Math 580
Supplemental Homework Problems #3

For the following problems, let G be a finite group and let a an element of G. When then get the following subsets of G:

(i) $C_a = \{gag^{-1} \mid g \in G\}$, the conjugacy class of a in G,
(ii) $C_G(a) = \{g \in G \mid ga = ag\}$, the centralizer of a in G,
(iii) $Z(G) = \{g \in G \mid gx = xg \text{ for all } x \in G\}$, the center of G.

Recall that in homework 4, problem N1.12, we showed that $C_G(a)$ is a subgroup of G and that $Z(G)$ is a normal subgroup of G.

1. Let $a, b \in G$. Show that if $C_a = C_b$, then $C_G(a)$ is conjugate to $C_G(b)$.

2. Let $a \in G$. Show that $a \in Z(G)$ if and only if $C_a = \{a\}$.

3. Let $G/C_G(a)$ be the set of left cosets for $C_G(a)$. Find a bijection $f : G/C_G(a) \to C_a$ and conclude that $|C_a| = [G, C_G(a)]$ (recall that $[G, C_G(a)]$ is the number of left cosets of $C_G(a)$ in G).

4. Let $f_a : G \to G$ be the function $f_a(g) = aga^{-1}$. Show that f_a is an isomorphism of G.