MATH 580–EXAM 1–SOLUTIONS

NAME: DATE:

OSU EMAIL (name.#):

1. Find the formula for the isometry of the plane $\mathbb{R}^2$ (having the Euclidean metric) that reflects points in the plane across the line $y = -4x + 7$. [20 pts]
So let L be the line $y = -4x + 7$ and let L' be the line $y = -4x$. Also let R_L and $R_{L'}$ be the reflections across the lines L and L' respectively. Then

$$R_L = T_{(0,7)} \circ R_{L'} \circ T_{(0,-7)}.$$

So for any $(x, y) \in \mathbb{R}^2$,

$$\begin{aligned}
R_L(x,y) &= T_{(0,7)} \circ R_{L'} \circ T_{(0,-7)}(x,y) \\
&= T_{(0,7)} \circ R_{L'}(x, y-7) \\
&= T_{(0,7)} \left(\frac{c^2-d^2}{c^2+d^2}x + \frac{2cd}{c^2+d^2}(y-7), \frac{2cd}{c^2+d^2}x + \frac{d^2-c^2}{c^2+d^2}(y-7) \right) \\
&= \left(\frac{c^2-d^2}{c^2+d^2}x + \frac{2cd}{c^2+d^2}(y-7), \frac{2cd}{c^2+d^2}x + \frac{d^2-c^2}{c^2+d^2}(y-7) + 7 \right)
\end{aligned}$$

where (c, d) is any point on L'. Since $(1, -4)$ is on L', we let $c = 1$ and $d = -4$.
So ultimately we have,

$$\begin{aligned}
R_L(x,y) &= \left(-\frac{15}{17}x + \frac{-8}{17}(y-7), \frac{-8}{17}x + \frac{15}{17}(y-7) + 7 \right) \\
&= \left(-\frac{15}{17}x + \frac{-8}{17}y + \frac{56}{17}, \frac{-8}{17}x + \frac{15}{17}y + \frac{14}{17} \right)
\end{aligned}$$

MATH 580–EXAM 1–SOLUTIONS

NAME: DATE:

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2. Let $\mathbb{R}^2$ the set of points in the real plane, let $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates a point θ radians counter clockwise about the origin, let $G = \{\rho_\theta \mid \theta \in \mathbb{R}\}$. [10 pts each]

(a) Let $\alpha, \beta \in \mathbb{R}$, show that $\rho_\alpha \circ \rho_\beta = \rho_{\alpha+\beta}$. Conclude that if $\rho_\alpha, \rho_\beta \in G$, then $\rho_\alpha \circ \rho_\beta \in G$ (you may use the identities $\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)$ and $\sin(\alpha+\beta) = \cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)$ as well as any other facts about trigonometric functions without proof).

Let $(x, y) \in \mathbb{R}^2$ and consider $\rho_\alpha \circ \rho_\beta(x, y)$.

$$\begin{aligned}
\rho_\alpha \circ \rho_\beta(x,y) &= \rho_\alpha(x\cos(\beta) - y\sin(\beta), x\sin(\beta) + y\cos(\beta)) \\
&= ((x\cos(\beta) - y\sin(\beta))\cos(\alpha) - (x\sin(\beta) + y\cos(\beta))\sin(\alpha), \\
&\qquad (x\cos(\beta) - y\sin(\beta))\sin(\alpha) - (x\sin(\beta) + y\cos(\beta))\cos(\alpha)) \\
&= ((x(\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)) - y(\cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)), \\
&\qquad x(\cos(\alpha)\sin(\beta) + \sin(\alpha)\cos(\beta)) + y(\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta))) \\
&= (x\cos(\alpha+\beta) - y\sin(\alpha+\beta), x\sin(\alpha+\beta) + y\cos(\alpha+\beta)) \\
&= \rho_{\alpha+\beta}(x,y)
\end{aligned}$$

So if $\alpha, \beta \in \mathbb{R}$, then $\alpha + \beta \in \mathbb{R}$. Thus $\rho_\alpha \circ \rho_\beta = \rho_{\alpha+\beta} \in G$.

(b) Let $I : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity function $I((x, y)) = (x, y)$. Find all angles $\alpha \in \mathbb{R}$ so that $I = \rho_\alpha$ Conclude that $I \in G$.

Suppose that $(x, y) \in \mathbb{R}^2$ and $\rho_\alpha(x, y) = (x, y)$. Then

$$(x,y) = \rho_\alpha(x,y) = (x\cos(\alpha) - y\sin(\alpha), x\sin(\alpha) + y\cos(\alpha))$$

so $\rho_\alpha(x, y) = (x, y)$ if and only if $\cos(\alpha) = 1$ and $\sin(\alpha) = 0$ if and only if $\alpha \in \{2\pi k \mid k \in \mathbb{Z}\}$.

For any $k \in \mathbb{Z}$, $2\pi k \in \mathbb{R}$, so $I = \rho_{2\pi k} \in G$ for any $k \in \mathbb{Z}$.

MATH 580–EXAM 1–SOLUTIONS

NAME: DATE:

OSU EMAIL (name.#):

2 (cont.). Let $\mathbb{R}^2$ the set of points in the real plane, let $\rho_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that rotates a point θ radians counter clockwise about the origin, let $G = \{\rho_\theta \,|\, \theta \in \mathbb{R}\}$. [10 pts each]

(c) For any $\alpha \in \mathbb{R}$, find all angles $\beta \in \mathbb{R}$ so that $\rho_\alpha^{-1} = \rho_\beta$. Conclude that $\rho_\alpha^{-1} \in G$ and that G is a group of functions. **Hint:** One rather parsimonious solution uses parts (a) and (b).

By 2(a) for any $\alpha, \beta \in \mathbb{R}$, $\rho_\alpha \circ \rho_\beta = \rho_{\alpha+\beta} = \rho_\beta \circ \rho_\alpha$. By 2(b), $\rho_{\alpha+\beta} = I$ if and only if $\alpha + \beta \in \{2\pi k \,|\, k \in \mathbb{Z}\}$. So if $\beta \in \{2\pi k - \alpha \,|\, k \in \mathbb{Z}\}$, then

$$\rho_\alpha \circ \rho_\beta = I = \rho_\beta \circ \rho_\alpha.$$

Thus $\rho_\beta = \rho_\alpha^{-1}$ if and only if $\beta \in \{2\pi k - \alpha \,|\, k \in \mathbb{Z}\}$. For any $\alpha \in \mathbb{R}$ and $k \in \mathbb{Z}$, $2\pi k - \alpha \in \mathbb{R}$, so $\rho_\alpha^{-1} = \rho_{2\pi k-\alpha} \in G$ for any $k \in \mathbb{Z}$.

Since G is closed under composition, has an identity element, and each ρ_α has an inverse in G, then G is a group of functions.

MATH 580–EXAM 1–SOLUTIONS

NAME: DATE:

OSU EMAIL (name.#):

3. In the following exercise, later parts of the problem might depend on earlier ones. You may always use a previous part of this problem in the proof a later part, even if you have yet to proof the prior part. [10 pts each]

(a) Consider the set of real number $\mathbb{R}$ and define a relation on $\mathbb{R}$ as follows: xRy if $x - y \in \mathbb{Z}$ (the set of integers). Show that this is an equivalence relation.

For any $x \in \mathbb{R}$, $x - x = 0 \in \mathbb{Z}$ so xRx and R is reflexive.

Suppose that xRy, then $x - y \in \mathbb{Z}$. For any $a \in \mathbb{Z}$, $-a \in \mathbb{Z}$, so $(-1)(x - y) \in \mathbb{Z}$. Notice that $(-1)(x - y) = y - x$, so $y - x \in \mathbb{Z}$ and yRx. Thus R is symmetric.

Suppose that xRy and yRz, then $x - y, y - z \in \mathbb{Z}$. Since $\mathbb{Z}$ is closed under addition, we find $x - z = (x - y) + (y - z) \in \mathbb{Z}$. So xRz and R is transitive.

So R satisfies the criteria for an equivalence relation.

(b) Let $\mathbb{R}/\mathbb{Z}$ be the set of equivalence classes for the equivalence relation in (a). We would like to define an addition of two equivalence classes $[a]$ and $[b]$ as follows: $[a] + [b] = [a + b]$. Show that such an operation is well-defined.

Suppose that $[a] = [a']$ and $[b] = [b']$. Then aRa' and bRb' so $(a - a'), (b - b') \in \mathbb{Z}$. Since $\mathbb{Z}$ is closed under addition, we have

$$(a + b) - (a' + b') = (a - a') + (b - b') \in \mathbb{Z}.$$

Thus $[a + b] = [a' + b']$ and our addition of equivalence classes is well defined.

MATH 580–EXAM 1–SOLUTIONS

NAME: DATE:

OSU EMAIL (name.#):

3 (cont.). In the following exercise, later parts of the problem might depend on earlier ones. You may always use a previous part of this problem in the proof a later part, even if you have yet to proof the prior part. [10 pts each]

(c) Given the operation in part (b), $\mathbb{R}/\mathbb{Z}$ is a group (you don't need to prove this). Let G be the group in exercise 2. We would like to define a function $f : \mathbb{R}/\mathbb{Z} \to G$ by $f([a]) = \rho_{2\pi a}$. Show that f is a well-defined homomorphism (you can use any of the results from exercise 2 in your proof).

To show that f is well defined, let $[a] = [a']$. Then aRa' which means that $a - a' \in \mathbb{Z}$. So

$$\rho_{2\pi a} = \rho_{2\pi(a'+(a-a'))} \overset{\text{by } 2(\text{a})}{=} \rho_{2\pi a'} \circ \rho_{2\pi(a-a')} \overset{\text{by } 2(\text{b})}{=} \rho_{2\pi a'} \circ I = \rho_{2\pi a'}.$$

Thus $\rho_{2\pi a} = \rho_{2\pi a'}$ and f is well defined.
To show that f is a homomorphism, let $[a], [b] \in \mathbb{R}/\mathbb{Z}$. Then

$$f([a] + [b]) = f([a + b]) = \rho_{2\pi(a+b)} \overset{\text{by } 2(\text{a})}{=} \rho_{2\pi a} \circ \rho_{2\pi b} = f([a]) \circ f([b]).$$

Thus f is a homomorphism (in fact, one can show that f is an isomorphism of groups).

MATH 580–EXAM 1–SOLUTIONS

NAME: DATE:

OSU EMAIL (name.#):

4. For each of the following sets having binary operations, determine whether they determine groups. If so, prove that the set and operation is a group. If not, use an example to show which group axiom fails. [10 pts each]

(a) Let S be the set of all subsets of $X = \{1, 2, 3, 4, 5\}$ (denoted $\mathcal{P}(X)$) with operation $U \cdot V = U \cap V$.

If S were a group, it would need an identity element. In particular, we would need a subset V of X so that $U \cap V = V \cap U = U$ for all $U \subset X$. Since $U \cap V = V \cap U \subset V$ for all $U \subset X$, we need must have that $U \subset V$ for all $U \subset X$. The only such subset is $V = X$.

However, if X is our identity element, then S does not contain inverses for any proper subset $U \subset X$. If U is a proper subset of X, then for any $V \subset X$, we have that $U \cap V = V \cap U \subset U \neq X$. So the S and operation $\cdot$ do not satisfy the axioms of a group.

(b) Let $S = \{(a, b) \,|\, a, b \in \mathbb{Q},\, a \neq 0\}$ with operation $(a, b) \cdot (c, d) = (ac, ad + b)$.

Let us first show that the operation is associative. So let $(a, b), (c, d), (f, g)$ be elements of S. Then

$$\begin{aligned}
(a, b) \cdot ((c, d) \cdot (f, g)) &= (a, b) \cdot (cf, cg + d) \\
&= (acf, a(cg + d) + b) \\
&= (acf, (ac)g + (ad + b)) \\
&= (ac, ad + b) \cdot (f, g) \\
&= ((a, b) \cdot (c, d)) \cdot (f, g)
\end{aligned}$$

So the operation on S is associative.
Consider the element $(1, 0) \in S$. For any $(a, b) \in S$, we have

$$\begin{aligned}
(1, 0) \cdot (a, b) &= ((1)a, (1)b + 0) = (a, b) \\
(a, b) \cdot (1, 0) &= (a(1), a(0) + b) = (a, b)
\end{aligned}$$

So the element $(1, 0)$ is an identity element.
Let $(a, b) \in S$ and consider $(a^{-1}, -ba^{-1})$. So we have

$$\begin{aligned}
(a, b) \cdot (a^{-1}, -ba^{-1}) &= (aa^{-1}, a(-ba^{-1}) + b) = (1, 0) \\
(a^{-1}, -ba^{-1}) \cdot (a, b) &= (a^{-1}a, a^{-1}b + (-ba^{-1})) = (1, 0)
\end{aligned}$$

So for any $(a, b) \in S$, the element $(a^{-1}, ba^{-1}) \in S$ is its inverse.
So the set S and operation $\cdot$ satisfies the axioms of a group.