THE BERGMAN KERNEL AND MASS EQUIDISTRIBUTION ON THE SIEGEL MODULAR VARIETY $Sp_{2n}(\mathbb{Z})\backslash\mathfrak{H}_n$

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Abstract. We study the mass equidistribution of holomorphic cuspidal Hecke eigenforms on the Siegel modular varieties, and show that equidistribution holds on average, by means of the Bergman kernel.

1. Introduction

In 1924 Artin showed that the geodesic flow on the open modular Riemann surface $SL_2(\mathbb{Z})\backslash\mathfrak{H}$ was ergodic [1]; interestingly, his proof was an analysis of the geodesic arcs via continued fractions. In 1939 Hopf showed that the geodesic flow on any manifold of constant negative curvature was ergodic [6]. Over time the geodesic flows on negatively curved manifolds X have become models for studying chaotic dynamics. The quantization of these chaotic systems corresponds to the eigenvalue problem for the Laplacian on X, with the eigenfucntions corresponding to quantum states. The behavior of these eigenfunctions as the eigenvalue grows corresponds to the semi-classical limit of the quantum states of this chaotic system. One problem of interest is that of quantum ergodicity. Let us order the eigenfunctions φ_λ of the Laplacian Δ on X according to increasing eigenvalue λ. To each L^2-normalized eigenfunction we can associate a probability measure

$$d\mu_{\varphi_\lambda} = |\varphi_\lambda|^2 \, d\mu$$

with $d\mu$ the normalized volume form. The question of quantum ergodicity or mass distribution asks for the possible limit measures as the eigenvalue grows, i.e., the understanding of the possible limits

$$\lim_{\lambda\to\infty} d\mu_{\varphi_\lambda}.$$

This phenomenon was analyzed in the context of compact negatively curved manifolds by the work of Schnirelman [19], Zelditch [27] and Colin de Verdière [2] among others; they showed, in the context they considered, that for a density one subsequence the only possible limit is the normalized volume form $d\mu$.

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Renewed number theoretic interest followed the work of Rudnick and Sarnak [17] where they concentrated on congruence quotients of the upper half-plane $\Gamma\backslash\mathfrak{H}$ and similar congruence quotients in higher dimensions. Here one has a preferred basis of eignefunctions for the Laplacian, namely those that are simultaneously eigenfunctions of all Hecke operators. For such bases they conjectured Arithmetic Quantum Unique Ergodicity (AQUE), namely that the only possible limit of the $d\mu_{\varphi_\lambda}$ is the normalized invariant volume $d\mu$. Interest was heightened further when Watson related AQUE to the triple product L-function for Maass forms [26]. Eventually, AQUE for the upper half-plane was established by Lindenstrauss via ergodic methods [11] for compact quotients; in the noncompact case he showed that there was a unique limit measure of the form $cd\mu$ for some $0 < c \leq 1$ but could not control mass escaping from the cusps. However, quite recently AQUE for Maass forms for $SL_2(\mathbb{Z})$ has been completed by Soundararajan [25] and the case of holomorphic Hecke eigenforms for $SL_2(\mathbb{Z})$ has now been completed by Holowinsky and Soundararajan [5], in both cases by arithmetic methods.

In this paper we consider a natural generalization of AQUE. We first will deal with a higher dimensional arithmetic variety, namely the arithmetic quotient of the Siegel upper half space $\mathfrak{H}_n$ for $n \geq 2$ by the full modular group $\Gamma_n = Sp_{2n}(\mathbb{Z})$. The quotient $X_n = \Gamma_n\backslash\mathfrak{H}_n$ is non-compact but of finite volume, and we let $d\mu$ be the normalized volume form so that $\mu(X_n) = 1$. Instead of Maass forms, we will consider holomorphic Siegel modular forms of integral weight k. The space of holomorphic Siegel cusp forms for Γ_n, denoted $S_k(\Gamma_n)$, is finite dimensional and has a basis $\{f_{k,j}\}$ consisting of Hecke eigenforms which are orthonormal for the Petersson inner product on $S_k(\Gamma_n)$. To each $f_{k,j}$ is attached a probability measure $d\mu_{k,j}$ and the analogue of AQUE in this context is that as $k \to \infty$ these measures approach $d\mu$. This seems at least as difficult as AQUE for classical Maass forms. However, we have found that if one approaches this conjecture *on average*, that is, considers the measures

$$d\mu_k = \frac{1}{\dim S_k(\Gamma_n)} \sum_j d\mu_{k,j},$$

we can obtain a version of mass equidistribution for these measures (Theorems 1 and 2 below) by relatively elementary techniques. It is well known that the reproducing kernel for $S_k(\Gamma_n)$ has two natural forms: one obtained from the normalized Hecke eigenbasis and another obtained as an average over Γ_n of the Bergman kernel on $\mathfrak{H}_n$. The equality of these two expressions lies at the heart of trace formula methods of computing the dimension of $S_k(\Gamma_n)$. Here we use this equality and show how a simple observation on the Bergman kernel lead to mass equidistribution (or AQUE) on average.

The later sections of the paper contain a discussion of the expected rate of equidistribution in this situation and a discussion of the statistical fluctuations of a mean observable for these measures.

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2. Statement of results.

Let $\mathfrak{H}_n = \{Z \in M_n(\mathbb{C}),\ Z = {}^tZ,\ \mathrm{Im}Z > 0\}$ be the Siegel upper half-space of degree n, and $\Gamma_n = Sp_{2n}(\mathbb{Z})$ be the Siegel modular group acting discontinuously on $\mathfrak{H}_n$. We restrict ourselves to the case of $n \geq 2$. It is known (see for example [10]) that the Siegel modular variety $X_n = \Gamma_n \backslash \mathfrak{H}_n$ is a complex manifold of dimension $n(n+1)/2$. The classical invariant symplectic measure on X_n is $\frac{dXdY}{(\det(Y))^{n+1}}$, where we write $Z = X + iY$ and $dX = \prod_{j\leq l} dx_{jl},\ dY = \prod_{j\leq l} dy_{jl}$ for $X = (x_{jl})$ and $Y = (y_{jl})$. The volume formula of Siegel [22] gives

$$\mathrm{vol}(\Gamma_n\backslash\mathfrak{H}_n) = 2\prod_{i=1}^{n} \frac{\Gamma(i)\zeta(2i)}{\pi^i}.$$

Denote by $S_k(\Gamma_n)$ the space of Siegel cusp forms of weight k with respect to Γ_n, i.e. the space of holomorphic functions f on $\mathfrak{H}_n$ satisfying

$$f(\gamma Z) = \det(CZ+D)^k f(Z), \text{ for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n,$$

and such that $(\det \mathrm{Im}(Z))^{k/2}|f(Z)|$ is bounded on $\mathfrak{H}_n$. $S_k(\Gamma_n)$ is a finite-dimensional Hilbert space equipped with the Petersson scalar product. Hashimoto [4] has shown

$$\dim_{\mathbb{C}} S_k(\Gamma_n) = 2^{n(n-1)/2}\frac{\mathrm{vol}(\Gamma_n\backslash\mathfrak{H}_n)}{(4\pi)^{n(n+1)/2}} k^{n(n+1)/2} + O(k^{n(n+1)/2-1}),$$

as $k \to \infty$ with $(-1)^{nk} = 1$. See also Mumford's more general results ([14], Corollary 3.5) in this direction. Denote $J_{n,k} = \dim_{\mathbb{C}} S_k(\Gamma_n)$. We assume in the following that $(-1)^{nk} = 1$, since otherwise $S_k(\Gamma_n) = \{0\}$.

Analogous to the elliptic case [18], it is expected that the mass of Hecke eigenforms should be equidistributed on the Siegel modular variety $X_n = \Gamma_n\backslash\mathfrak{H}_n$ as the weight increases. Let $\{f_{k,j}\}$ be an orthonormal Hecke eigenbasis for $S_k(\Gamma_n)$ with respect to $\frac{dXdY}{(\det(Y))^{n+1}}$. Then for $K \subset X_n$ compact, Arithmetic Quantum Unique Erdodicity, or mass equidistribution, in this context would lead us to expect

$$\lim_{k\to\infty} \int_K (\det \mathrm{Im}(Z))^k |f_{k,j}(Z)|^2 \frac{dXdY}{(\det(Y))^{n+1}} = \frac{1}{\mathrm{vol}(\Gamma_n\backslash\mathfrak{H}_n)} \int_K \frac{dXdY}{(\det(Y))^{n+1}}. \tag{1}$$

On X_n, denote by $d\mu(Z) = \frac{1}{\mathrm{vol}(\Gamma_n\backslash\mathfrak{H}_n)}\frac{dXdY}{(\det(Y))^{n+1}}$ the normalized invariant symplectic measure. For each $f_{k,j}$ define a new probability measure $d\mu_{k,j}(Z)$ on X_n by

$$d\mu_{k,j}(Z) = \det \mathrm{Im}(Z)^k |f_{k,j}(Z)|^2 \frac{dXdY}{(\det(Y))^{n+1}}.$$

Then (1) can be simply rephrased: for any compact $K \subset X_n$,

$$\int_K d\mu_{k,j}(Z) \to \int_K d\mu(Z)$$

as $k \to \infty$. Note that the Siegel modular variety X_n is *noncompact*. In the compact case the corresponding result is known to be true at least for a full density subsequence, which is a consequence of a general theorem of Shiffman-Zelditch [20] for holomorphic sections of tensor powers of any ample Hermitian line bundle L on a compact Kähler manifold X.

To investigate mass distribution *on average*, we associate a probability measure $d\mu_k$ on X_n to each integer $k > 2n$ by averaging over the Hecke eigenbasis for $S_k(\Gamma_n)$

$$d\mu_k(Z) = \frac{1}{J_{n,k}} \sum_{j=1}^{J_{n,k}} d\mu_{k,j} = \frac{1}{J_{n,k}} \sum_{j=1}^{J_{n,k}} \det \mathrm{Im}(Z)^k |f_{k,j}(Z)|^2 \frac{dXdY}{(\det(Y))^{n+1}}. \tag{2}$$

We now ask for the limiting behavior of these averaged measures as the weight increases.

By Minkowski's reduction theory [10], the following set F_n is a fundamental domain of Γ_n:

$$F_n = \{Z \in \mathfrak{H}_n, \ \text{ satisfying the following conditions (i), (ii) and (iii)}\},$$

where

(i) $-1/2 \le x_{jl} \le 1/2, \ j, l = 1, \cdots, n;$

(ii) ${}^t gYg \ge y_{jj}$ for all integral $g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix} \in \mathbb{Z}^n$ with $gcd(g_j, \cdots, g_n) = 1, \ 1 \le j \le n;$
$y_{j,j+1} \ge 0, \ 1 \le j \le n-1;$

(iii) $|\det(CZ + D)| \ge 1$ for any $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_n$.

In particular $Z \in F_n$ implies that

(i) $y_{jj} \le y_{j+1,j+1}, \ 1 \le j \le n-1;$

(ii) $y_{jj} \ge \sqrt{3}/2, \ \ 1 \le j \le n;$

(iii) $\det Y \le \prod_{\nu=1}^{n=1} y_{\nu\nu} \le c \det Y$, for some $c \ge 1$.

We will fix such a fundamental F_n as a realization of X_n.

Our main result on mass equidistribution on average is then the following.

Theorem 1. *For any compact domain $K \subset \ int(F_n)$ (the interior of F_n), we have*

$$\int_K d\mu_k = \int_K d\mu + O(k^{-1}), \tag{3}$$

where the constant implicit in the O-symbol depends on K.

More generally, we have the following result.

Theorem 2. *For any compact domain $K \subset \ F_n$, and any $\eta > 0$, we have*

$$\int_K d\mu_k = \int_K d\mu + O(k^{-1/2+\eta}), \tag{4}$$

where the constant implicit in the O-symbol depends on K and η.

3. The Bergman kernel.

The Bergman kernel for the Siegel modular variety $X_n = \Gamma_n \backslash \mathfrak{H}_n$ has been studied by Godement [3] to derive the dimension formula using ideas from the theory of the trace formula. The Bergman kernel function for $\mathfrak{H}_n$ is obtained from the Bergman kernel on the generalized unit disc $\mathbf{D}_n = \{Z \in M_n(\mathbb{C}),\ {}^tZ = Z,\ I_n - Z\overline{Z} > 0\}$, one of the four types of the irreducible bounded symmetric domains studied by E. Cartan, via the Cayley transform. Then the Bergman kernel for the modular variety X_n is obtained by averaging [3, 10].

Fix $n \geq 2$ and define

$$h_k(Z,\ Z') = \sum_{\gamma \in \Gamma_n} \det\left(\frac{Z - \gamma\overline{Z'}}{2i}\right)^{-k} \det(C\overline{Z'} + D)^{-k},$$

where $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. By Godement's theorem (cf. Section 5 of [3] or Proposition III.6.2 of [10]) $h_k(Z, Z')$ is cuspidal in both variables as long as $k > 2n$. For any $f \in S_k(\Gamma_n)$ we have

$$2^{-1}a(n,k)\int_{\Gamma_n\backslash\mathfrak{H}_n} \det \mathrm{Im}(Z')^k f(Z')h_k(Z,Z')\frac{dX'dY'}{(\det(Y'))^{n+1}} = f(Z), \tag{5}$$

where the constant

$$a(n,k) = 2^{-n(n+3)/2}\pi^{-n(n+1)/2}\prod_{\nu=1}^{n}\frac{\Gamma\left(k - \frac{\nu-1}{2}\right)}{\Gamma\left(k - \frac{\nu+n}{2}\right)}$$

([7, 10]). Thus it follows that

$$B_k(Z, Z') = 2^{-1}a(n,k)h_k(Z,\ Z') \tag{6}$$

is the Bergman kernel, or reproducing kernel, for $S_k(\Gamma_n)$ on the modular variety X_n. On the other hand, we have a natural expression for this reproducing kernel in terms of our orthonormal basis, namely as

$$\sum_{j=1}^{J_{n,k}} f_{k,j}(Z)\overline{f_{k.j}(Z')}.$$

Equating these expressions we find the often used equality

$$\sum_{j=1}^{J_{n,k}} f_{k,j}(Z)\overline{f_{k.j}(Z')} = B_k(Z, Z') = 2^{-1}a(n,k)h_k(Z,\ Z').$$

4. Proof of Theorem 1.

Utilizing the expression of the Bergman kernel as an average over Γ we can rewrite our averaged measure as

$$d\mu_k = \frac{\det \mathrm{Im}(Z)^k}{J_{n,k}} B_k(Z,Z)\frac{dXdY}{(\det(Y))^{n+1}} = \frac{2^{-1}a(n,k)}{J_{n,k}} R_k(Z,Z)\frac{dXdY}{(\det(Y))^{n+1}},$$

where

$$R_k(Z,Z) = \det \operatorname{Im}(Z)^k h_k(Z,Z)$$
$$= \sum_{\gamma=\left(\begin{smallmatrix} A & B \\ C & D\end{smallmatrix}\right)\in\Gamma_n} \det \operatorname{Im}(Z)^k \det\left(\frac{Z-\gamma\overline{Z}}{2i}\right)^{-k} \det(C\overline{Z}+D)^{-k}.$$

If we let

$$h_\gamma(Z) = \frac{\det \operatorname{Im}(Z)}{\det\left(\frac{Z-\gamma\overline{Z}}{2i}\right)\cdot \det(C\overline{Z}+D)}$$

we can rewrite the above as

$$R_k(Z,Z) = \sum_{\gamma\in\Gamma_n} h_\gamma(Z)^k.$$

The following is the key observation for proving our theorems.

Lemma 1. *For $Z \in \mathfrak{H}_n$ and $\gamma \in \Gamma_n$, we have*

$$|h_\gamma(Z)| \le 1$$

with equality if and only if $\gamma Z = Z$. Moreover, there exists a constant c, depending only on K, such that for $0 < \epsilon < 1$ we have

$$|h_\gamma(Z)| \le (1+\epsilon)^{-1/2}$$

unless $\|\gamma Z - Z\| \le c\sqrt{\epsilon}$.

Before we begin the proof, let us recall that for $Z \in M_n(\mathbb{C})$ with ${}^tZ = Z$ the norm $\|Z\|$ is defined by

$$\|Z\|^2 = \operatorname{tr}(\overline{Z}\cdot Z) = \sum_i |z_{i,i}|^2 + 2\sum_{i<j}|z_{i,j}|^2$$

and that if we write $Z = X + iY$ with X and Y real symmetric matrices then $\|Z\|^2 = \|X\|^2 + \|Y\|^2$.

Proof: Let $Y = \operatorname{Im}(Z)$ and $Y' = \operatorname{Im}(Z')$, where $Z' = \gamma Z = (AZ+B)(CZ+D)^{-1}$. Since $\operatorname{Im}(\gamma Z) = {}^t(C\overline{Z}+D)^{-1}\operatorname{Im}(Z)(CZ+D)^{-1}$, we can write

$$|h_\gamma(Z)| = \left|\frac{\det \operatorname{Im}(Z)}{\det\left(\frac{Z-\gamma\overline{Z}}{2i}\right)\cdot\det(C\overline{Z}+D)}\right| = \left|\frac{\sqrt{\det(Y)\det(Y')}}{\det\left(\frac{Z-\overline{Z}'}{2i}\right)}\right|,$$

so we need to show that

$$\left|\frac{\sqrt{\det(Y)\det(Y')}}{\det\left(\frac{Z-\overline{Z}'}{2i}\right)}\right| \le 1.$$

We begin with the following two observations.

(i) For any pair of real symmetric matrices S, T such that $T > 0$, we have, on writing $T = {}^tPP$,

$$|\det(T+iS)| = |\det(T)\det(I + i^tP^{-1}SP^{-1})| \geq \det(T),$$

and the equality holds if and only if $S = 0$. In fact if U is an orthogonal matrix such that ${}^tU\,{}^tP^{-1}SP^{-1}U = \Lambda$ is a diagonal matrix with eigenvalues $\{\lambda_i\}$, we see that

$$|\det(T+iS)| = \left(\prod \sqrt{1+\lambda_i^2}\right)\det(T) \geq \sqrt{1+\epsilon^2}\det(T),$$

unless the eigenvalues lie in the range $-\epsilon < \lambda_i < \epsilon$, that is, unless $-\epsilon I < \Lambda < \epsilon I$, or equivalently, $-\epsilon T < S < \epsilon T$. Hence

$$\frac{1}{|\det(T+iS)|} \leq (1+\epsilon)^{-1/2}\frac{1}{\det(T)} \quad \text{unless} \quad -\sqrt{\epsilon}T < S < \sqrt{\epsilon}T.$$

(ii) Moreover if $T, T' > 0$,

$$\frac{\det(T)\det(T')}{\det(\frac{T+T'}{2})^2} = \frac{\det({}^tP^{-1}T'P^{-1})}{\det(\frac{I+{}^tP^{-1}T'P^{-1}}{2})^2} = \frac{\prod_\lambda \lambda}{\prod_\lambda(\frac{1+\lambda}{2})^2} \leq 1,$$

where λ runs over all eigenvalues of the matrix ${}^tP^{-1}T'P^{-1}$, and equality holds if and only if ${}^tP^{-1}T'P^{-1} = I$, i.e. $T' = T$. In addition we note that in fact

$$\frac{\det(T)\det(T')}{\det(\frac{T+T'}{2})^2} = \frac{1}{\prod_\lambda(1+\frac{(\lambda^{1/4}-\lambda^{-1/4})^2}{2})^2} < (1+\epsilon)^{-2},$$

unless $|\lambda^{1/4} - \lambda^{-1/4}|^2 \leq 2\epsilon$ for all eigenvalues λ. If an eigenvalue $\lambda > 1$ we see that $|\lambda^{1/4} - \lambda^{-1/4}|^2 \leq 2\epsilon$ implies that $0 < \lambda^{1/4} - 1 < 2\sqrt{\epsilon}$ which in turn implies that $1 < \lambda < 1 + c'\sqrt{\epsilon}$ with c' an absolute constant. Similarly, if $\lambda < 1$ we arrive at $1 < \lambda^{-1} < 1 + c'\sqrt{\epsilon}$ or $1 > \lambda > 1 - c''\sqrt{\epsilon}$. Hence if $|\lambda^{1/4} - \lambda^{-1/4}|^2 \leq 2\epsilon$ we can find an absolute constant c_1 such that $1 - c_1\sqrt{\epsilon} < \lambda < 1 + c_1\sqrt{\epsilon}$ for all λ and all ϵ in the range $0 < \epsilon < 1$. As above, this can be rewritten as $T - c_1\sqrt{\epsilon}T < T' < T + c_1\sqrt{\epsilon}T$. Hence

$$\frac{\det(T)\det(T')}{\det(\frac{T+T'}{2})^2} < (1+\epsilon)^{-2} \quad \text{unless} \quad -c_1\sqrt{\epsilon}T < T' - T < c_1\sqrt{\epsilon}T.$$

We first infer that

$$\left|\frac{\sqrt{\det(Y)\det(Y')}}{\det\left(\frac{Z-\overline{Z}'}{2i}\right)}\right| = \left|\frac{\sqrt{\det(Y)\det(Y')}}{\det\left(\frac{Y+Y'}{2} - i\frac{X-X'}{2}\right)}\right| \leq \left|\frac{\sqrt{\det(Y)\det(Y')}}{\det\left(\frac{Y+Y'}{2}\right)}\right| \leq 1,$$

and equalities hold if and only if $X' = X$ for the first from (i) and $Y' = Y$ for the second from (ii). This gives the first statement of the lemma.

Moreover we see that

$$\left|\frac{\det \mathrm{Im}(Z)}{\det\left(\frac{Z-\gamma\overline{Z}}{2i}\right)\cdot\det(C\overline{Z}+D)}\right| = \left|\frac{\sqrt{\det(Y)\det(Y')}}{\det\left(\frac{Y+Y'}{2} - i\frac{X-X'}{2}\right)}\right| \leq (1+\epsilon)^{-1/2},$$

unless

$$-\sqrt{\epsilon}(Y'+Y) < X'-X < \sqrt{\epsilon}(Y'+Y)$$

from (i), and

$$-c_1\sqrt{\epsilon}Y < Y'-Y < c_1\sqrt{\epsilon}Y$$

from (ii) with c_1 an absolute constant. We next wish to turn these two conditions into the condition of the form $\|Z'-Z\| \leq c\sqrt{\epsilon}$ or equivalent conditions on $\|X'-X\|$ and $\|Y'-Y\|$.

Let us begin with the second condition. Let P be an orthogonal matrix diagonalizing Y, i.e., such that ${}^tPYP = D$ with D diagonal. Since Y is positive definite, its eigenvalues, which are the diagonal entries of D, are positive. Let $S = {}^tP(Y'-Y)P$ so that our condition is $-c_1\sqrt{\epsilon}D < S < c_1\sqrt{\epsilon}D$. Thus $c_1\sqrt{\epsilon}D - S > 0$. Note that the matrix $c_1\sqrt{\epsilon}D - S$ is actually symmetric and positive definite. Thus all diagonal entries and all 2×2 minors centered on the main diagonal are positive (see, for example, Section 11.4 of [15]). Thus for each i we have $c_1 d_{ii}\sqrt{\epsilon} > s_{ii}$ and $(c_1 d_{ii}\sqrt{\epsilon} - s_{ii})(c_1 d_{jj}\sqrt{\epsilon} - s_{jj}) > s_{ij}^2$. Similarly, from $S + c_1\sqrt{\epsilon}D > 0$ we conclude that $s_{ii} > -c_1 d_{ii}\sqrt{\epsilon}$, so that $c_1 d_{ii}\sqrt{\epsilon} > |s_{ii}|$ for all i. We can also conclude that $4c_1^2 d_{ii} d_{jj}\epsilon > s_{ij}^2$. If we let $d = d(Y) = \max\{d_{ii}\}$ we have uniformly $c_1^2 d^2\epsilon > s_{ii}^2$, and $c_1^2 d^2 \epsilon > s_{ij}^2$. Thus $\|S\|^2 < c_2 d^2\epsilon$. Now, $Y'-Y = PS^tP$ with P orthogonal, so the entries of P are all of bounded absolute value at most 1. Hence $\|Y'-Y\|^2 < c_2 d(Y)^2\epsilon$. If we let Z vary in a compact set K with $Y = \mathrm{Im}(Z)$ we see that $d(Y)$ is bounded above uniformly for $Z\in K$ (either by a continuity argument or by a diagonalization argument). So there exists a constant c_3, depending on K, such that $\|Y'-Y\| < c_3\sqrt{\epsilon}$ for all $Z \in K$. This is one half of our desired condition.

The argument for bounding $\|X'-X\|$ is exactly the same. We first note that since $-c_1\sqrt{\epsilon}Y < Y'-Y < c_1\sqrt{\epsilon}Y$ we can conclude $Y'+Y < (2+c_1\sqrt{\epsilon})Y = c_4 Y$. Then the condition $-\sqrt{\epsilon}(Y'+Y) < X'-X < \sqrt{\epsilon}(Y'+Y)$ implies $-c_4\sqrt{\epsilon}Y < X'-X < c_4\sqrt{\epsilon}Y$. We now proceed as in the previous paragraph.

Combining these two estimates, we see that there exists a constant c, depending only on K, so that for $Z\in K$

$$|h_\gamma(Z)| = \left| \frac{\det \mathrm{Im}(Z)}{\det\left(\frac{Z-\gamma\overline{Z}}{2i}\right)\cdot \det(C\overline{Z}+D)} \right| \leq (1+\epsilon)^{-1/2}, \tag{7}$$

unless the norm

$$\|\gamma Z - Z\| \leq c\sqrt{\epsilon}. \tag{8}$$

This establishes the last statement of the lemma. □

If $Z \in K \subset int(F_n)$ and $\gamma \neq \pm I$, we have $\|\gamma Z - Z\| \geq d_K > 0$, where d_K is the distance from the compact set K to the boundary $\partial(F_n)$ of F_n as measured by the norm, i.e.,

$$d_K = \min_{\substack{Z\in K\\ W\in\partial(F_n)}} \|Z-W\|.$$

Hence by our lemma we have a uniform estimate for $Z \in K$

$$|h_\gamma(Z)| = \left| \frac{\det \mathrm{Im}(Z)}{\det\left(\frac{Z-\gamma\overline{Z}}{2i}\right) \cdot \det(C\overline{Z}+D)} \right| \le e^{-\delta}, \tag{9}$$

where $\delta > 0$ depends only on K.

Let $k_0 > 2n$ be fixed and k sufficiently large. We have

$$\begin{aligned} R_k(Z,Z) &= \sum_{\gamma\in\Gamma_n} h_\gamma(Z)^k = 2 + O(\sum_{\gamma\neq\pm I} |h_\gamma(Z)|^k) \\ &= 2 + O(\max_{\gamma\neq\pm I,\, Z\in K} |h_\gamma(Z)|^{k-k_0} \sum_{\gamma\in\Gamma_n} |h_\gamma(Z)|^{k_0}) \\ &= 2 + O(e^{-\delta k} \sum_{\gamma\in\Gamma_n} |h_\gamma(Z)|^{k_0}), \end{aligned}$$

by using (9). Now by Godement's theorem (see Proposition III.6.2 in [10]),

$$(\det Y)^{-k_0/2} \sum_{\gamma\in\Gamma_n} |h_\gamma(Z)|^{k_0}$$

is uniformly bounded on K and so is $\det Y$. Hence

$$\sum_{\gamma\in\Gamma_n} |h_\gamma(Z)|^{k_0} \ll 1,$$

and thus

$$R_k(Z,Z) = 2 + O(e^{-\delta k}).$$

This estimate then gives

$$\int_K d\mu_k = \frac{a(n,k)}{2J_{n,k}} \int_K R_k(Z,Z) \frac{dX\,dY}{\det(Y)^{n+1}} = \frac{a(n,k)\mathrm{vol}(\Gamma_n\backslash\mathfrak{H}_n)}{2J_{n,k}} \mu(K)(2+O(e^{-\delta k})).$$

A direct computation using the explicit formulas for $a(n,k)$ and $J_{n,k}$ gives

$$\frac{a(n,k)\mathrm{vol}(\Gamma_n\backslash\mathfrak{H}_n)}{J_{n,k}} = 1 + O(k^{-1})$$

(see also Hashimoto's first asymptotic formula in [4]) so that

$$\int_K d\mu_k = \mu(K) + O(k^{-1}).$$

This completes the proof of Theorem 1.

5. Proof of Theorem 2.

Since Γ_n acts discontinuously on $\mathfrak{H}_n$, for sufficiently small $\delta_0 > 0$ we have

$$\#\{\gamma \in \Gamma_n,\ \gamma K \cap K_{\delta_0} \neq \emptyset\} < \infty,$$

where

$$K_{\delta_0} = \{Z' \in \mathfrak{H}_n,\ \|Z' - Z\| \le \delta_0, \text{ for some } Z \in K\}$$

is compact in $\mathfrak{H}_n$. Let

$$\{\gamma \in \Gamma_n,\ \gamma K \cap K_{\delta_0} \neq \emptyset\} = \{\gamma_1, \cdots, \gamma_r\},$$

say. Thus if $\gamma \neq \gamma_j,\ 1 \le j \le r$, we have

$$\|\gamma Z - Z\| > \delta_0, \text{ for all } Z \in K.$$

As in the proof of Theorem 1 the contribution from these elements are exponentially small by (7) and (8).

Now suppose $\gamma = \gamma_j$. If γ doesn't have any fixed points in K, then by changing δ_0 if necessary, we also have, by the compactness of K,

$$\|\gamma Z - Z\| > \delta_0, \text{ for all } Z \in K.$$

Hence its contribution is negligible. On the other hand if $\gamma \neq \pm I$ does have fixed points in K, the set of fixed points of γ in K, $F_\gamma = \{Z \in K,\ \gamma Z = Z\}$, lies on the boundary of F_n ([10] Proposition 3.2), $\partial(F_n)$. For $\epsilon > 0$ define

$$A_\epsilon = \{Z \in K,\ \|Z' - Z\| \le c\epsilon \text{ implies } Z' \in \text{int}(F_n)\},$$

where c is as in (8), and $B_\epsilon = K \backslash A_\epsilon \subset \cup_{Z \in \partial(K)} B(Z, \epsilon)$, where $B(Z,\epsilon)$ is the ball with center at Z and radius ϵ. From (7) and (8), we see that

$$\left| \frac{\det \text{Im}(Z)}{\det\left(\frac{Z - \gamma \overline{Z}}{2i}\right) \cdot \det(C\overline{Z} + D)} \right| \le (1+\epsilon^2)^{-1/2} \tag{10}$$

whenever $Z \in A_\epsilon$ since $\|\gamma_j Z - Z\| > c\epsilon$, where $\gamma_j = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Moreover it is easy to see that $\mu(B_\epsilon) \ll_K \epsilon$. (In fact, $\mu(B_\epsilon) \ll \epsilon \text{Area}(\partial K)$, assuming that ∂K is piecewise smooth.)

We now balance the contribution $(1+\epsilon^2)^{-k/2}$ coming from the finitely many terms from the Bergman kernel corresponding to elliptic terms and the boundary contribution of ϵ. Note $(1+\epsilon^2)^{-k/2} = \exp(-k/2 \log(1+\epsilon^2)) = \exp(-k\epsilon^2/2 + O(k\epsilon^4))$. If we take $\epsilon \ll k^{-1/2}$, then this term would be larger than a positive constant. On the other hand, we want to choose ϵ as small as possible due to the second term. Thus the optimal choice is $\epsilon = k^{-1/2+\eta}$, and we conclude

$$\int_K d\mu_k = \int_K d\mu + O(k^{-1/2+\eta}).$$

This completes the proof of Theorem 2.

6. Rate of equidistribution

Finally we look into the question on what possible decay rate for the mass equidistribution we would expect, following the suggestion by Sarnak. The difficulty for the case $n > 1$, in contrast to the case $n = 1$, lies in the apparent lack of suitable L-function interpretation. Let $f_k \in S_k(\Gamma_n)$ be an Hecke eigenform, and $E(Z, s)$ be the Siegel-Eisenstein series

$$E(Z,\ s) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_\infty \backslash \Gamma_n} \frac{\det \operatorname{Im}(Z)^s}{|\det(CZ+D)|^{2s}} \ ,$$

where

$$\Gamma_\infty = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \in \Gamma_n \right\} \ .$$

Denote

$$R_n^+ = \left\{ (n_{ij}) \in M_n(\mathbb{Q}),\ N = {}^tN > 0,\ n_{ii}, 2n_{ij} \in \mathbb{Z} \right\},$$

and $\widehat{R}_n^+$ the set of equivalence classes of $N \in R_n^+$ with respect to the action of the group $SL_n(\mathbb{Z})$ on $R_n^+ :\quad N \to\ {}^t\gamma N \gamma$, for $N \in R_n^+,\ \gamma \in SL_n(\mathbb{Z})$. Denote by e_N the order of the isotropy group of N,

$$e_N = \#\{\gamma \in SL_n(\mathbb{Z}),\ {}^t\gamma N \gamma = N\},$$

and define

$$R(s,\ f_k \times f_k) = \sum_{N \in \widehat{R_n^+}} \frac{|f_k(N)|^2}{e_N} (\det N)^{-s}.$$

where $f_k(N)$ is the Fourier coefficient of f_k,

$$f_k(z) = \sum_{N \in R_n^+} f_k(N) \exp(2\pi i \operatorname{tr}(NZ)).$$

Note $f_k(N)$ is invariant under the action of $SL_n(\mathbb{Z})$.

We have the integral representation [9]

$$\gamma(s) R(s,\ f_k \times f_k) = \int_{\Gamma_n \backslash \mathfrak{H}_n} E(Z,\ s + (n+1)/2 - k) \det \operatorname{Im}(Z)^k |f_k(Z)|^2 \frac{dXdY}{(\det(Y))^{n+1}} \ ,$$

where

$$\gamma(s) = 2^{1-2sn} \pi^{-sn+n(n-1)/4} \prod_{j=1}^{n} \Gamma(s - (j-1)/2) \ .$$

Moreover, if we let

$$\Lambda(s,\ f_k \times f_k) = \xi(2s + n - 2k + 1) \prod_{j=1}^{[n/2]} \xi(4s + 2n - 4k + 2 - 2j) \gamma(s) R(s,\ f_k \times f_k),$$

where $\xi(s) = \pi^{-s/2} \Gamma(s/2) \zeta(s)$, then we have

$$\Lambda(s,\ f_k \times f_k) = \Lambda(2k - (n+1)/2 - s,\ f_k \times f_k),$$

and $\Lambda(s,\ _k \times f_k)$ has a simple pole at $s = k$ with residue

$$\prod_{j=1}^{[n/2]} \xi(2j+1) \langle f_k,\ f_k \rangle_k,$$

where

$$\langle f_k,\ f_k\rangle_k \ = \ \int_{\Gamma_n\backslash\mathfrak{H}_n} \det \mathrm{Im}(Z)^k |f_k(Z)|^2 \frac{dXdY}{(\det(Y))^{n+1}}$$

is the Petersson scalar product on $S_k(\Gamma_n)$. Thus, the critical line of $R(s,\ f_k \times f_k)$ is on $\mathrm{Re}(s) = k-(n+1)/4$. For the heuristic on the decay rate, we assume the Lindelöf conjecture for $R(s,\ f_k \times f_k)$ in the k-aspect, although $R(s,\ f_k \times f_k)$ in general doesn't have an Euler product,

$$R(k-(n+1)/4+it,\ f_k \times f_k) \ \ll_\epsilon k^\epsilon,$$

for any $\epsilon > 0$. One may normalize the zeta function by rescaling s so that the critical strip is on $\mathrm{Re}(s) = 1/2$ as usual by considering $R(k-(n+1)/2+s(n+1)/2,\ f_k \times f_k)$ instead. Moreover it follows by Tauberian theorem as in [9] (we correct some obvious misprints there), that

$$\frac{1}{T}\sum_{r\le T} r^{-(k-1)} \sum_{N\in\widehat{R}_n^+,\, \det N = r} \frac{|f_k(N)|^2}{e_N} \ \sim \ \mathrm{res}_{s=k} R(s,\ f_k\times f_k) \ = \ C^{-1}\langle f_k,\ f_k\rangle_k,$$

as $T \to \infty$, where

$$C = \frac{\xi(n+1)\ \prod_{j=1}^n \Gamma(k-(j-1)/2)\ \prod_{j=1}^{[n/2]} \xi(2n+2-2j)}{2^{2kn-1}\ \pi^{kn-n(n-1)/4}\ \prod_{j=1}^{[n/2]} \xi(2j+1)}.$$

Let r_0 be the smallest positive integer such that

$$\sum_{N\in\widehat{R}_n^+,\, \det N = r_0} \frac{|f_k(N)|^2}{e_N} \neq 0\ ,$$

and we normalize f_k so that

$$\sum_{N\in\widehat{R}_n^+,\, \det N = r_0} \frac{|f_k(N)|^2}{e_N} = 1\ .$$

In analogy with the case $n = 1$ in which we have $\mathrm{res}_{s=k}R(s,\ f_k \times f_k) = \frac{L(k,\, \mathrm{sym}^2(f_k))}{\zeta(2)}$ and k is at the edge of the critical strip of $L(s,\ \mathrm{sym}^2(f_k))$, we *assume*

$$\mathrm{res}_{s=k} R(s,\ f_k \times f_k) \ll k^\epsilon,$$

which is a consequence of the Lindelöf conjecture for $R(s,\ f_k \times f_k)$. Therefore, under this conjecture, we would have that

$$\begin{aligned}\int_{\Gamma_n\backslash\mathbf{H}_n} & E(Z,\ (n+1)/4+it) \det \mathrm{Im}(Z)^k \frac{|f_k(Z)|^2}{< f_k,\ f_k >_k} \frac{dXdY}{(\det(Y))^{n+1}} \\ & \ll C^{-1}\gamma(k-(n+1)/4+it) R(k-(n+1)/4+it,\ f_k\otimes f_k) k^\epsilon \\ & \ll_{\epsilon,n} \frac{(4\pi)^{-kn}\ \prod_{j=1}^n \Gamma(k-(n+1)/4+it-(j-1)/2)}{(4\pi)^{-kn}\ \prod_{j=1}^n \Gamma(k-(j-1)/2)} k^\epsilon \\ & \ll_{\epsilon,n,t} \prod_{j=1}^n k^{-(n+1)/4} k^\epsilon \ll_{\epsilon,n,t} k^{-n(n+1)/4+\epsilon}.\end{aligned}$$

This indicates that the expected decay rate in the equidistribution would be $k^{-n(n+1)/4+\epsilon}$.

7. Statistical fluctuations

We next make some observations about the statistical fluctuations of our measures $d\mu_{k,j}$ for a mean zero test function. Let $C_{c,0}^{\infty}(X)$ be the space of smooth functions of compact support with mean zero on X with respect to normalized invariant measure $d\mu$, i.e, $\int_{X_n} \phi(Z)d\mu(Z) = 0$. Then for the measures $d\mu_{k,j}$ and the expected decay rate of Section 6, the first moment is expected to satisfy

$$\frac{1}{J_{n,k}} \sum_j \frac{1}{k^{-n(n+1)/4}} \mu_{k,j}(\phi) = o(1)$$

or

$$\frac{\mu_k(\phi)}{k^{-n(n+1)/4}} = o(1). \tag{11}$$

Here, as usual, $\mu_{k,j}(\phi) = \int_{X_n} \phi(Z)d\mu_{k,j}(Z)$.

First, let us assume that $supp(\phi) \subset K$ where K is a compact subset of X_n with $F \subset int(F_n)$. Then we are in the situation of Theorem 1. Then from the proof of Theorem 1, we see that

$$\mu_k(\phi) = \int_K \phi(Z)d\mu_k(Z) = \frac{a(n,k)}{2J_{n,k}} \int_K \phi(Z)R_k(Z,Z)\frac{dXdY}{\det(Y)^{n+1}}.$$

In the case of Theorem 1, we have the uniform estimate

$$R_k(Z,Z) = 2 + O(e^{-\delta k})$$

for $Z \in K$. Substituting this in the integral above, since ϕ is mean zero with respect to $d\mu$ the main term vanishes and the remaining term can be estimated to give

$$\mu_k(\phi) = \frac{a(n,k)\mathrm{vol}(\Gamma_n\backslash\mathfrak{H}_n)}{2J_{n,k}} \mu(K)\sup(|\phi|)O(e^{-\delta k}).$$

Once again using that

$$\frac{a(n,k)\mathrm{vol}(\Gamma_n\backslash\mathfrak{H}_n)}{J_{n,k}} = 1 + O(k^{-1})$$

we have

$$\mu_k(\phi) = O_\phi(e^{-\delta k}).$$

As the negative exponential dominates the division by the expected decay rate, this is indeed $o(k^{-n(n+1)/4})$, and we obtain (11) in this situation.

If the support of the observable ϕ is not bounded away from the fixed point set of the elliptic elements of Γ_n, as in Theorem 2, our Bergman kernel method seems to fail. We cannot control the estimates coming from elliptic elements. One way to avoid this would be to pass to a finite index (congruence) subgroup $\Gamma_n(N)$ which is torsion free. It always exists, essentially by an argument of Minkowski [21]. While we gain the avoidance of the elliptic elements in the Bergman kernel estimates, we may lose control over $J_{n,k}(N) = \dim S_k(\Gamma_n(N))$. Ibukiyama and Saito [8] have a conjectural formula for $J_{n,k}(N)$ which implies

$$\frac{a(n,k)\mathrm{vol}(\Gamma_n(N)\backslash\mathfrak{H}_n)}{J_{n,k}(N)} = 1 + O(k^{-1}). \tag{12}$$

(Note the constant $a(n,k)$ is independent of the level N.) The formula is known to be valid in case of $n=2$ and $n=3$. In these cases, our previous analysis is valid and we obtain both the equidistribution result Theorem 1 and the statistical fluctuation result (11) for $\Gamma_n(N)$ for N sufficiently large, with no conditions on K or the support of ϕ. For $n \geq 4$ these become valid when the dimension formula of Ibukiyama and Saito is established, or a weaker version which suffices for (12).

8. Concluding remarks.

For *compact complex* manifolds an analogue of the above theorems would follow from the asymptotic expansion of the Bergman kernel proved by S. Zelditch [28]. However the arithmetically defined variety X_n is more involved and delicate, in view of the presence of unipotent elements and elliptic elements in Γ_n.

In the case that one has a *compact* arithmetic quotient of Siegel space, and for the *real analytic* Hecke-Maass forms, Silberman and Venkatesh [23, 24], using Lindenstrauss' methods [11], have proved a stronger equidistribution result.

Our analysis of the Bergman kernel would work for any subgroup Γ of finite index in Γ_n. Thus our equidistribution result would also be valid in these cases *provided* we had the corresponding dimension formula for these groups. On the other hand, it seems quite possible that one could obtain an estimate for the quantity

$$\frac{a(\Gamma,k)\mathrm{vol}(\Gamma\backslash\mathfrak{H}_n)}{J_{\Gamma,k}}$$

without explicit knowledge of the individual pieces from a purely geometric argument.

Our method seems quite robust. In his Ph.D. dissertation, Sheng-Chi Liu has obtained an analogue of the above result for the Hilbert modular varieties $\Gamma\backslash\mathfrak{H}_1^n$ for Γ a congruence subgroup of $SL_2(\mathcal{O}_F)$ where F is a totally real number field of degree n and $\mathcal{O}_F$ is the ring of integers in F [12].

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