Math 580–EXAM 1–Review Problems

I. Here are some problems about equivalence relations.
   1. Let $\mathbb{R}^\times$ be the set of **non-zero** real numbers and define a relation aRb if $ab^{-1} \in \mathbb{Q}^\times$ where $\mathbb{Q}^\times$ are the **non-zero** rational numbers (equivalently $R = \{(a,b) \in \mathbb{R}^\times \times \mathbb{R}^\times \mid ab^{-1} \in \mathbb{Q}^\times\}$).
      (a) Show that R defines an equivalence relation.
      (b) Let $\mathbb{R}^\times/\mathbb{Q}^\times$ be the set of equivalence classes for R. Suppose that we want to define an operation on $\mathbb{R}^\times/\mathbb{Q}^\times$ by $[a]\cdot[b] = [ab]$. Show such an operation is well-defined.
      (c) Show that the set $\mathbb{R}^\times/\mathbb{Q}^\times$ with the operation as is in (b) defines a group.
   2. Let $\mathrm{Isom}(\mathbb{R}^2)$ be the set of all isometries of the plane $\mathbb{R}^2$ (with the Euclidean metric) and define a relation $f_1 R f_2$ if there exists an invertible element $g \in \mathrm{Isom}(\mathbb{R}^2)$ so that $f_1 = g \circ f_2 \circ g^{-1}$. Show that R defines an equivalence relation.
   3. Let S be the set of all people (living or deceased) and define a relation xRy if x has the same biological father as y.
      (a) Show that R defines an equivalence relation.
      (b) In a sentence or two, describe an equivalence class for this equivalence relation.
      (c) Let S/R be the set of equivalence classes of our equivalence relation. Suppose we wish to define a function $f : S/R \to S$ by

$$f([x]) = \text{the biological mother of } x.$$

      Is such a function well-defined? Explain your answer in a sentence or two.

II. Here are some problems about isometries.
   1. Let $\mathbb{R}$ be the set of real numbers and let $d(x,y) = |x-y|$.
      (a) Determine if $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$ is an isometry.
      (b) Determine if $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 7 - x$ is an isometry.
      (c) For $a \in \mathbb{R}$, let $T_a(x) = x + a$. Show that the set $\{T_a \mid a \in \mathbb{Z}\}$ is a subgroup of $\mathrm{Isom}(\mathbb{R})$.
      (d) For $a \in \mathbb{R}$, let $R_a(x) = a - x$. Determine whether the set $\{R_a \mid a \in \mathbb{R}\}$ is a subgroup of $\mathrm{Isom}(\mathbb{R})$.
   2. Let $\mathbb{R}^2$ be the real plane and let $d((x,y),(z,w)) = |x-z| + |y-w|$ be a metric (don't worry about proving this, you can assume that d is a metric).
      (a) Show that any translation $T_{(a,b)} : \mathbb{R}^2 \to \mathbb{R}^2$ (given by $T_{(a,b)}(x,y) = (x+a, y+b)$) is an isometry (with respect to the metric d).
      (b) Let $r_L : \mathbb{R}^2 \to \mathbb{R}^2$ be reflection across the line $y = mx$. Show that r_L is not an isometry (with respect to the metric d) if $0 < m < 1$.

III. Here are some problems about abstract groups.
   1. Consider the following sets and binary operations. Determine which are groups. If it is a group, prove it. If it is not, show (by example) which group axiom fails.
      (a) Let $S = \mathcal{P}(\{1,2,3,4\})$ (the power set of $\{1,2,3,4\}$) and let $U \cdot V = U \cup V$.
      (b) Let $S = \mathbb{Z}$ and $a \cdot b = a - b$.
      (c) Let $S = \mathbb{R}$ and let $x \cdot y = x^2 - y^2$.

2. Let G be the group of integers ($\mathbb{Z}$) under the addition operation and let H be the group Isom($\mathbb{R}$) under the composition operation.
   (a) Show that the function $f : G \to H$ given by $f(a) = T_a$ (where $T_a : \mathbb{R} \to \mathbb{R}$ is given by $T_a(x) = x + a$) is a homomorphism.
   (b) Determine if the function $f : G \to H$ given by $f(a) = R_a$ (where $R_a : \mathbb{R} \to \mathbb{R}$ is given by $R_a(x) = a - x$) a homomorphism.
   (c) Show that for any $m \in \mathbb{Z}$, the function $f_m : G \to G$ given by $f_m(a) = ma$ is a homomorphism. For which m is f_m injective? For which m is f_m surjective?
3. Let G be a group and let $g \in G$.
   (a) Show that the function $f_g : G \to G$ given by $f_g(x) = gxg^{-1}$ is an isomorphism.
   (b) Assume that $g \neq e$. Determine if the function $L_g : G \to G$ given by $L_g(x) = gx$ is a homomorphism (**Hint:** If L_g it is a homomorphism, what would I know about $\ker(L_g)$.)