steps then reduce the local Langlands conjecture to the question of existence of some correspondence satisfying (i)–(v). It is this existence problem that was solved by Laumon, Rapoport, and Stuhler in positive characteristic and by Harris and Taylor and then Henniart in the characteristic zero case.

4. Existence: Global Geometric Constructions. In all cases, the local existence is based on establishing certain instances of a *global* correspondence of Galois representations and automorphic representations. Note that we now work with ℓ-adic representations on both the Galois and automorphic side.

For k of characteristic $p > 0$, Laumon, Rapoport, and Stuhler begin with a local representation $\pi \in \mathcal{A}_n^0(k)$. They realize k as a local component of a global field K of characteristic p, so $k = K_v$ for some place v of K, and then embed π as the local component at v of a cuspidal representation Π of a global division algebra $D(\mathbb{A})$ of rank n such that $D^\times(K_v) = \mathrm{GL}_n(k)$ and $\Pi_v = \pi$. They globally realize an action of $\mathcal{G}_K \times D^\times(\mathbb{A})$ on the ℓ-adic cohomology of the moduli space of $\mathcal{D}$-elliptic modules ($\mathcal{D}$ an order in D) such that in the decomposition of this cohomology a representation $R \otimes \Pi$ of $\mathcal{G}_K \times D^\times(\mathbb{A})$ occurs. By construction $\Pi_v = \pi$ and they take $R_v = \rho_\pi$. By the nature of their construction they are able to verify that (i)–(v) are satisfied. Thus they establish the needed local existence statement via a global geometric construction and a limited global correspondence.

The proof of Harris and Taylor of the local Langlands conjecture for non-archimedean fields of characteristic 0 is similar in spirit to that of Laumon, Rapoport, and Stuhler in characteristic p. They replace the moduli space of $\mathcal{D}$-elliptic modules with certain "simple Shimura varieties" associated to unitary groups U_n of Kottwitz. They realize k as a local component of a number field K, so $k = K_v$ for some place v of K, and then embed π as the local component at v of a cuspidal representation Π of a certain (twisted) unitary group of rank n. They then realize a global correspondence between these cuspidal representations and global Galois representations in the ℓ-adic cohomology of the associated Shimura varieties. By studying the resulting correspondence locally at a place of *bad* reduction they find a local representation on which they have an irreducible action of $\mathrm{GL}_n(k) \times D_n^\times(k) \times W_k$, where D is the division algebra over k of rank n and Hasse invariant $1/n$, by $\pi \otimes\ JL(\pi) \otimes \rho_\pi$, where $JL(\pi)$ is the image of π under the local Jacquet-Langlands correspondence and ρ_π is thus defined. Again, from their construction they can verify that this correspondence satisfies conditions (i)–(v). Note that not only do they get a geometric realization of the local Langlands correspondence, they get a simultaneous realization of the local Jacquet-Langlands correspondence.

Henniart, in his proof of the local Langlands conjecture for non-archimedean fields of characteristic 0, again uses a global construction but in a far less serious way. In particular, he does not give a geometric realization of the correspondence. For Henniart, both the statement and the proof most naturally give a bijection from the Galois side to the automorphic side $\rho \mapsto \pi_\rho$. Henniart begins with an irreducible representation ρ of $\mathcal{G}_k$ with finite order determinant. This then factors through a representation of $Gal(F/k)$ for a finite dimensional extension F of k. Using Brauer induction, he writes ρ as a sum of monomial representations. The characters can be lifted to the automorphic side by local class field theory, and so he must show that the corresponding sum of automorphically induced representations exists and is supercuspidal. The resulting supercuspidal representation is then π_ρ. This he does again by embedding the local situation into a global one and then appealing to certain weak cases of global automorphic induction that had been earlier established by Harris [25]. Harris's result relies on the theory of base change and the association of ℓ-adic representations to automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ by Clozel [12], which in turn relies on the work of Kottwitz on the good reduction of certain unitary Shimura varieties. So at the bottom there is in fact a global geometric construction, but it is of a simpler type than used by Harris and Taylor. Henniart's proof makes more use of L-functions and less use of geometry. His proof is shorter and more analytic, but does not give a geometric realization of the correspondence.

A more complete synopsis of these results can be found in the Séminaire Bourbaki reports of Carayol [9, 10].

3.3. **Complements.** In order to complete the local Langlands correspondence one needs to consider all suitable representations of the Weil-Deligne group on the Galois theoretic side and all irreducible admissible representations of $\mathrm{GL}_n(k)$ on the automorphic side. In order to do this, the first step is to remove the condition of finite-order on the central character. This is obtained by simply replacing the Galois group by the Weil group on the Galois side of the correspondence. On the automorphic side one still has supercuspidal representations. Then to pass to all admissible representations of $\mathrm{GL}_n(k)$ one uses the representations of the Weil-Deligne group. Representation theoretically, the passage from representations of the Weil group to representations of the Weil-Deligne group on the Galois side mirrors the passage from supercuspidal representations to irreducible representations on the automorphic side, as was shown by Bernstein and Zelevinsky (see [1, 60], particularly Section 10 of [60], or [47]). Thus from the results of Laumon, Rapoport, Stuhler, Harris, Taylor, and Henniart the full local Langlands correspondence follows.

In spite of these results, the work on the local Langlands conjecture continues. The proofs above give the existence of the correspondence and in some cases provides an explicit geometric model. It provides a matching of certain invariants, like the conductor, and the local L-functions. However, for applications, it would be desirable to have an explicit version of the local Langlands correspondence, particularly for the supercuspidal representations of GL_n in terms of the Bushnell-Kutzko compact induction data [6]. The search for an explicit local Langlands correspondence is currently being pursued by Bushnell, Henniart, Kutzko, and others.

4. The Global Langlands Conjecture

As in the local case, in its most basic form, the global Langlands conjecture should be a non-abelian generalization of (abelian) global class field theory. When Deligne pointed out the necessity of introducing the Weil-Deligne group in the local non-archimedean regime, it was realized that there seemed to be no natural global version of the Weil-Deligne group. This lead to a search for a global group to replace the Weil-Deligne group. This was one of the purposes of Langlands' article [39]. It is now believed that this group, which Ramakrishnan calls the conjectural Langlands group $\mathcal{L}_k$, should be related to the equally conjectural motivic Galois group of k, $\mathcal{M}_k$ [46].

4.1. k **a global field of characteristic** $p > 0$**.** In spite of these difficulties, Drinfeld formulated and proved a version of the global Langlands conjecture for global function fields [23] which related the irreducible 2-dimensional representations of the Galois group itself with the irreducible cuspidal representations of $\mathrm{GL}_2(\mathbb{A})$. This is the global analogue of the local theorem of Laumon, Rapoport, and Stuhler for which the Weil-Deligne group was not needed. We should emphasize that the results of Drinfeld were obtained in the 1970's, though published only later, and so predate those of Laumon, Rapoport, and Stuhler by several years. Recently the work of Drinfeld has been extended by L. Lafforgue to give a proof of the global Langlands conjecture for GL_n over a function field [35].

The formulation of the global Langlands conjecture established by Drinfeld and Lafforgue is essentially the same as in the local non-archimedean case above with a few modifications that we would now like to explain. Take k to be the function field of a smooth, projective, geometrically connected curve X over a finite field $\mathbb{F}$ of characteristic p. Fix a prime ℓ which is different from p and an isomorphism $\iota : \overline{\mathbb{Q}}_\ell \to \mathbb{C}$.

On the Galois side, they consider isomorphism classes of irreducible continuous ℓ-adic representations $\rho : \mathcal{G}_k \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ which are unramified outside a finite number of places, as described in Section 1, and whose determinant is of finite order. We will denote these by $\mathrm{Rep}_n^0(\mathcal{G}_k; \overline{\mathbb{Q}}_\ell)_f$. On the automorphic side they consider the space of $\overline{\mathbb{Q}}_\ell$-valued cuspidal representations whose central character is of finite order $\mathcal{A}_n^0(k; \overline{\mathbb{Q}}_\ell)_f$, as described in Section 2. A reasonable formulation of a global Langlands conjecture in analogy with what we have in the local situation is the following.

Global Langlands Conjecture in characteristic p: *For each* $n \geq 1$ *there exists a bijective map* $\mathcal{A}_n^0(k;\overline{\mathbb{Q}}_\ell)_f \to \mathrm{Rep}_n^0(\mathcal{G}_k;\overline{\mathbb{Q}}_\ell)_f$ *denoted* $\pi \mapsto \rho_\pi$ *with the following properties.*

(i) *For* $n = 1$ *the bijection is given by global class field theory.*
(ii) *For any* $\pi \in \mathcal{A}_n^0(k;\overline{\mathbb{Q}}_\ell)_f$ *and* $\pi' \in \mathcal{A}_{n'}^0(k;\overline{\mathbb{Q}}_\ell)_f$ *we have*

$$L(s,\rho_\pi \otimes \rho_{\pi'}) = L(s,\pi\times\pi') \qquad \varepsilon(s,\rho_\pi\otimes\rho_{\pi'}) = \varepsilon(s,\pi\times\pi').$$

(iii) *For any* $\pi \in \mathcal{A}_n^0(k;\overline{\mathbb{Q}}_\ell)_f$ *the determinant of* ρ_π *corresponds to the central character of* π *under global class field theory.*
(iv) *For any* π *in* $\mathcal{A}_n^0(k:\overline{\mathbb{Q}}_\ell)_f$ *we have* $\rho_{\tilde{\pi}} = \widetilde{\rho_\pi}$.
(v) *for any* $\pi \in \mathcal{A}_n^0(k;\overline{\mathbb{Q}}_\ell)_f$ *and any character* χ *of* $k^\times$ *of finite order we have* $\rho_{\pi\otimes\chi} = \rho_\pi \otimes \chi$;
(vi) *the global bijections should be compatible with the local bijections of the local Langlands conjecture.*

If we take a cuspidal representation $\pi \in \mathcal{A}_n^0(k;\overline{\mathbb{Q}}_\ell)_f$, let $S(\pi)$ denote the finite set of places $x \in |X|$ such that π_x is unramified for all $x \notin S(\pi)$. For $x \notin S(\pi)$ the local representation π_x is completely determined by a semi-simple conjugacy class A_{π_x} in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$, which we identify with $\mathrm{GL}_n(\mathbb{C})$ via ι, called the Satake class or Satake parameter of π_x [4]. As this parameter determines and is determined by a character of the associated unramified Hecke algebra $\mathcal{H}_x$ at x [50] the eigenvalues of A_{π_x}, denoted $z_1(\pi_x),\dots,z_n(\pi_x)$, are also called the *Hecke eigenvalues* of π_x. These Hecke eigenvalues completely determine π_x. Then by the strong multiplicity one theorem for GL_n [13] the collection of Hecke eigenvalues $\{z_1(\pi_x),\dots,z_n(\pi_x)\}$ for almost all $x \notin S(\pi)$ completely determine π.

If we take a Galois representation $\rho \in \mathrm{Rep}_n^0(\mathcal{G}_k;\overline{\mathbb{Q}}_\ell)_f$ then we also have a finite set of places $S(\rho)$ such that ρ is unramified at all $x \notin S(\rho)$. For $x \notin S(\rho)$ the image $\rho(\Phi_x)$ of a geometric Frobenius Φ_x at x is a well defined semi-simple conjugacy class in $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell) \simeq \mathrm{GL}_n(\mathbb{C})$. The eigenvalues of $\rho(\Phi_x)$, denoted $z_1(\rho_x),\dots,z_n(\rho_x)$, are called the *Frobenius eigenvalues* of ρ_x. These Frobenius eigenvalues completely determine ρ_x and by the Chebotarev density theorem the collection of Frobenius eigenvalues $\{z_1(\rho_x),\dots,z_n(\rho_x)\}$ for almost all $x \notin S(\rho)$ completely determine ρ itself.

The result established by Drinfeld for $n = 2$ [23] and Lafforgue for $n \geq 3$ [35], which as we will outline below is equivalent to the statement above, is the following.

Theorem 4.1. *Let* k *be a global function field of characteristic* p *as above. For each* $n \geq 1$ *there exists a unique bijective map* $\mathcal{A}_n^0(k;\overline{\mathbb{Q}}_\ell)_f \to \mathrm{Rep}_n^0(\mathcal{G}_k;\overline{\mathbb{Q}}_\ell)_f$ *denoted* $\pi \mapsto \rho_\pi$ *such that for every cuspidal* $\pi \in \mathcal{A}_n^0(k;\overline{\mathbb{Q}}_\ell)_f$ *we have the equality of Hecke and Frobenius eigenvalues*

$$\{z_1(\pi_x),\dots,z_n(\pi_x)\} = \{z_1(\rho_{\pi,x}),\dots,z_n(\rho_{\pi,x})\}$$

for all $x \notin S$, *a finite set of places containing* $S(\pi)\cup S(\rho_\pi)$, *or equivalently we have the equality of the partial complex analytic* L*-functions*

$$L^S(s,\pi) = L^S(s,\rho_\pi).$$

To see that this does indeed give the statements of the conjecture as presented above, note first that for all $x \notin S$, the equality of the equality of the associated Hecke and Frobenius eigenvalues at these places is consistent with the local Langlands conjecture at the unramified places. Hence (vi) is satisfied at these places. Next, for $\pi \in \mathcal{A}_n^0(k;\overline{\mathbb{Q}}_\ell)_f$ we have an Euler product factorization for the L-function $L(s,\pi) = \prod_{x\in|X|} L(s,\pi_x)$ and for all $x \notin S(\pi)$ the local L-function is given by $L(s,\pi_x) = \det(1-\iota(A_{\pi_x})q^{-s})^{-1}$ [13]. Hence the Hecke eigenvalues for π_x, $x \notin S(\pi)$, are determined by the local L-factor at these places and conversely. Similarly, for $\rho \in \mathrm{Rep}_n^0(\mathcal{G}_k;\overline{\mathbb{Q}}_\ell)_f$ we again have a factorization of the global L-function $L(s,\rho) = \prod_{x\in|X|} L(s,\rho_x)$ where now $L(s,\rho_x) = \det(1-\iota(\rho(\Phi_x))q_x^{-s})^{-1}$. Hence now the local eigenvalues of Frobenius for $\rho(\Phi_x)$, $x \notin S(\rho)$, are determined by the local L-factor at these places and conversely. Hence we do have the equality of partial L-functions $L^S(s,\pi) = L^S(s,\rho_\pi)$ for a finite set $S \supset S(\pi)\cup S(\rho_\pi)$ as stated. Using the global functional equation for both $L(s,\pi)$ and $L(s,\rho_\pi)$ and the local factorization of the global ε–factors, standard L-function techniques give that in fact $S(\pi) = S(\rho_\pi)$, that $L(s,\pi_x) = L(s,\rho_{\pi,x})$ at these places, and that in general the restriction of ρ_π to the local Galois group of k at x corresponds to π_x under the local Langlands conjecture. Thus (vi) is satisfied in general and from this (ii)–(v) follow from the Euler product factorizations and the analogous statements from the local conjecture.

4.1.1. **Reductions and Constructions.** As in the local situation, the proof passes through a chain of reductions that reduces one to proving the existence of a single map having the desired properties. The existence is then established by a global construction using the cohomology of a certain moduli scheme on which GL_n acts.

1. Uniqueness and bijectivity. Given the existence of one global bijection as above, the uniqueness of the bijection has long been known to follow from an application of the strong multiplicity one theorem on the automorphic side [13] and the Chebotarev density theorem on the Galois side. These strong uniqueness principles also imply that any such maps $\pi \mapsto \rho_\pi$ and $\rho \mapsto \pi_\rho$ satisfying the conditions of the theorem must be reciprocal bijections.

2. The inductive procedure of Piatetski-Shapiro and Deligne. This inductive principle was outlined by Deligne in an IHES seminar in 1980 and then later recorded in [41]. It reduces the proof of the theorem to the following seemingly weaker existence statement.

Theorem 4.2. *For each $n \geq 1$ there exists a map $\mathcal{A}_n^0(k;\overline{\mathbb{Q}}_\ell)_f \to \mathrm{Rep}_n^0(\mathcal{G}_k;\overline{\mathbb{Q}}_\ell)_f$ denoted $\pi \mapsto \rho = \rho_\pi$ such that we have the equality of Hecke and Frobenius eigenvalues*

$$\{z_1(\pi_x),\ldots,z_n(\pi_x)\} = \{z_1(\rho_{\pi,x}),\ldots,z_n(\rho_{\pi,x})\}$$

for almost all $x \notin S(\pi)\cup S(\rho_\pi)$.

Indeed, suppose one has established the existence of the map $\pi \mapsto \rho_\pi$ for $\pi \in A_r^0(k;\overline{\mathbb{Q}}_\ell)_f$ for $r = 1,\ldots,n-1$. Then utilizing the global functional equation of Grothendieck, the factorization of the global Galois ε-factor [41], and the converse theorem for GL_n [13, 15, 16] one obtains for free the inverse map $\rho \mapsto \pi_\rho$ for $\rho \in \mathrm{Rep}_r^0(\mathcal{G}_k;\overline{\mathbb{Q}}_\ell)_f$ for $r = 1,\ldots,n$.

3. Existence [23, 35]. It had been known since Weil that there is a natural moduli problem associated to GL_n over a function field, namely the set of isomorphism classes of rank n vector bundles on the curve X are parameterized by the double cosets $\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})/\mathrm{GL}_n(\mathcal{O})$. To obtain the maps in question, one needs a bit more structure, and so Drinfeld and then Lafforgue considered the (compactified) Deligne-Mumford stack V of rank n shtukas (with level structure), which is actually a stack over $X \times X$. There is a natural action of the global Hecke algebra $\mathcal{H}$ on this stack by correspondences and the corresponding ℓ-adic cohomology $H_c^*(\overline{k}\otimes_k V; \overline{\mathbb{Q}}_\ell)$ then affords a simultaneous representation of $\mathcal{H}$ and $\mathcal{G}_k \times \mathcal{G}_k$. One then uses the geometric Grothendieck-Lefshetz trace formula to compute the trace of this representation. One then compares this with the output of the Arthur-Selberg trace formula to prove that indeed the derived representation $\pi \otimes \rho_\pi \otimes \widetilde{\rho_\pi}$ of $\mathcal{H} \times \mathcal{G}_k \times \mathcal{G}_k$ occurs in this cohomology. The construction is inductive and essentially uses everything.

4.1.2. **Complements.** An immediate consequence of this result is the Ramanujan-Petersson conjecture for GL_n. This had been earlier established by Drinfeld for $n = 2$ [22] and partially by Lafforgue for $n \geq 3$ [34]. The complete solution follows from the global Langlands conjecture.

Theorem 4.3. *For every $\pi \in A_n^0(k)_f$ and every place $x \notin S(\pi)$ we have $|z_i(\pi_x)| = 1$.*

In addition, Lafforgue [35] is able to deduce the following conjecture of Deligne [20].

Theorem 4.4. *Every irreducible local system ρ over a curve whose determinant is of finite order is pure of weight 0; moreover the symmetric polynomials in the eigenvalues of Frobenius generate a finite extension of $\mathbb{Q}$.*

In addition, Lafforgue is able to conclude that over a curve the notion of an irreducible local ℓ-adic system does not depend on the choice of ℓ and to verify the assertion of *descent* in the "geometric Langlands correspondence".

A more complete synopsis of these results can be found in the reports of Laumon [42, 43].

4.2. k **a global field of characteristic** 0. There is very little known of a general nature in the number field case. However, there are some rather spectacular examples of such global correspondences.

4.2.1. **General conjectures.** Recall that for $n = 1$ from global class field theory we have a canonical bijection between the continuous characters of $\mathcal{G}_k$ and characters of finite order of $k^\times\backslash\mathbb{A}^\times$. To obtain all characters of $k^\times\backslash\mathbb{A}^\times$ we must again replace the Galois group by the global Weil group W_k.

For $n \geq 2$, by analogy with the local Langlands conjecture, we need a global analogue of the Weil-Deligne group. But unfortunately no such analogue is available. Instead the conjectures are envisioned in terms of a conjectural *Langlands group* $\mathcal{L}_k$ [46]. At best, one hopes that $\mathcal{L}_k$ fits into an exact sequence

$$1 \longrightarrow \mathcal{L}_k^0 \longrightarrow \mathcal{L}_k \longrightarrow \mathcal{G}_k \longrightarrow 1$$

with $\mathcal{L}_k^0$ complex pro-reductive. This should fit into a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathcal{L}_k^0 & \longrightarrow & \mathcal{L}_k & \longrightarrow & \mathcal{G}_k & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & \mathcal{M}_k^0 & \longrightarrow & \mathcal{M}_k & \longrightarrow & \mathcal{G}_k & \longrightarrow & 1 \end{array}$$

where $\mathcal{M}_k$ is the equally conjectural *motivic Galois group* [46].

In these terms, in general one expects/conjectures the following types of global correspondences [11, 46].

(i) The irreducible n-dimensional representations of $\mathcal{G}_k$ should be in bijective correspondence with the cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$ of *Galois type*. (This is a restriction on π_∞.)
(ii) The irreducible n-dimensional representations of $\mathcal{M}_k$ should be in bijective correspondence with the *algebraic* cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$. These are the analogues of algebraic Hecke characters.
(iii) The irreducible n-dimensional representations of $\mathcal{L}_k$ should be in bijective correspondence with all cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$.

Of course, all of these correspondences should satisfy properties similar to those on the local conjectures, particularly the preservation of L- and ε- factors (with twists), compatibility with the local correspondences, etc.

In reality, very little is known of a truly general nature. One problem for the current methods seems to be that there is no natural moduli problem for GL_n over a number field.

4.2.2. **Known results.** There are many partial results of a general nature if one starts on the automorphic side and tries to construct the associated Galois representation.

When $n = 2$ and $k = \mathbb{Q}$ we have the fundamental result of Deligne [17], based on foundational work of Eichler and Shimura, which associates to every cuspidal representation π of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ which corresponds to a holomorphic new form of weight ≥ 2 a compatible system of ℓ-adic representations $\rho = \rho_\pi$ such that $L(s,\pi) = L(s,\rho)$. The Ramanujan-Petersson conjecture for such forms followed. This was extended to weight one forms over $\mathbb{Q}$ in the classical context by Deligne and Serre [21]. These results were extended to totally real fields k, still with $n = 2$, by a number of people, including Rogawski-Tunnell [48], Ohta [45], Carayol [8], Wiles [58], Taylor [52, 54], and Blasius-Rogawski [3]. For imaginary quadratic fields there is the work of Harris-Soudry-Taylor [26] and Taylor [53]. For more complete surveys, the reader can consult the surveys of Blasius [2] and Taylor [55].

For general GL_n and k a totally real number field Clozel has been able to attach a compatible system of ℓ-adic representations to cuspidal, algebraic, regular, self-dual representation of $\mathrm{GL}_n(\mathbb{A}_k)$ having local components of a certain type at one or two finite places [12].

More spectacular are the results which go in the opposite direction, that is, starting with a specific Galois representation and showing that it is modular. The results we have in mind are those of Langlands [37] and Tunnell [57], with partial results by Taylor, et. al, [7, 56], on the modularity of degree 2 complex Galois representations (the strong Artin conjecture) and the results of Wiles [59] and then Breuil, Conrad, Diamond, and Taylor [5] on the modularity of (the two dimensional Galois representation on the ℓ-adic Tate module of) elliptic curves over $\mathbb{Q}$.

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