An equation without the Painleve property

```
> d20:=diff(y(x), x, x)-y(x)^2-x^2;
```

$$d20 := \frac{d^2}{dx^2} y(x) - y(x)^2 - x^2 \tag{1}$$

Since it is more convenient to work with power series at zero, we let c be the singular point and substitute x=s+c (I keep it x, by abuse of notation);

```
> subs(x^2=(x+c)^2,%);
```

$$\frac{d^2}{dx^2} y(x) - y(x)^2 - (x+c)^2 \tag{2}$$

```
> expand(%);
```

$$\frac{d^2}{dx^2} y(x) - y(x)^2 - x^2 - 2\,x\,c - c^2 \tag{3}$$

```
> d2:=%;
```

$$d2 := \frac{d^2}{dx^2} y(x) - y(x)^2 - x^2 - 2\,x\,c - c^2 \tag{4}$$

The leading order singularity should be the same as in Painleve, the balance is between y'' and y^2

```
> subs(y(x)=6/x^2+d(x),d2);
```

$$\frac{d^2}{dx^2}\left(\frac{6}{x^2} + d(x)\right) - \left(\frac{6}{x^2} + d(x)\right)^2 - x^2 - 2\,x\,c - c^2 \tag{5}$$

```
> expand(%);
```

$$\frac{d^2}{dx^2} d(x) - \frac{12\,d(x)}{x^2} - d(x)^2 - x^2 - 2\,x\,c - c^2 \tag{6}$$

Try power series; it should not work

```
> d4:=%;
```

$$d4 := \frac{d^2}{dx^2} d(x) - \frac{12\,d(x)}{x^2} - d(x)^2 - x^2 - 2\,x\,c - c^2 \tag{7}$$

```
> subs(d(x)=sum(c[k]*x^k,k=-1..8),d4):series(%,x,10):convert(%,
  polynom);
```

$$-\frac{10\,c_{-1}}{x^3} + \frac{-c_{-1}^2 - 12\,c_0}{x^2} + \frac{-2\,c_{-1}\,c_0 - 12\,c_1}{x} - 2\,c_1\,c_{-1} - c_0^2 - 10\,c_2 - c^2 + (-6\,c_3 - 2\,c - 2\,c_2\,c_{-1} - 2\,c_1\,c_0)\,x + (-1 - 2\,c_2\,c_0 - c_1^2 - 2\,c_3\,c_{-1})\,x^2 + (8\,c_5 - 2\,c_2\,c_1 - 2\,c_3\,c_0 - 2\,c_{-1}\,c_4)\,x^3 + (-2\,c_3\,c_1 - c_2^2 - 2\,c_4\,c_0 - 2\,c_5\,c_{-1} + 18\,c_6)\,x^4 + (-2\,c_2\,c_3 - 2\,c_1\,c_4 - 2\,c_5\,c_0 - 2\,c_{-1}\,c_6 + 30\,c_7)\,x^5 + (44\,c_8 - 2\,c_5\,c_1 - c_3^2 - 2\,c_{-1}\,c_7 - 2\,c_6\,c_0 - 2\,c_2\,c_4)\,x^6 + (-2\,c_2\,c_5 - 2\,c_3\,c_4 - 2\,c_0\,c_7 - 2\,c_1\,c_6 - 2\,c_{-1}\,c_8)\,x^7 + (-2\,c_5\,c_3 - 2\,c_7\,c_1 - 2\,c_0\,c_8 - 2\,c_2\,c_6 - c_4^2)\,x^8 + (-2\,c_3\,c_6 - 2\,c_7\,c_2 - 2\,c_8\,c_1 - 2\,c_4\,c_5)\,x^9 \tag{8}$$

Solve coefficient by coefficient.

```
> sr2:=%;
```

$$sr2 := -\frac{10\,c_{-1}}{x^3} + \frac{-c_{-1}^2 - 12\,c_0}{x^2} + \frac{-2\,c_{-1}\,c_0 - 12\,c_1}{x} - 2\,c_1\,c_{-1} - c_0^2 - 10\,c_2 - c^2 + (-6\,c_3 - 2\,c - 2\,c_2\,c_{-1} - 2\,c_1\,c_0)\,x + (-1 - 2\,c_2\,c_0 - c_1^2 - 2\,c_3\,c_{-1})\,x^2 + (8\,c_5 - 2\,c_2\,c_1 - 2\,c_3\,c_0 - 2\,c_{-1}\,c_4)\,x^3 + (-2\,c_3\,c_1 - c_2^2 - 2\,c_4\,c_0 - 2\,c_5\,c_{-1} + 18\,c_6)\,x^4 + (-2\,c_2\,c_3 - 2\,c_1\,c_4 - 2\,c_5\,c_0 - 2\,c_{-1}\,c_6 + 30\,c_7)\,x^5 + (44\,c_8 - 2\,c_5\,c_1 - c_3^2 - 2\,c_{-1}\,c_7 - 2\,c_6\,c_0 - 2\,c_2\,c_4)\,x^6 + (-2\,c_2\,c_5 - 2\,c_3\,c_4 - 2\,c_0\,c_7 - 2\,c_1\,c_6 - 2\,c_{-1}\,c_8)\,x^7 + (-2\,c_5\,c_3 - 2\,c_7\,c_1 - 2\,c_0\,c_8 - 2\,c_2\,c_6 - c_4^2)\,x^8 + (-2\,c_3\,c_6 - 2\,c_7\,c_2 - 2\,c_8\,c_1 - 2\,c_4\,c_5)\,x^9 \tag{9}$$

The coefficient of the lowest power, (−3) must vanish.

```
> subs(c[-1]=0,%);
```

$$-\frac{12\,c_0}{x^2} - \frac{12\,c_1}{x} - c_0^2 - 10\,c_2 - c^2 + (-6\,c_3 - 2\,c - 2\,c_1\,c_0)\,x + (-1 - 2\,c_2\,c_0 - c_1^2)\,x^2 + (8\,c_5 - 2\,c_2\,c_1 - 2\,c_3\,c_0)\,x^3 + (-2\,c_3\,c_1 - c_2^2 - 2\,c_4\,c_0 + 18\,c_6)\,x^4 + (-2\,c_2\,c_3 - 2\,c_1\,c_4 - 2\,c_5\,c_0 + 30\,c_7)\,x^5 + (44\,c_8 - 2\,c_5\,c_1 - c_3^2 - 2\,c_6\,c_0 - 2\,c_2\,c_4)\,x^6 + (-2\,c_2\,c_5 - 2\,c_3\,c_4 - 2\,c_0\,c_7 - 2\,c_1\,c_6)\,x^7 + (-2\,c_5\,c_3 - 2\,c_7\,c_1 - 2\,c_0\,c_8 - 2\,c_2\,c_6 - c_4^2)\,x^8 + (-2\,c_3\,c_6 - 2\,c_7\,c_2 - 2\,c_8\,c_1 - 2\,c_4\,c_5)\,x^9 \tag{10}$$

```
> subs(c[0]=0,%);
```

$$-\frac{12\,c_1}{x} - 10\,c_2 - c^2 + (-6\,c_3 - 2\,c)\,x + (-1 - c_1^2)\,x^2 + (8\,c_5 - 2\,c_2\,c_1)\,x^3 + (-2\,c_3\,c_1 - c_2^2 + 18\,c_6)\,x^4 + (-2\,c_2\,c_3 - 2\,c_1\,c_4 + 30\,c_7)\,x^5 + (44\,c_8 - 2\,c_5\,c_1 - c_3^2 - 2\,c_2\,c_4)\,x^6 + (-2\,c_2\,c_5 - 2\,c_3\,c_4 - 2\,c_1\,c_6)\,x^7 + (-2\,c_5\,c_3 - 2\,c_7\,c_1 - 2\,c_2\,c_6 - c_4^2)\,x^8 + (-2\,c_3\,c_6 - 2\,c_7\,c_2 - 2\,c_8\,c_1 - 2\,c_4\,c_5)\,x^9 \tag{11}$$

```
> subs(c[1]=0,%);
```

$$-10\,c_2 - c^2 + (-6\,c_3 - 2\,c)\,x - x^2 + 8\,c_5\,x^3 + (-c_2^2 + 18\,c_6)\,x^4 + (-2\,c_2\,c_3 + 30\,c_7)\,x^5 + (44\,c_8 - c_3^2 - 2\,c_2\,c_4)\,x^6 + (-2\,c_2\,c_5 - 2\,c_3\,c_4)\,x^7 + (-2\,c_5\,c_3 - 2\,c_2\,c_6 - c_4^2)\,x^8 + (-2\,c_3\,c_6 - 2\,c_7\,c_2 - 2\,c_4\,c_5)\,x^9 \tag{12}$$

```
> sr4:=%:coeff(%,x,0);
```

$$-10\,c_2 - c^2 \tag{13}$$

```
> solve(%,c[2]);
```

$$-\frac{1}{10}\,c^2 \tag{14}$$

```
> subs(c[2]=%,sr4);
```

$$(-6\,c_3 - 2\,c)\,x - x^2 + 8\,c_5\,x^3 + \left(-\frac{1}{100}\,c^4 + 18\,c_6\right)x^4 + \left(\frac{1}{5}\,c^2\,c_3 + 30\,c_7\right)x^5 + \left(44\,c_8 - c_3^2\right. \tag{15}$$

$$+\frac{1}{5}c^2c_4\Big)x^6+\Big(\frac{1}{5}c^2c_5-2c_3c_4\Big)x^7+\Big(-2c_5c_3+\frac{1}{5}c^2c_6-c_4^2\Big)x^8+\Big(-2c_3c_6+\frac{1}{5}c_7c^2-2c_4c_5\Big)x^9$$

```
> sr5:=%;
```

$$sr5:=(-6c_3-2c)x-x^2+8c_5x^3+\Big(-\frac{1}{100}c^4+18c_6\Big)x^4+\Big(\frac{1}{5}c^2c_3+30c_7\Big)x^5+\Big(44c_8-c_3^2+\frac{1}{5}c^2c_4\Big)x^6+\Big(\frac{1}{5}c^2c_5-2c_3c_4\Big)x^7+\Big(-2c_5c_3+\frac{1}{5}c^2c_6-c_4^2\Big)x^8+\Big(-2c_3c_6+\frac{1}{5}c_7c^2-2c_4c_5\Big)x^9 \quad (16)$$

So far so good.

```
> coeff(sr5,x,1);
```

$$-6c_3-2c \quad (17)$$

```
> solve(%,c[3]);
```

$$-\frac{1}{3}c \quad (18)$$

```
> subs(c[3]=%,sr5):%;
```

$$-x^2+8c_5x^3+\Big(-\frac{1}{100}c^4+18c_6\Big)x^4+\Big(-\frac{1}{15}c^3+30c_7\Big)x^5+\Big(44c_8-\frac{1}{9}c^2+\frac{1}{5}c^2c_4\Big)x^6+\Big(\frac{1}{5}c^2c_5+\frac{2}{3}cc_4\Big)x^7+\Big(\frac{2}{3}c_5c+\frac{1}{5}c^2c_6-c_4^2\Big)x^8+\Big(\frac{2}{3}cc_6+\frac{1}{5}c_7c^2-2c_4c_5\Big)x^9 \quad (19)$$

```
> sr6:=%;
```

$$sr6:=-x^2+8c_5x^3+\Big(-\frac{1}{100}c^4+18c_6\Big)x^4+\Big(-\frac{1}{15}c^3+30c_7\Big)x^5+\Big(44c_8-\frac{1}{9}c^2+\frac{1}{5}c^2c_4\Big)x^6+\Big(\frac{1}{5}c^2c_5+\frac{2}{3}cc_4\Big)x^7+\Big(\frac{2}{3}c_5c+\frac{1}{5}c^2c_6-c_4^2\Big)x^8+\Big(\frac{2}{3}cc_6+\frac{1}{5}c_7c^2-2c_4c_5\Big)x^9 \quad (20)$$

```
> coeff(%,x,2);
```

$$-1 \quad (21)$$

We cannot cancel this coefficient, we have no free constant. The power series does not solve the equation.

Is there no expansion at this singular point? There must be one. We return to the equation.

```
> d4;
```

$$\frac{d^2}{dx^2}d(x)-\frac{12\,d(x)}{x^2}-d(x)^2-x^2-2xc-c^2 \quad (22)$$

We see that d/x^2>>d^2 since d<<1/x^2;

But d'' can b of the same order of magnitude as d/x^2; we keep them both on the left side, move the rest on the right side,
as we do in any contraction mapping method;

```
> dsolve(diff(d(x), x, x)-12*d(x)/x^2=R(x));
```

$$d(x) = _C2\, x^4 + \frac{_C1}{x^3} + \frac{1}{7}\,\frac{-\left(\int x^4 R(x)\,dx\right) + \left(\int \frac{R(x)}{x^3}\,dx\right) x^7}{x^3} \tag{23}$$

Here, R(x)=$d(x)^2 + x^2 + 2\,x\,c + c^2$;

```
> subs( R(x)=-(-d(x)^2-x^2-2*x*c-c^2),%);
> %;
```

$$d(x) = _C2\, x^4 + \frac{_C1}{x^3} + \frac{1}{7}\,\frac{-\left(\int x^4 \left(d(x)^2 + x^2 + 2\,x\,c + c^2\right) dx\right) + \left(\int \frac{d(x)^2 + x^2 + 2\,x\,c + c^2}{x^3}\,dx\right) x^7}{x^3}$$

$$d(x) = _C2\, x^4 + \frac{_C1}{x^3} + \frac{1}{7}\,\frac{-\left(\int x^4 \left(d(x)^2 + x^2 + 2\,x\,c + c^2\right) dx\right) + \left(\int \frac{d(x)^2 + x^2 + 2\,x\,c + c^2}{x^3}\,dx\right) x^7}{x^3} \tag{24}$$

```
> expand(%,int);
```

$$d(x) = _C2\, x^4 + \frac{_C1}{x^3} - \frac{1}{7}\,\frac{\int x^4 d(x)^2\,dx}{x^3} - \frac{1}{7}\,\frac{\int x^6\,dx}{x^3} - \frac{2}{7}\,\frac{c\left(\int x^5\,dx\right)}{x^3} - \frac{1}{7}\,\frac{c^2\left(\int x^4\,dx\right)}{x^3} + \frac{1}{7}\,x^4\left(\int \frac{d(x)^2}{x^3}\,dx\right) + \frac{1}{7}\,x^4\left(\int \frac{1}{x}\,dx\right) + \frac{2}{7}\,x^4 c\left(\int \frac{1}{x^2}\,dx\right) + \frac{1}{7}\,x^4 c^2\left(\int \frac{1}{x^3}\,dx\right) \tag{25}$$

```
> subs(Int=int,%);
```

$$d(x) = _C2\, x^4 + \frac{_C1}{x^3} - \frac{1}{7}\,\frac{\int x^4 d(x)^2\,dx}{x^3} - \frac{1}{7}\,\frac{\int x^6\,dx}{x^3} - \frac{2}{7}\,\frac{c\left(\int x^5\,dx\right)}{x^3} - \frac{1}{7}\,\frac{c^2\left(\int x^4\,dx\right)}{x^3} + \frac{1}{7}\,x^4\left(\int \frac{d(x)^2}{x^3}\,dx\right) + \frac{1}{7}\,x^4\left(\int \frac{1}{x}\,dx\right) + \frac{2}{7}\,x^4 c\left(\int \frac{1}{x^2}\,dx\right) + \frac{1}{7}\,x^4 c^2\left(\int \frac{1}{x^3}\,dx\right) \tag{26}$$

Some manipulations...

```
> eqd2:=%;
```

$$eqd2 := d(x) = _C2\, x^4 + \frac{_C1}{x^3} - \frac{1}{7}\,\frac{\int x^4\, d(x)^2\, dx}{x^3} - \frac{1}{49}\, x^4 - \frac{1}{3}\, x^3\, c - \frac{1}{10}\, x^2\, c^2 + \frac{1}{7}\, x^4 \left(\int \frac{d(x)^2}{x^3}\, dx \right) + \frac{1}{7}\, x^4 \ln(x) \tag{27}$$

We have decided d is smaller. So, 1/x^3 has no room here. _C1 must be zero.

```
> e2:=subs(_C1=0,eqd2);
```

$$e2 := d(x) = _C2\, x^4 - \frac{1}{7}\,\frac{\int x^4\, d(x)^2\, dx}{x^3} - \frac{1}{49}\, x^4 - \frac{1}{3}\, x^3\, c - \frac{1}{10}\, x^2\, c^2 + \frac{1}{7}\, x^4 \left(\int \frac{d(x)^2}{x^3}\, dx \right) + \frac{1}{7}\, x^4 \ln(x) \tag{28}$$

Now it looks OK.

Note that we have decided that d^2 is a "small" term.

So we iterate in the usual way, starting with d=0, etc.

Take the rhs of the equation first.

```
> rs:=rhs(e2);
```

$$rs := _C2\, x^4 - \frac{1}{7}\,\frac{\int x^4\, d(x)^2\, dx}{x^3} - \frac{1}{49}\, x^4 - \frac{1}{3}\, x^3\, c - \frac{1}{10}\, x^2\, c^2 + \frac{1}{7}\, x^4 \left(\int \frac{d(x)^2}{x^3}\, dx \right) + \frac{1}{7}\, x^4 \ln(x) \tag{29}$$

```
> subs(d=0,%);
```

$$_C2\, x^4 - \frac{1}{7}\,\frac{\int x^4\, 0(x)^2\, dx}{x^3} - \frac{1}{49}\, x^4 - \frac{1}{3}\, x^3\, c - \frac{1}{10}\, x^2\, c^2 + \frac{1}{7}\, x^4 \left(\int \frac{0(x)^2}{x^3}\, dx \right) + \frac{1}{7}\, x^4 \ln(x) \tag{30}$$

```
> %;
```

$$_C2\, x^4 - \frac{1}{49}\, x^4 - \frac{1}{3}\, x^3\, c - \frac{1}{10}\, x^2\, c^2 + \frac{1}{7}\, x^4 \ln(x) \tag{31}$$

We now use this as the new d;

```
> subs(d=unapply(%,x),rs);
```

(32)

$$_C2\,x^4 - \frac{1}{7}\,\frac{\int x^4 \left(x \to _C2\,x^4 - \frac{1}{49}\,x^4 - \frac{1}{3}\,x^3\,c - \frac{1}{10}\,x^2\,c^2 + \frac{1}{7}\,x^4 \ln(x)\right)(x)^2\,dx}{x^3} - \frac{1}{49}\,x^4 \tag{32}$$
$$- \frac{1}{3}\,x^3\,c - \frac{1}{10}\,x^2\,c^2 + \frac{1}{7}\,x^4 \left(\int \frac{\left(x \to _C2\,x^4 - \frac{1}{49}\,x^4 - \frac{1}{3}\,x^3\,c - \frac{1}{10}\,x^2\,c^2 + \frac{1}{7}\,x^4 \ln(x)\right)(x)^2}{x^3}\,dx \right) + \frac{1}{7}\,x^4 \ln(x)$$

```
> %;
```

$$_C2\,x^4 - \frac{1}{7}\,\frac{1}{x^3}\left(-\frac{1}{18}\,_C2\,c\,x^{12} - \frac{1}{55}\,_C2\,c^2\,x^{11} + \frac{2}{91}\,_C2\,x^{13}\ln(x) - \frac{40}{8281}\,_C2\,x^{13}\right. \tag{33}$$
$$- \frac{1}{126}\,c\,x^{12}\ln(x) + \frac{19}{10584}\,c\,x^{12} - \frac{1}{385}\,c^2\,x^{11}\ln(x) + \frac{2857}{266805}\,c^2\,x^{11}$$
$$- \frac{40}{57967}\,x^{13}\ln(x) + \frac{449}{5274997}\,x^{13} + \frac{1}{13}\,_C2^2\,x^{13} + \frac{1}{150}\,c^3\,x^{10} + \frac{1}{637}\,x^{13}\ln(x)^2$$
$$\left. + \frac{1}{900}\,c^4\,x^9\right) - \frac{1}{49}\,x^4 - \frac{1}{3}\,x^3\,c - \frac{1}{10}\,x^2\,c^2 + \frac{1}{7}\,x^4\left(-\frac{2}{15}\,_C2\,c\,x^5 - \frac{1}{20}\,_C2\,c^2\,x^4\right.$$
$$+ \frac{1}{21}\,_C2\,x^6\ln(x) - \frac{13}{882}\,_C2\,x^6 - \frac{2}{105}\,c\,x^5\ln(x) + \frac{8}{1225}\,c\,x^5 - \frac{1}{140}\,c^2\,x^4\ln(x)$$
$$+ \frac{1079}{35280}\,c^2\,x^4 - \frac{13}{6174}\,x^6\ln(x) + \frac{109}{259308}\,x^6 + \frac{1}{6}\,_C2^2\,x^6 + \frac{1}{45}\,c^3\,x^3$$
$$\left. + \frac{1}{294}\,x^6\ln(x)^2 + \frac{1}{200}\,c^4\,x^2\right) + \frac{1}{7}\,x^4\ln(x)$$

Too many terms. Most are irrelevant, since we have not gotten this level of accuracy.

```
> series(%,x,6):convert(%,polynom);
```

$$-\frac{1}{10}\,x^2\,c^2 - \frac{1}{3}\,x^3\,c + \left(-\frac{1}{49} + _C2 + \frac{1}{7}\ln(x)\right)x^4 \tag{34}$$

And repeat;

```
> subs(d=unapply(%,x),rs);
```

$$_C2\,x^4 - \frac{1}{7}\,\frac{\int x^4\left(x \to -\frac{1}{10}\,x^2\,c^2 - \frac{1}{3}\,x^3\,c + \left(-\frac{1}{49} + _C2 + \frac{1}{7}\ln(x)\right)x^4\right)(x)^2\,dx}{x^3} \tag{35}$$
$$- \frac{1}{49}\,x^4 - \frac{1}{3}\,x^3\,c - \frac{1}{10}\,x^2\,c^2 + \frac{1}{7}\,x^4\left(\right.$$

$$\int \frac{\left(x \to -\frac{1}{10}\,x^2 c^2 - \frac{1}{3}\,x^3 c + \left(-\frac{1}{49} + _C2 + \frac{1}{7}\ln(x)\right) x^4\right)(x)^2}{x^3}\,dx + \frac{1}{7}\,x^4 \ln(x)$$

```
> %:series(%,x,8):convert(%,polynom);
```

$$-\frac{1}{10}\,x^2 c^2 - \frac{1}{3}\,x^3 c + \left(-\frac{1}{49} + _C2 + \frac{1}{7}\ln(x)\right) x^4 \tag{36}$$

Stable, to x^4. We can move to higher orders.

```
> subs(d=unapply(%,x),rs):series(%,x,12):convert(%,polynom);
```

$$-\frac{1}{10}\,x^2 c^2 - \frac{1}{3}\,x^3 c + \left(-\frac{1}{49} + _C2 + \frac{1}{7}\ln(x)\right) x^4 + \frac{1}{1800}\,c^4 x^6 + \frac{1}{450}\,c^3 x^7 + \left(-\frac{1}{220}\,_C2\,c^2 - \frac{1}{1540}\,c^2 \ln(x) + \frac{12121}{4268880}\,c^2\right) x^8 \tag{37}$$

Repeat until stable

```
> subs(d=unapply(%,x),rs):series(%,x,12):convert(%,polynom);
```

$$-\frac{1}{10}\,x^2 c^2 - \frac{1}{3}\,x^3 c + \left(-\frac{1}{49} + _C2 + \frac{1}{7}\ln(x)\right) x^4 + \frac{1}{1800}\,c^4 x^6 + \frac{1}{450}\,c^3 x^7 + \left(-\frac{1}{220}\,_C2\,c^2 - \frac{1}{1540}\,c^2 \ln(x) + \frac{12121}{4268880}\,c^2\right) x^8 \tag{38}$$

Stable, to x^8. We can move to higher orders.

```
> subs(d=unapply(%,x),rs):series(%,x,16):convert(%,polynom);
```

$$-\frac{1}{10}\,x^2 c^2 - \frac{1}{3}\,x^3 c + \left(-\frac{1}{49} + _C2 + \frac{1}{7}\ln(x)\right) x^4 + \frac{1}{1800}\,c^4 x^6 + \frac{1}{450}\,c^3 x^7 + \left(-\frac{1}{220}\,_C2\,c^2 - \frac{1}{1540}\,c^2 \ln(x) + \frac{12121}{4268880}\,c^2\right) x^8 + \left(-\frac{1}{90}\,_C2\,c + \frac{179}{264600}\,c - \frac{1}{630}\,c\ln(x)\right) x^9 + \left(\frac{1}{273}\,_C2 \ln(x) + \frac{1}{3822}\ln(x)^2 + \frac{1}{78}\,_C2^2 + \frac{27283}{569699676} - \frac{211}{1043406}\ln(x) - \frac{211}{149058}\,_C2 - \frac{1}{702000}\,c^6\right) x^{10} - \frac{11}{1323000}\,c^5 x^{11} + \left(\frac{1}{59400}\,c^4\,_C2 - \frac{45407}{2561328000}\,c^4 + \frac{1}{415800}\,c^4 \ln(x)\right) x^{12} \tag{39}$$

```
> subs(d=unapply(%,x),rs):series(%,x,16):convert(%,polynom);
```

$$-\frac{1}{10}\,x^2 c^2 - \frac{1}{3}\,x^3 c + \left(-\frac{1}{49} + _C2 + \frac{1}{7}\ln(x)\right) x^4 + \frac{1}{1800}\,c^4 x^6 + \frac{1}{450}\,c^3 x^7 + \left(-\frac{1}{220}\,_C2\,c^2 - \frac{1}{1540}\,c^2 \ln(x) + \frac{12121}{4268880}\,c^2\right) x^8 + \left(-\frac{1}{90}\,_C2\,c + \frac{179}{264600}\,c - \frac{1}{630}\,c\ln(x)\right) x^9 + \left(\frac{1}{273}\,_C2 \ln(x) + \frac{1}{3822}\ln(x)^2 + \frac{1}{78}\,_C2^2 + \frac{27283}{569699676} - \frac{211}{1043406}\ln(x) - \frac{211}{149058}\,_C2 - \frac{1}{702000}\,c^6\right) x^{10} - \frac{11}{1323000}\,c^5 x^{11} + \left(\frac{1}{59400}\,c^4\,_C2 - \frac{45407}{2561328000}\,c^4 + \frac{1}{415800}\,c^4 \ln(x)\right) x^{12} \tag{40}$$

Works well. It stablizes quickly.

Note that the expansion contains $x^4(\ln(x))$, $x^8(\ln(x))$, **and** $x^{10}(\ln(x))^2$,

It is series in x and x ln x; It converges. How do you prove that?