```
> diff(y(x),x,x)-y(x)^2-x=0;
```

$$\frac{d^2}{dx^2}\,y(x)-y(x)^2-x=0 \tag{1}$$

This is the Painleve P1 equation.
We look for a singularity, say at x=c. It is more practical to work with series at zero, thus I take x=s+c. I am lazy, and write x=x+c.

```
> dp:=diff(y(x),x,x)-y(x)^2-x-c=0;
```

$$dp := \frac{d^2}{dx^2}\,y(x)-y(x)^2-x-c=0 \tag{2}$$

```
> int(1,x=0..infinity);
```

$$\infty \tag{3}$$

If y–> ∞, **then** $y(x)^2 >> x$;
Thus the balance is between y'' and y^2. We try y=Ax^p;

```
> subs(y(x)=A*x^p,lhs(dp)+x+c);
```

$$\frac{\partial^2}{\partial x^2}\,(A\,x^p)-A^2\,(x^p)^2 \tag{4}$$

```
> expand(%/x^p);
```

$$\frac{A\,p^2}{x^2}-\frac{A\,p}{x^2}-x^p\,A^2 \tag{5}$$

We see that p=–2

```
> subs(p=-2,%);
```

$$\frac{6\,A}{x^2}-\frac{A^2}{x^2} \tag{6}$$

```
> solve(%,A);
```

$$0,\,6 \tag{7}$$

We don't care about A=0, we are looking for singular solutions.

What comes next? We substitute $y(x)=\frac{6}{x^2}+d(x)$ **in** *the equation.*

```
> subs(y(x)=6/x^2+d(x),dp):
> d3:=expand(%);
```

$$d3 := \frac{d^2}{dx^2}\,d(x)-\frac{12\,d(x)}{x^2}-d(x)^2-x-c=0 \tag{8}$$

$d(x)^2 << \frac{d(x)}{x^2}$ *since* $d(x) << \frac{1}{x^2}$;

Thus, we place $-\left(-d(x)^2-x-c\right)$ on the rhs and solve as though it was known.

```
> diff(d(x), x, x)-12*d(x)/x^2=u(x);
```

$$\frac{d^2}{dx^2} d(x) - \frac{12\, d(x)}{x^2} = u(x) \tag{9}$$

```
> dsolve(%);
```

$$d(x) = \frac{_C2}{x^3} + x^4 _C1 + \frac{1}{7} \frac{-\left(\int x^4 u(x)\, dx\right) + \left(\int \frac{u(x)}{x^3}\, dx\right) x^7}{x^3} \tag{10}$$

```
> subs(u(x)=d(x)^2+x+c,%);
```

$$d(x) = \frac{_C2}{x^3} + x^4 _C1 + \frac{1}{7} \frac{-\left(\int x^4 \left(d(x)^2 + x + c\right) dx\right) + \left(\int \frac{d(x)^2 + x + c}{x^3}\, dx\right) x^7}{x^3} \tag{11}$$

```
> expand(%);
```

$$d(x) = \frac{_C2}{x^3} + x^4 _C1 - \frac{1}{7} \frac{\int x^4 d(x)^2\, dx}{x^3} - \frac{1}{7} \frac{\int x^5\, dx}{x^3} - \frac{1}{7} \frac{c \left(\int x^4\, dx\right)}{x^3} + \frac{1}{7} x^4 \left(\int \frac{d(x)^2}{x^3}\, dx\right) + \frac{1}{7} x^4 \left(\int \frac{1}{x^2}\, dx\right) + \frac{1}{7} x^4 c \left(\int \frac{1}{x^3}\, dx\right) \tag{12}$$

```
> di:=subs(Int=int,%);
```

$$di := d(x) = \frac{_C2}{x^3} + x^4 _C1 - \frac{1}{7} \frac{\int x^4 d(x)^2\, dx}{x^3} - \frac{1}{7} \frac{\int x^5\, dx}{x^3} - \frac{1}{7} \frac{c \left(\int x^4\, dx\right)}{x^3} + \frac{1}{7} x^4 \left(\int \frac{d(x)^2}{x^3}\, dx\right) + \frac{1}{7} x^4 \left(\int \frac{1}{x^2}\, dx\right) + \frac{1}{7} x^4 c \left(\int \frac{1}{x^3}\, dx\right) \tag{13}$$

```
> %;
```

$$d(x) = \frac{_C2}{x^3} + x^4 _C1 - \frac{1}{7} \frac{\int x^4 d(x)^2\, dx}{x^3} - \frac{1}{6} x^3 - \frac{1}{10} x^2 c + \frac{1}{7} x^4 \left(\int \frac{d(x)^2}{x^3}\, dx\right) \tag{14}$$

Note that 1/x^3 is not an acceptable behavior. d(x)<<x^(−2). So _C2 must be zero.

```
> di2:=subs(_C2=0,%);
```

$$di2 := d(x) = x^4 _C1 - \frac{1}{7} \frac{\int x^4 d(x)^2\, dx}{x^3} - \frac{1}{6} x^3 - \frac{1}{10} x^2 c + \frac{1}{7} x^4 \left(\int \frac{d(x)^2}{x^3}\, dx\right) \tag{15}$$

Now iterate. Start with discarding d altogether. It is supposed to be dominated by other terms. We extract the rhs of the equation and iterate accordingly.

```
> rr:=rhs(di2);
```

$$rr := x^4 _C1 - \frac{1}{7} \frac{\int x^4 d(x)^2\, dx}{x^3} - \frac{1}{6} x^3 - \frac{1}{10} x^2 c + \frac{1}{7} x^4 \left(\int \frac{d(x)^2}{x^3}\, dx\right) \tag{16}$$

```
> subs(d=0,rr):expand(%);
```

$$x^4 _C1 - \frac{1}{6} x^3 - \frac{1}{10} x^2 c \tag{17}$$

Take now this as d, and insert it.

```
> subs(d=unapply(%,x),rr):expand(%);
```

$$x^4 _C1 + \frac{1}{78} x^{10} _C1^2 - \frac{1}{180} x^9 _C1 - \frac{1}{220} x^8 c _C1 + \frac{1}{1584} x^8 + \frac{1}{900} x^7 c + \frac{1}{1800} x^6 c^2 - \frac{1}{6} x^3 - \frac{1}{10} x^2 c \quad (18)$$

We don't have yet such a precision, after just one iteration. Take a power series, say with 6 terms and repeat
until those first terms don't change anymore. Then we can increase the precision.

```
> series(%,x,6):convert(%,polynom):subs(d=unapply(%,x),rr):series
  (%,x,6):convert(%,polynom);
```

$$x^4 _C1 - \frac{1}{6} x^3 - \frac{1}{10} x^2 c \quad (19)$$

```
> subs(d=unapply(%,x),rr):series(%,x,6):convert(%,polynom);
```

$$x^4 _C1 - \frac{1}{6} x^3 - \frac{1}{10} x^2 c \quad (20)$$

Ok, stable already, we can increase the precision.

```
> subs(d=unapply(%,x),rr):series(%,x,12):convert(%,polynom);
```

$$-\frac{1}{10} x^2 c - \frac{1}{6} x^3 + x^4 _C1 + \frac{1}{1800} c^2 x^6 + \frac{1}{900} c x^7 + \left(-\frac{1}{220} c _C1 + \frac{1}{1584} \right) x^8 \quad (21)$$

```
> subs(d=unapply(%,x),rr):series(%,x,12):convert(%,polynom);
```

$$-\frac{1}{10} x^2 c - \frac{1}{6} x^3 + x^4 _C1 + \frac{1}{1800} c^2 x^6 + \frac{1}{900} c x^7 + \left(-\frac{1}{220} c _C1 + \frac{1}{1584} \right) x^8 \quad (22)$$

We can increase again the precision.

```
> subs(d=unapply(%,x),rr):series(%,x,16):convert(%,polynom);
```

$$-\frac{1}{10} x^2 c - \frac{1}{6} x^3 + x^4 _C1 + \frac{1}{1800} c^2 x^6 + \frac{1}{900} c x^7 + \left(-\frac{1}{220} c _C1 + \frac{1}{1584} \right) x^8 - \frac{1}{180} _C1 x^9 + \left(-\frac{1}{702000} c^3 + \frac{1}{78} _C1^2 \right) x^{10} - \frac{11}{2646000} c^2 x^{11} + \left(-\frac{1}{324000} c + \frac{1}{525} \left(-\frac{1}{220} c _C1 + \frac{1}{1584} \right) c + \frac{1}{108000} _C1 c^2 + \frac{1}{11088000} (180 c _C1 - 25) c \right) x^{12} \quad (23)$$

```
> subs(d=unapply(%,x),rr):series(%,x,16):convert(%,polynom);
```

$$-\frac{1}{10} x^2 c - \frac{1}{6} x^3 + x^4 _C1 + \frac{1}{1800} c^2 x^6 + \frac{1}{900} c x^7 + \left(-\frac{1}{220} c _C1 + \frac{1}{1584} \right) x^8 - \frac{1}{180} _C1 x^9 + \left(-\frac{1}{702000} c^3 + \frac{1}{78} _C1^2 \right) x^{10} - \frac{11}{2646000} c^2 x^{11} + \left(-\frac{1}{324000} c + \frac{1}{525} \left(-\frac{1}{220} c _C1 + \frac{1}{1584} \right) c + \frac{1}{108000} _C1 c^2 - \frac{1}{2542700160000} (5733000 - 41277600 c _C1) c \right) x^{12} \quad (24)$$

```
> subs(d=unapply(%,x),rr):series(%,x,16):convert(%,polynom);
```

$$-\frac{1}{10}x^2c-\frac{1}{6}x^3+x^4_C1+\frac{1}{1800}c^2x^6+\frac{1}{900}cx^7+\left(-\frac{1}{220}c_C1+\frac{1}{1584}\right)x^8 \tag{25}$$
$$-\frac{1}{180}_C1\,x^9+\left(-\frac{1}{702000}c^3+\frac{1}{78}_C1^2\right)x^{10}-\frac{11}{2646000}c^2x^{11}+\left(-\frac{1}{324000}c$$
$$+\frac{1}{525}\left(-\frac{1}{220}c_C1+\frac{1}{1584}\right)c+\frac{1}{108000}_C1\,c^2-\frac{1}{2542700160000}(5733000$$
$$-41277600\,c\,_C1)\,c\Bigg)x^{12}$$

Stable already. But we get bored.

Now we have to prove all this rigorously.

```
> subs(d=(X->X^2*H(X)),di2);
```

$$(X\to X^2H(X))(x)=x^4_C2-\frac{1}{7}\frac{\int x^4\,(X\to X^2H(X))(x)^2\,dx}{x^3}-\frac{1}{6}x^3-\frac{1}{10}x^2c+\frac{1}{7}x^4\left(\int\frac{(X\to X^2H(X))(x)^2}{x^3}\,dx\right) \tag{26}$$

```
> %;
```

$$x^2H(x)=x^4_C2-\frac{1}{7}\frac{\int x^8H(x)^2\,dx}{x^3}-\frac{1}{6}x^3-\frac{1}{10}x^2c+\frac{1}{7}x^4\left(\int xH(x)^2\,dx\right) \tag{27}$$

```
> expand(%/x^2);
```

$$H(x)=x^2_C2-\frac{1}{7}\frac{\int x^8H(x)^2\,dx}{x^5}-\frac{1}{6}x-\frac{1}{10}c+\frac{1}{7}x^2\left(\int xH(x)^2\,dx\right) \tag{28}$$

```
> H1:=%;
```

$$H1:=H(x)=x^2_C2-\frac{1}{7}\frac{\int x^8H(x)^2\,dx}{x^5}-\frac{1}{6}x-\frac{1}{10}c+\frac{1}{7}x^2\left(\int xH(x)^2\,dx\right) \tag{29}$$

```
> r1:=rhs(%);
```

$$r1:=x^2_C2-\frac{1}{7}\frac{\int x^8H(x)^2\,dx}{x^5}-\frac{1}{6}x-\frac{1}{10}c+\frac{1}{7}x^2\left(\int xH(x)^2\,dx\right) \tag{30}$$

```
> subs(H(x)=0,r1):expand(%);
```

$$x^2_C2-\frac{1}{6}x-\frac{1}{10}c \tag{31}$$

```
> subs(int=Int,%):expand(%,int):subs(Int=int,%);
```

$$x^2_C2-\frac{1}{6}x-\frac{1}{10}c \tag{32}$$

```
> subs(H=unapply(%,x),r1):sr(%,x,12);
```

$$-\frac{1}{10}c-\frac{1}{6}x+x^2_C2+\frac{1}{1800}x^4c^2+\frac{1}{900}x^5c+\left(-\frac{1}{220}c_C2+\frac{1}{1584}\right)x^6 \quad (33)$$

```
> subs(H=unapply(%,x),r1):sr(%,x,12);
```

$$-\frac{1}{10}c-\frac{1}{6}x+x^2_C2+\frac{1}{1800}x^4c^2+\frac{1}{900}x^5c+\left(-\frac{1}{220}c_C2+\frac{1}{1584}\right)x^6 \quad (34)$$

```
> diff(y(x), x, x)-y(x)^2-x^2;
```

$$\frac{d^2}{dx^2}y(x)-y(x)^2-x^2 \quad (35)$$

```
> subs(x^2=(x+c)^2,%);
```

$$\frac{d^2}{dx^2}y(x)-y(x)^2-(x+c)^2 \quad (36)$$

```
> expand(%);
```

$$\frac{d^2}{dx^2}y(x)-y(x)^2-x^2-2xc-c^2 \quad (37)$$

```
> d2:=%;
```

$$d2:=\frac{d^2}{dx^2}y(x)-y(x)^2-x^2-2xc-c^2 \quad (38)$$