2. $y'' - 2y' - 8y = 4e^{2x} - 21e^{-3x}$.

3. $y'' + 2y' + 5y = 6 \sin 2x + 7 \cos 2x$.

4. $y'' + 2y' + 2y = 10 \sin 4x$.

5. $y'' + 2y' + 4y = 13 \cos 4x$.

6. $y'' - 3y' - 4y = 16x - 12e^{2x}$.

7. $y'' + 6y' + 5y = 2e^{x} + 10e^{5x}$.

8. $y'' + 2y' + 10y = 5xe^{-2x}$.

9. $2y'' + 3y' - 2y = 6x^2e^{x} - 4x^2 + 12$.

10. $y'' + 6y' + 8y = 6xe^{2x} + 8x^2$.

11. $y'' + 4y = 4 \sin 2x + 8 \cos 2x$.

12. $y'' - 4y = 16xe^{2x}$.

13. $4y'' - 4y' + y = e^{x/2} + e^{-x/2}$.

14. $y'' - 6y' + 9y = 6e^{3x} + 5xe^{4x}$.

15. $y''' + 4y'' + y' - 6y = -18x^2 + 1$.

16. $y''' + 2y'' - 3y' - 10y = 8xe^{-2x}$.

17. $y''' + y'' + 3y' - 5y = 5 \sin 2x + 10x^2 + 3x + 7$.

18. $4y''' - 4y'' - 5y' + 3y = 3x^3 - 8x$.

19. $y'' + y' - 6y = 10e^{2x} - 18e^{3x} - 6x - 11$.

20. $y'' + y' - 2y = 6e^{-2x} + 3e^{x} - 4x^2$.

21. $y'' - 6y' + 5y = 24x^2e^{x} + 8e^{5x}$.

22. $y'' - 4y' + 5y = 6e^{2x} \cos x$.

23. $y''' - 3y'' + 4y = 4e^{x} - 18e^{-x}$.

24. $y''' - 2y'' - y' + 2y = 9e^{2x} - 8e^{3x}$.

25. $y''' + y' = 2x^2 + 4 \sin x$.

26. $y^{iv} - 3y''' + 2y'' = 3e^{-x} + 6e^{2x} - 6x$.

27. $y''' - 6y'' + 11y' - 6y = xe^{x} - 4e^{2x} + 6e^{4x}$.

28. $y''' - 4y'' + 5y' - 2y = 3x^2e^{x} - 7e^{x}$.

29. $y''' - 4y'' + 4y' = 24xe^{2x} + 16 + 9e^{3x}$.

30. $y''' - 4y' = 32xe^{2x} - 24x^2$.

31. $y'' + y = x \sin x$.

32. $y'' + 4y = 12x^2 - 16x \cos 2x$.

33. $y^{iv} + 2y''' - 3y'' = 18x^2 + 16xe^x + 4e^{3x} - 9.$

34. $y^{iv} - 5y''' + 7y'' - 5y' + 6y = 5 \sin x - 12 \sin 2x.$

Solve the initial-value problems in Exercises 35–50:

35. $y'' - 4y' + 3y = 9x^2 + 4, \quad y(0) = 6, \quad y'(0) = 8.$

36. $y'' + 5y' + 4y = 16x + 20e^x, \quad y(0) = 0, \quad y'(0) = 3.$

37. $y'' - 8y' + 15y = 9xe^{2x}, \quad y(0) = 5, \quad y'(0) = 10.$

38. $y'' + 7y' + 10y = 4xe^{-3x}, \quad y(0) = 0, \quad y'(0) = -1.$

39. $y'' + 8y' + 16y = 8e^{-2x}, \quad y(0) = 2, \quad y'(0) = 0.$

40. $y'' + 6y' + 9y = 27e^{-6x}, \quad y(0) = -2, \quad y'(0) = 0.$

41. $y'' + 4y' + 13y = 18e^{-2x}, \quad y(0) = 0, \quad y'(0) = 4.$

42. $y'' - 10y' + 29y = 8e^{5x}, \quad y(0) = 0, \quad y'(0) = 8.$

43. $y'' - 4y' + 13y = 8 \sin 3x, \quad y(0) = 1, \quad y'(0) = 2.$

44. $y'' - y' - 6y = 8e^{2x} - 5e^{3x}, \quad y(0) = 3, \quad y'(0) = 5.$

45. $y'' - 2y' + y = 2xe^{2x} + 6e^x, \quad y(0) = 1, \quad y'(0) = 0.$

46. $y'' - y = 3x^2e^x, \quad y(0) = 1, \quad y'(0) = 2.$

47. $y'' + y = 3x^2 - 4 \sin x, \quad y(0) = 0, \quad y'(0) = 1.$

48. $y'' + 4y = 8 \sin 2x, \quad y(0) = 6, \quad y'(0) = 8.$

49. $y''' - 4y'' + y' + 6y = 3xe^x + 2e^x - \sin x,$
$$y(0) = \frac{33}{40}, \quad y'(0) = 0, \quad y''(0) = 0.$$

50. $y''' - 6y'' + 9y' - 4y = 8x^2 + 3 - 6e^{2x},$
$$y(0) = 1, \quad y'(0) = 7, \quad y''(0) = 10.$$

For each of the differential equations in Exercises 51–64 *set up* the correct linear combination of functions with undetermined literal coefficients to use in finding a particular integral by the method of undetermined coefficients. (Do not actually find the particular integrals.)

51. $y'' - 6y' + 8y = x^3 + x + e^{-2x}.$

52. $y'' + 9y = e^{3x} + e^{-3x} + e^{3x} \sin 3x.$

53. $y'' + 4y' + 5y = e^{-2x}(1 + \cos x).$

54. $y'' - 6y' + 9y = x^4e^x + x^3e^{2x} + x^2e^{3x}.$

55. $y'' + 6y' + 13y = xe^{-3x} \sin 2x + x^2e^{-2x} \sin 3x.$

56. $y''' - 3y'' + 2y' = x^2e^x + 3xe^{2x} + 5x^2.$

57. $y''' - 6y'' + 12y' - 8y = xe^{2x} + x^2e^{3x}.$

58. $y^{iv} + 3y''' + 4y'' + 3y' + y = x^2e^{-x} + 3e^{-x/2}\cos\dfrac{\sqrt{3}}{2}x.$

59. $y^{iv} - 16y = x^2 \sin 2x + x^4e^{2x}.$

60. $y^{vi} + 2y^{v} + 5y^{iv} = x^3 + x^2e^{-x} + e^{-x}\sin 2x.$

61. $y^{iv} + 2y'' + y = x^2 \cos x.$

62. $y^{iv} + 16y = xe^{\sqrt{2}x}\sin\sqrt{2}x + e^{-\sqrt{2}x}\cos\sqrt{2}x.$

63. $y^{iv} + 3y'' - 4y = \cos^2 x - \cosh x.$

64. $y^{iv} + 10y'' + 9y = \sin x \sin 2x.$

4.4 VARIATION OF PARAMETERS

A. The Method

While the process of carrying out the method of undetermined coefficients is actually quite straightforward (involving only techniques of college algebra and differentiation), the method applies in general to a rather small class of problems. For example, it would not apply to the apparently simple equation

$$y'' + y = \tan x.$$

We thus seek a method of finding a particular integral that applies in all cases (including variable coefficients) in which the complementary function is known. Such a method is the method of *variation of parameters*, which we now consider.

We shall develop this method in connection with the general second-order linear differential equation with variable coefficients

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = F(x). \tag{4.51}$$

Suppose that y_1 and y_2 are linearly independent solutions of the corresponding homogeneous equation

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = 0. \tag{4.52}$$

Then the complementary function of Equation (4.51) is

$$c_1y_1(x) + c_2y_2(x),$$

where y_1 and y_2 are linearly independent solutions of (4.52) and c_1 and c_2 are arbitrary constants. The procedure in the method of variation of parameters is to replace the arbitrary constants c_1 and c_2 in the complementary function by respective *functions* v_1 and v_2 which will be determined so that the resulting function, which is defined by

$$v_1(x)y_1(x) + v_2(x)y_2(x), \tag{4.53}$$

will be a particular integral of Equation (4.51) (hence the name, *variation* of parameters).