Mathematics of the first year program.

In 1979, Kohn [Koh] introduced a fundamental idea while studying certain estimates on partial differential equations arising in several complex variables.

Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary, $b\Omega$. Consider the Cauchy-Riemann complex

$$L_2^{p,0}(\Omega) \xrightarrow{\bar\partial} L_2^{p,1}(\Omega) \xrightarrow{\bar\partial} L_2^{p,2}(\Omega) \xrightarrow{\bar\partial} \dots,$$

where $L_2^{p,q}(\Omega)$ denotes the square-integrable (p,q)-forms and $\bar\partial$ denotes the Cauchy-Riemann operators, extended to act on L^2 in the natural way. One of the most basic problems in complex analysis is to solve the equation

$$\bar\partial v = \alpha, \tag{1}$$

with estimates, of various kinds, on v in terms of α. The equation (1) is over-determined, i.e., solving (1) requires that $\bar\partial\alpha = 0$ and $\operatorname{Ker}(\bar\partial)$ is not trivial, or even finite dimensional. One way to deal with this difficulty is to pass to the Laplacian associated to this complex. The adjoint operator, $\bar\partial^*$, is defined by the $L_2^{p,q}(\Omega)$ structure, and this then allows one to consider the complex Laplacian $\Box = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$ at each form level; the equation $\Box u = \alpha$ has the huge advantage of being determined. However, $\Box$ is an unbounded operator on $L_2^{p,q}$ and there are boundary conditions which must hold for $u \in \operatorname{Dom}(\Box)$. The *$\bar\partial$-Neumann problem* is the following boundary-value problem: given $\alpha \in L_2^{p,q}(\Omega)$ find $u \in L_2^{p,q}(\Omega)$ such that

$$\begin{cases} \Box u = f \\ u \in \operatorname{Dom}(\Box). \end{cases} \tag{2}$$

When (2) can be solved, (1) can be solved whenever $\bar\partial\alpha = 0$ by setting $v_0 = \bar\partial^*\left(\Box^{-1}\alpha\right)$. In fact, this solution is special: amongst all solutions v to (1), v_0 has the smallest L^2 norm.

We emphasize that condition $u \in \operatorname{Dom}(\Box)$ in (2) is an explicit set of boundary conditions on the components of u and its first derivatives. These boundary conditions, however, are not *coercive* and therefore (2) is not solvable by standard elliptic methods. This makes the $\bar\partial$-Neumann problem highly interesting as a PDE problem but, more importantly for the story at hand, brings geometry into the picture — (2) is not always solvable; the geometry of $b\Omega$ controls the solvability and regularity of he solutions to (2).

Since (2) is not coercive, elliptic estimates on it do not hold. A *subelliptic estimate* may hold, depending on the geometry of $b\Omega$, and such an estimate turns out to imply many elliptic-like properties of (2), including solvability and regularity. A subelliptic estimate of order $\epsilon > 0$ holds if

$$\|u\|_\epsilon^2 \le C\left(\left\|\bar\partial u\right\|^2 + \left\|\bar\partial^* u\right\|\right), \qquad u \in \operatorname{Dom}\left(\bar\partial^*\right), \tag{3}$$

where $\|\cdot\|_\epsilon$ denotes the Sobolev norm of order ϵ. It was estimate (3) that Kohn studied in [Koh].

As already mentioned, (3) does not always hold. Kohn had the truly broad-reaching idea of studying the functions $m \in C^\infty\left(\overline{\Omega}\right)$ such that

$$\|m \cdot u\|_\epsilon^2 \leq C\left(\left\|\bar{\partial} u\right\|^2 + \left\|\bar{\partial}^* u\right\|\right), \qquad u \in \text{Dom}\ \left(\bar{\partial}^*\right). \tag{4}$$

Call m a *subelliptic multiplier* if (4) holds, and denote the set of all subelliptic multipliers by $\mathcal{J}(\Omega)$.

By itself, this idea does not seem particularly remarkable. However, Kohn went on to note that several non-trivial functions always belong to $\mathcal{J}(\Omega)$, including the determinant of the Levi form associated to $b\Omega$. He then noted the very useful fact that $\mathcal{J}(\Omega)$ always forms an algebraic *ideal*. Most importantly, Kohn found an algorithm *which builds new multipliers from known multipliers*, and this is what brings geometry (and algebra) into play.

Kohn's algorithm involves taking derivatives of a given m, forming matrices from these derivatives, taking determinants, then taking radicals of certain, explicit sub-ideals generated by the previous steps. In this way, he constructs an increasing chain of ideals $I_1 \subset I_2 \subset \dots$, all contained in $\mathcal{J}(\Omega)$, and then formulates the following condition: say Ω is of *finite type* if there exists an N such that

$$\begin{cases} I_k = I_N, \quad k \geq N, & \text{and} \\ I_N \text{ if maximal.} \end{cases}$$

Since I_N is maximal $\iff 1 \in \mathcal{J}(\Omega)$, it holds that the estimate (3) happens if and only if Ω is of finite type. However, this reformulation allowed Kohn to prove an utterly remarkable theorem, in the case that $b\Omega$ is real-analytic; namely,

$$\Omega \text{ is of finite type} \Longleftrightarrow \begin{array}{l}\text{the order of contact of } b\Omega \text{ with} \\ \text{holomorphic curves is bounded}\end{array} \tag{5}$$

Thus, the analytic estimate (3) is completely characterized by the curvature condition on $b\Omega$ given by the right half of (5).

Inspired by Kohn's idea, Nadel [Nad] introduced *multiplier ideal sheaves* into algebraic geometry. The problem Nadel considered was how to construct a Kähler-Einstein metric on a (certain class of) Fano manifolds.[1]

In order to construct a metric which satisfies the Einstein condition, it is natural to consider the complex Monge-Ampere equation and the variation of its solutions. Prior to Nadel's work, a version of the continuity method on the Monge-Ampere equation had been developed by Siu, Tian, and Yau, [Siu4, Tia, Tia-Yau], and it was this approach to the problem that Nadel followed. Roughly, this method goes as follows: start with an *arbitrary* Kähler metric, $g_{k\bar{l}}$ and (after normalization) try to find a function $u = u_t$ which solves the equation

$$\frac{\text{Det}\ \left(g_{k\bar{l}} + \partial_k \bar{\partial}_l u_t\right)}{\text{Det}\ \left(g_{k\bar{l}}\right)} = e^{-t\, u_t} \tag{6}$$

for each $t \in [0,1]$. Yau's solution to the Calabi conjecture implies that (6) can be solved for $t = 0$. However, if (6) can also be solved for $t = 1$, it follows (by elementary manipulations) that the metric

$$\tilde{g}_{k\bar{l}} = g_{k\bar{l}} + \partial_k \bar{\partial}_l u_1$$

[1] A Kähler metric is a hermitian metric satisfying a "locally euclidean to 2nd order" condition. It forces a strong relationship between holomorphic and harmonic forms on the given manifold. A Kähler-Einstein metric is a Kähler metric whose Ricci form is proportional to the metric itself. And a Fano manifold is a compact, complex (connected) manifold with positive first Chern class.

satisfies the Einstein condition. One, of course, needs an a priori inequality to pass from $t = 0$ to $t = 1$ and, although it is a quite different inequality than the subelliptic estimate (3), Nadel sought to mimic Kohn's approach to (3) for the estimate he needed.

Instead of recalling the particular estimate Nadel dealt with, or his original definition of a multiplier ideal sheaf, we want to first note the parallels with Kohn's idea and indicate the scope of Nadel's successful application of this idea. Like Kohn, Nadel "tamed" his estimate by introducing certain multipliers into the inequality. Then, again as Kohn had done, he established certain algebraic-geometric properties of this ideal of multipliers. Nadel then established a vanishing theorem on the cohomology groups associated to his (coherent) sheaf of multiplier ideals and, from this deduced that his needed inequality held if certain algebraic conditions — not dissimilar to finite type — were satisfied. Because Nadel's conditions were easier to check in certain circumstances, he was able to show that his basic estimate held (and, thus, Kähler-Einstein metrics existed) on Fano manifolds without the restrictions on codimension or symmetry needed in the earlier work of Siu, Tian, and Yau. (In retrospect, this was what Kohn had done too, for his estimate, though in this case the flow of viewpoint was from Nadel to Kohn rather than vice-versa.)

The basic idea of a multiplier ideal has been expanded and simplified since Nadel's work. Perhaps the most elementary version of this idea, staying within the category of polynomials, is the following. Let f be a polynomial in $\mathbb{C}^n$ and let $s > 0$ be a fixed real number. The *multiplier ideal of f of order* (or weight) s is

$$\mathcal{J}\left(f^s\right) = \left\{\text{polynomials } g : \frac{|g|}{|f|^s} \in L^2_{\text{loc}}(z_0), \text{ for all } z) \in \mathbb{C}^n\right\},$$

where $L^2_{\text{loc}}(z_0)$ denotes the locally square-integrable functions in a neighborhood of z_0. Similarly, if I is an ideal generated by polynomials, $I = (f_1, \ldots, f_k)$,

$$\mathcal{J}\left(I^s\right) = \left\{\text{polynomials } g : \frac{|g|^2}{\left(\sum |f_j|^2\right)^s} \in L^1_{\text{loc}}(z_0), \quad z_0 \in \mathbb{C}^n\right\}$$

is the multiplier ideal of I of order s.

Whether viewed attached to a single function f or to an ideal, the multiplier ideal offers a powerful approach to the study of singularities. For instance, the singularity exponent of a polynomial f is defined

$$e(f) = \inf_{z_0 \in \mathbb{C}^n} \left(\sup\left\{t > 0 : |f|^{-t} \in L^2_{\text{loc}}(z_0)\right\}\right)$$

and measures how singular f is on its zero-set, $V(f)$. Clearly, if $s > e(f)$, then $|f|^{-s} \notin L^2_{\text{loc}}(z_0)$ for some $z_0 \in \mathbb{C}^n$. But, if g vanishes to high enough order on $V(f)$, $g \in \mathcal{J}(f)$. Therefore algebraic properties of $\mathcal{J}(f)$ — which are often easy to establish — give geometric consequences, such as the behavior of $e(f)$ — which are neither easy nor obvious to check directly.

There have been many excellent applications of the multiplier ideal notion to problems in algebraic geometry, some of them startling in their depth. A partial list would have to include Siu's work on the Fujita conjecture [Siu1, 2], Siu's proof of the invariance of plurigenera [Siu3], Demailly and Kollar's work on the existence of Kähler-Einstein metrics [Dem-Kol], and several works by Lazarsfeld, some in collaboration with Ein, Mustata, Smith, and Varolin, to which we refer the reader to Lazarsfeld's books [Laz] and its bibliography.

We should mention that our account of the development of multiplier ideals is not complete, as we focused only on those aspects growing out of [Koh]. There were some parallel developments on the algebraic side, especially the work of Esnault-Viehweg [Esn-Vie] and Lipman [Lip], which introduced essentially the same notion in different mathematical language.