Separation Properties for Graph-Directed Self-Similar Fractals

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Abstract

Examples of "separation properties" for iterated function systems of similitudes include: the open set condition, the weak separation property, finite type. Alternate descriptions for these properties and relations among these properties have been worked out. Here we consider the same situation for "graph-directed" iterated function systems, and provide the definitions and proofs for that setting. We deal with the case of strongly connected graphs. In many cases the definitions (and proofs) are much like the one-node case. But sometimes we have found changes were needed.

Key words: Iterated function system, IFS, Graph-directed, Self-similar, Separation, Weak separation, Finite type
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1 The Setting

Directed multigraph. Begin with a directed multigraph $G = (V, E)$. So V is a finite set (of "vertices" or "nodes"), E is a finite set (of "edges"), for each $u, v \in V$, $E_{uv} \subseteq E$ is the set of edges from u to v. For convenience we assume that E is the disjoint union of the sets E_{uv}. If $e \in E_{uv}$ then e has **initial vertex** u and **final vertex** v. Again for convenience we assume that every node u is the initial vertex for at least one edge. Write $E^{(k)}_{uv}$ for paths of

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length k, say $\sigma = e_1 e_2 \cdots e_k$ where e_1 has initial vertex u, the final vertex of each e_i matches the initial vertex of the next one e_{i+1}, and the final vertex of e_k is v. Then $E^{(*)}_{uv} = \bigcup_{k=0}^{\infty} E^{(k)}_{uv}$ is the forest of all paths in G, ordered by the "prefix" relation. If $\sigma = e_1 e_2 \cdots e_k$, then its **parent** is $\sigma^- = e_1 e_2 \cdots e_{k-1}$. We say that $G = (V, E)$ is **strongly connected** if $E^{(*)}_{uv} \neq \varnothing$ for all $u, v \in V$.

The IFS. For each $u \in V$ we have a metric space X_u. For now we will let all $X_u = \mathbb{R}^d$ for a certain d. (But it still helps to think of X_u as separate spaces.) For each $e \in E_{uv}$ we have a similitude $S_e \colon X_v \to X_u$, with contraction ratio ρ_e:

$$|S_e(x) - S_e(y)| = \rho_e |x - y|.$$

Assume $0 < \rho_e < 1$. Write $\rho_{\min} = \min\{\, \rho_e : e \in E \,\}$, $\rho_{\max} = \max\{\, \rho_e : e \in E \,\}$. For $\sigma = e_1 e_2 \cdots e_k$ write $S_\sigma = S_{e_1} \circ S_{e_2} \circ \cdots \circ S_{e_k}$ and $\rho(\sigma) = \rho_{e_1} \cdots \rho_{e_k}$. This formulation is found in [5,3].

The original version of an IFS, where no graph is specified, can be fit into this scheme by using a graph $G = (V, E)$ where V has exactly one element. Then all edges are loops from that node to itself. To emphasize this case, we will sometimes call it the **one-node** case.

The family $(S_e)_{e \in E}$ is known as a (graph-directed) **iterated function system** or **IFS**. There is a unique family $\{\, K_u : u \in V \,\}$ of nonempty compact sets such that

$$K_u = \bigcup_{v \in V} \bigcup_{e \in E_{uv}} S_e(K_v)$$

for all $u \in V$ [3, Theorem (4.3.5)]. These are the **attractors** or **invariant sets** defined by the IFS (S_e).

If R is a similitude, write $\rho(R)$ for its contraction ratio. So in our setting, $\rho(S_\sigma) = \rho(\sigma)$.

Definitions. Here are a few additional definitions formulated in terms of a graph-directed iterated function system. Let $u, v \in V$, $0 < a < b$, $I \subseteq \mathbb{R}$ an interval, $0 < r < 1$, $U \subseteq X_u$ bounded, $M \subseteq X_v$ nonempty. Define

$$\begin{aligned}
&\mathcal{R}_{uv} = \{\, R : R \text{ is a similitude from } X_v \text{ to } X_u \,\} \\
&\mathcal{R}_{uv}(I) = \{\, R \in \mathcal{R}_{uv} : \rho(R) \in I \,\} \\
&E^{(*)}_{uv}(]a, b]) = \left\{\, \sigma \in E^{(*)}_{uv} : S_\sigma \in \mathcal{R}_{uv}(]a, b]) \,\right\}
\end{aligned}$$

$$
\begin{aligned}
&F_{uv}(]a,b]) = \left\{\, S_\sigma : \sigma \in E_{uv}^{(*)}(]a,b]) \,\right\} \\
&\mathcal{F}_{uv}(]a,b]) = \left\{\, T^{-1}\circ S : T,S \in F_{uv}(]a,b]) \,\right\} \quad \text{(These map } X_v \text{ to itself.)} \\
&\mathcal{F}_{uv}(r) = \bigcup_{b>0} \mathcal{F}_{uv}(]rb,b]) \\
&\mathcal{F}_{uv} = \bigcup_{0<a<b} \mathcal{F}_{uv}(]a,b]) = \bigcup_{0<r<1} \mathcal{F}_{uv}(r) = \left\{\, S_\tau^{-1}\circ S_\sigma : \tau,\sigma \in E_{uv}^{(*)} \,\right\} \\
&F_{uv}(]a,b],U,M) = \{\, T \in F_{uv}(]a \ \operatorname{diam} U, b \ \operatorname{diam} U]) : T(M)\cap U \neq \varnothing \,\} \\
&\gamma_{uv}(]a,b],M) = \sup\{\, \# F_{uv}(]a,b],U,M) : U \subseteq X_u \text{ bounded} \,\}
\end{aligned}
$$

Proposition 1.1 *If $E_{wu}^{(*)} \neq \varnothing$ then $\mathcal{F}_{uv}(r) \subseteq \mathcal{F}_{wv}(r)$ for all r. So if G is strongly connected, then $\mathcal{F}_{uv}(r)$ is independent of u.*

PROOF. Let $\sigma \in E_{wu}^{(*)}$. Any element of $\mathcal{F}_{uv}(r)$ belongs to $\mathcal{F}_{uv}(]rb,b])$ for some $b > 0$. So it has the form $T^{-1}\circ S$ where $T,S \in F_{uv}(]rb,b])$. Then $S_\sigma\circ T, S_\sigma\circ S \in F_{wv}(]rb\rho(\sigma), b\rho(\sigma)])$, and $(S_\sigma\circ T)^{-1}\circ(S_\sigma\circ S) = T^{-1}\circ S$, so $T^{-1}\circ S \in \mathcal{F}_{wv}(]rb\rho(\sigma), b\rho(\sigma)]) \subseteq \mathcal{F}_{wv}(r)$. $\square$

Proposition 1.2 $\mathcal{F}_{uv}(r) = \mathcal{F}_{uv} \cap \mathcal{R}_{uv}(]r, r^{-1}[\,)$.

PROOF. Let $R \in \mathcal{F}_{uv}(r)$. Then there is b so that $R = T^{-1}\circ S$ with $T,S \in F_{uv}(]rb,b])$. So $\rho(R) = \rho(T)^{-1}\rho(S) < (rb)^{-1}b = r^{-1}$ and $\rho(R) = \rho(T)^{-1}\rho(S) > b^{-1}(rb) = r$. So $R \in \mathcal{F}_{uv} \cap \mathcal{R}_{uv}(]r,r^{-1}[)$.

Conversely, let $R \in \mathcal{F}_{uv} \cap \mathcal{R}_{uv}(]r,r^{-1}[)$. Say $R = T^{-1}\circ S$. First take the case $\rho(T) \le \rho(S)$. Let $b = \rho(S)$ so that $T,S \in F_{uv}(]rb,b])$ and $R \in \mathcal{F}_{uv}(r)$. For the other case $\rho(T) > \rho(S)$, let $b = \rho(T)$ and again $R \in \mathcal{F}_{uv}(r)$. $\square$

2 The Weak Separation Property

The **weak separation property** was formulated by Lau and Ngai [4] and studied by Zerner [9]. Here we adapt [9] for the graph-directed setting.

Let us say that a set $Y \subseteq \mathbb{R}^d$ is in **general position** iff it is not contained in a hyperplane. So if Y is in general position, then the only similitude R with $R(y) = y$ for all $y \in Y$ is the identity.

Equivalent conditions. In [9], Zerner gave many equivalent formulations for the definition of "weak separation property". Here we have adapted them for the graph-directed case. Let $r \in]0, \rho_{\min}]$. Consider these conditions:

(1a) For all $v \in V$, there exist $x \in K_v$ and $\varepsilon > 0$ such that for all $u \in V$ and all $R \in \mathcal{F}_{uv}(r)$, either R is the identity or $|R(x) - x| \geq \varepsilon$.

(1b) For all $u \in V$ there exist $x \in X_u$ and $\varepsilon > 0$ such that for all $R \in \mathcal{F}_{uu}(r)$, either $R(x) = x$ or $|R(x) - x| \geq \varepsilon$.

(2a) For all $u \in V$ there are $\{x_0, \cdots, x_d\} \subseteq X_u$ in general position and $\varepsilon > 0$ such that for all $R \in \mathcal{F}_{uu}(r)$ and all j, either $R(x_j) = x_j$ or $|R(x_j) - x_j| \geq \varepsilon$.

(2b) For all $u \in V$ there are $\{x_0, \cdots, x_d\} \subseteq X_u$ in general position and $\varepsilon > 0$ such that for all $R \in \mathcal{F}_{uu}(r)$, either R is the identity or $|R(x_j) - x_j| \geq \varepsilon$ for some j.

(3a) For all $u \in V$, the identity is an isolated point of $\mathcal{F}_{uu}$.

(3b) For all $u \in V$, the identity is an isolated point of $\mathcal{F}_{uu}(r)$.

(4a) For all $u, v \in V$, all bounded $M \subseteq X_v$, and all $b > 0$, we have

$$\gamma_{uv}(]rb, b], M) < \infty.$$

(4b) For all $u, v \in V$ there exist nonempty $M \subseteq X_v$ and $b > 0$ such that

$$\gamma_{uv}(]rb, b], M) < \infty.$$

(5a) For all $u, v, w \in V$ and $z \in X_w$, there exists $l \in \mathbb{N}$ such that for any $\tau \in E^{(*)}_{vw}$ and any $b > 0$, every ball in X_u with radius b contains at most l elements of

$$\left\{ S_{\sigma\tau}(z) : \sigma \in E^{(*)}_{uv}(]rb, b]) \right\}.$$

(5b) For all $u, v \in V$, there exist $w \in V$, $z \in X_w$ and $l \in \mathbb{N}$ such that for any $\tau \in E^{(*)}_{vw}$ and any $b > 0$, every ball in X_u with radius b contains at most l elements of

$$\left\{ S_{\sigma\tau}(z) : \sigma \in E^{(*)}_{uv}(]rb, b]) \right\}.$$

Next we will prove that these conditions are equivalent for strongly connected graphs G. For the most part, our proof follows [9] with appropriate changes for the graph case. Note that [9] cites [1,8] as sources for some of these arguments.

Lemma 2.1 *Let $u, v \in V$. Assume* (5b) *holds, G is strongly connected, and K_v is in general position in $X_v = \mathbb{R}^d$. Let w, z, l be as in* (5b)*. Then there is* *a constant C and $\tau \in E_{vw}^{(*)}$ such that for all $y \in X_u$, and all $b > 0$,*

$$\#\{\, T \in F_{uv}(]br, b]) : T(S_\tau(z)) = y \,\} \le C.$$

PROOF. Because G is strongly connected, K_v is contained in the closure of the set $A = \left\{ S_\tau(z) : \tau \in E_{vw}^{(*)} \right\}$. Since K_v is in general position, so is A. Let $x_0, \cdots, x_d \in A$ be such that a similitude defined on X_v is uniquely determined by its values on $x_0, \cdots, x_d$. Say $x_j = S_{\tau_j}(z)$ $(0 \le j \le d)$. Let $t = \max\{\, |x_j - x_0| : 0 \le j \le d \,\}$, let c_t be the number of balls of radius 1 required to cover a ball of radius t, write $m = (d+1)c_t l$ and $C = m(m-1)(m-2)\cdots(m-d+1)$.

Now let $y \in X_u$ and $b > 0$ be given. The ball $\overline{B}(y, bt)$ is covered by c_t balls of radius b, so for each $j \in \{0, \cdots, d\}$

$$\begin{aligned} &\#\left\{\, T(x_j) : T \in F_{uv}(]br, b]), T(x_j) \in \overline{B}(y, bt) \,\right\} \\ &\qquad = \#\left(\left\{\, S_{\sigma\tau_j}(z) : \sigma \in E_{uv}^{(*)}(]rb, b]) \,\right\} \cap \overline{B}(y, bt)\right) \le c_t l. \end{aligned}$$

If $T \in F_{uv}(]br, b])$ and $T(x_0) = y$, then $|T(x_j) - y| = |T(x_j) - T(x_0)| \le bt$ for all j. So

$$\#\{\, T(x_j) : T \in F_{uv}(]br, b]), j \in \{0, \cdots, d\}, T(x_0) = y \,\} \le (d+1)c_t l = m.$$

And a similitude is determined by its values on $\{x_0, \cdots, x_d\}$, so

$$\#\{\, T \in F_{uv}(]br, b]) : T(x_0) = y \,\} \le m(m-1)\cdots(m-d+1) = C.$$

Since x_0 has the form $S_\tau(z)$, this completes the proof. □

Lemma 2.2 *Let K be a nonempty closed set in Euclidean space $\mathbb{R}^d$. Suppose K is contained in the union of countably many hyperplanes. Then for some $x \in K$, there is a neighborhood U of x such that $K \cap U$ is contained in a single hyperplane.*

PROOF. Say $K \subseteq \bigcup_{n=1}^\infty L_n$, for hyperplanes L_n. Note K is itself a complete metric space, so by the Baire Category Theorem K is not a countable union of sets nowhere dense in K. If no neighborhood in K is contained in $K \cap L_n$, then (since it is closed) $K \cap L_n$ is nowhere dense in K. □

Corollary 2.3 *Suppose no K_v is contained in a hyperplane. Then for all $v \in V$ there is $x \in K_v$ such that for all $R \in \bigcup_u \mathcal{F}_{uv}$, either R is the identity or $R(x) \ne x$.*

PROOF. Since no K_v is contained in a hyperplane, and every neighborhood in every K_v contains a similar image of some K_u, by Lemma 2.2, K_v is not contained in a countable union of hyperplanes. The sets $\mathcal{F}_{uv}$ are countable, and for each $R \in \mathcal{F}_{uv}$ other than the identity, $\{\, x : R(x) = x \,\}$ is contained in a hyperplane. So we may choose $x \in K_v$ such that $R(x) = x$ for $R \in \bigcup_u \mathcal{F}_{uv}$, only if R is the identity. $\square$

We say $x \in X_v$ is **generic** for the IFS (S_e) iff for all $R \in \bigcup_u \mathcal{F}_{uv}$, either R is the identity or $R(x) \neq x$.

The following proof is adapted from [9], where parts of it are attributed to [1,8,4].

Theorem 2.4 *Suppose G is strongly connected, and all K_u are in general position. Let $r \in]0, \rho_{\min}]$. Then* (1a)–(5b) *are equivalent. Since* (3a) *in independent of r, so are the others.*

PROOF. (1a) $\Longrightarrow$ (1b), (2a) $\Longrightarrow$ (2b), (3a) $\Longrightarrow$ (3b), (4a) $\Longrightarrow$ (4b), (5a) $\Longrightarrow$ (5b) are trivial.

(4a) $\Longrightarrow$ (5a): Assume (4a). Let $u, v, w \in V$ and $z \in X_w$ be given. Then the set

$$M = \left\{\, S_\tau(z) : \tau \in E_{vw}^{(*)} \,\right\}$$

is bounded. Let $l = \gamma_{uv}(]r/2, 1/2], M) < \infty$. Then for any $\tau \in E_{vw}^{(*)}$, any $b > 0$, and any ball U in X_u of radius b (and diameter $2b$):

$$\begin{aligned} \#\Big(\Big\{\, S_{\sigma\tau}(z) : \sigma \in E_{uv}^{(*)}(]rb, b]) \,\Big\} \cap U\Big) & \\ \leq \#\,\{\, T \in F_{uv}(]rb, b]) : T(S_\tau(z)) \in U \,\} & \\ \leq \#\,\{\, T \in F_{uv}(]rb, b]) : T(M) \cap U \neq \varnothing \,\} & \\ = \#F_{uv}(]r/2, 1/2], U, M) \leq l. & \end{aligned}$$

(5b) $\Longrightarrow$ (4b): Assume (5b). Let $u, v \in V$ be given. Apply (5b) to get $w \in V$, $z \in X_w$, and l; then apply Lemma 2.1 to get $x_0 = S_\tau(z)$ and $C > 0$. Let c be the number of balls of radius 1 required to cover a set of diameter 2. We claim that $\gamma_{uv}(]r/2, 1/2], \{x_0\}) \leq cCl$. Indeed, let $U \subseteq X_u$ be a bounded set. Write $b = \operatorname{diam} U$. Now let B be a ball in X_u of radius $b/2$. Write $Q = \{\, T(x_0) : T \in F_{uv}(]rb/2, b/2]) \,\} \cap B$. Then $\#Q \leq l$, and

$$\begin{aligned}
&\#\{\, T \in F_{uv}(]rb/2, b/2]) : T(x_0) \in B \,\} \\
&\qquad = \sum_{y \in Q} \#\{\, T \in F_{uv}(]rb/2, b/2]) : T(x_0) = y \,\} \le Cl.
\end{aligned}$$

Then since U can be covered by at most c balls of radius $b/2$,

$$\#F_{uv}(]r/2, 1/2], U, \{x_0\}) \le cCl.$$

This is true for all U, so $\gamma_{uv}(]r/2, 1/2], \{x_0\}) \le cCl$.

(4b) $\Longrightarrow$ (4a): Assume (4b). Let $u, v \in V$. There exist $M_0 \neq \varnothing$ and $b_0 > 0$ with $\gamma_{uv}(]rb_0, b_0], M_0) < \infty$. Then since $M_0 \neq \varnothing$, there is $y_0 \in M_0$ with $\gamma_{uv}(]rb_0, b_0], \{y_0\}) < \infty$.

We claim now that $\gamma_{uv}(]rb, b], \{y_0\}) < \infty$ for *all* $b > 0$. Indeed, let c be the number of balls of diameter b required to cover a set of diameter b_0. Given a bounded set $U \subseteq X_u$, write $k = \operatorname{diam} U$, cover it by c balls V_i of diameter kb/b_0. Then

$$\begin{aligned}
F_{uv}(]rb, b], U, \{y_0\}) &= \{\, T \in F_{uv}(]rbk, bk]) : T(y_0) \in U \,\} \\
&= \bigcup_i \left\{ T \in F_{uv}\left(\left]rb_0 \frac{kb}{b_0}, b_0 \frac{kb}{b_0}\right]\right) : T(y_0) \in V_i \right\},
\end{aligned}$$

so $\#F_{uv}(]rb, b], U, \{y_0\}) \le c\gamma_{uv}(]rb_0, b_0], \{y_0\})$. Taking supremum on U, we conclude

$$\gamma_{uv}(]rb, b], \{y_0\}) \le c\gamma_{uv}(]rb_0, b_0], \{y_0\}).$$

Now we are ready to prove (4a). Let $M \subseteq X_v$ be bounded, and let $b > 0$. We claim there exists $b' > 0$ such that

$$\gamma_{uv}(]rb, b], M) \le \gamma_{uv}(]rb', b'], \{y_0\}).$$

To see this: let $k = \operatorname{diam}(M \cup \{y_0\})$, and $b' = b/(1 + 2bk)$. Let $U \subseteq X_u$ be a bounded set. Define $U' = B(U, bk \operatorname{diam} U)$, the open set of all points within distance less than $bk \operatorname{diam} U$ of the set U. So $\operatorname{diam} U' = \operatorname{diam} U + 2bk \operatorname{diam} U = (1 + 2bk) \operatorname{diam} U$. We claim that

$$F_{uv}(]rb, b], U, M) \subseteq F_{uv}(]rb', b'], U', \{y_0\}).$$

Indeed, let $T \in F_{uv}(]rb, b], U, M)$. So $\rho(T) \in]rb \operatorname{diam} U, b \operatorname{diam} U]$ and $T(M) \cap U \neq \varnothing$. So there exists $y \in M$ with $T(y) \in U$. Now $|y - y_0| \le k$, and

$|T(y) - T(y_0)| \leq bk \operatorname{diam} U$, so $T(y_0) \in U'$. Also

$$\rho(T) \in \left] \frac{rb}{1+2bk} \operatorname{diam} U', \frac{b}{1+2bk} \operatorname{diam} U' \right].$$

Thus $T \in F_{uv}(]rb', b'], U', \{y_0\})$, as required. Now we have

$$\# F_{uv}(]rb, b], U, M) \leq \# F_{uv}(]rb', b'], U', \{y_0\}) \leq \gamma_{uv}(]rb', b'], \{y_0\}).$$

This is true for all U, so

$$\gamma_{uv}(]rb, b], M) \leq \gamma_{uv}(]rb', b'], \{y_0\}) < \infty.$$

This completes the proof of (4a).

(2b) $\Longrightarrow$ (4a): Let $M \subseteq X_v$ be bounded, and let $b > 0$. Then apply (2b) with node v to get $\{x_0, \cdots, x_d\}$ in general position in X_v and $\varepsilon > 0$. Let $k = \operatorname{diam}(M \cup \{x_0, \cdots, x_d\})$. We must show $\gamma_{uv}(]rb, b], M) < \infty$. Let $U \subseteq X_u$ be bounded. Recall $\mathcal{F}_{uv}(r) = \mathcal{F}_{vv}(r)$ by Proposition 1.1. Now if $T, S \in F_{uv}(]rb, b], U, M)$, and $T \neq S$, then there exists $j = j(S,T) \in \{0, \cdots, d\}$ with $|T^{-1}(S(x_j)) - x_j| \geq \varepsilon$, and thus $|S(x_j) - T(x_j)| \geq rb\varepsilon \operatorname{diam} U$. This choice of $j(S,T)$ is a "coloring" of all pairs from $F_{uv}(]rb, b], U, M)$ in $d+1$ colors. Ramsey's Theorem asserts that if $\sup_U \# F_{uv}(]rb, b], U, M) = \infty$, then $\sup_U \# F'_U = \infty$ as well, for some choice of $F'_U \subseteq F_{uv}(]rb, b], U, M)$ such that all pairs $T, S \in F'_U$ have the same color. But suppose all pairs in F'_U have color j. Then the balls $B(T(x_j), (rb\varepsilon/2) \operatorname{diam} U)$, $T \in F'_U$, are disjoint, and all their centers have distance at most $bk \operatorname{diam} U$ from U. So these balls are all contained in a ball of radius $(1 + bk + rb\varepsilon/2) \operatorname{diam} U$. Comparing volumes we see that $\# F'_U \leq (1 + bk + rb\varepsilon/2)^d / (rb\varepsilon/2)^d$, a bound independent of U. So in fact $\gamma_{uv}(]rb, b], M) = \sup_U \# F_{uv}(]rb, b], U, M) < \infty$.

(4a) $\Longrightarrow$ (1a): Assume (4a). For each $v \in V$, apply Corollary 2.3 to get generic $y_v \in K_v$. Now by (4a), for all $u, v \in V$ we have $\gamma_{uv}(]r, 1], \{y_v\}) < \infty$. There are finitely many u, v, so there is a single bound for all u, v. Now for $v \in V$ write $U_v = B(y_v, 1/2)$; for $S \in F_{uv}$ write $U_S = S(U_v) = B(S(y_v), \rho(S)/2)$. For $u, v \in V$ and $S \in F_{uv}$, define

$$I_{uv}(S) = \{\, T \in F_{uv}(]r\rho(S), \rho(S)]) : T(y_v) \in U_S \,\}.$$

Note $I_{uv}(S) = F_{uv}(]r, 1], U_S, \{y_v\})$, so $\sup_{u,v,S} \# I_{uv}(S) < \infty$.

Choose $u_0, v_0 \in V$, $T_0 \in F_{u_0 v_0}$ so that

$$\# I_{u_0 v_0}(T_0) = \sup \{\, \# I_{uv}(T) : u, v \in V, T \in F_{uv} \,\}. \tag{1}$$

For all $u \in V$ and all $T \in F_{uu_0}$ we claim $I_{uv_0}(TT_0) = TI_{u_0v_0}(T_0)$. We first prove $\supseteq$. Let $S \in I_{u_0v_0}(T_0)$, so that $r\rho(T_0) < \rho(S) \leq \rho(T_0)$ and $S(y_{v_0}) \in U_{T_0}$. Then $TS \in F_{uv_0}$, $r\rho(TT_0) < \rho(TS) \leq \rho(TT_0)$, and $T(S(y_{v_0})) \in T(U_{T_0}) = U_{TT_0}$. Thus $TS \in I_{uv_0}(TT_0)$. So we have $TI_{u_0v_0}(T_0) \subseteq I_{uv_0}(TT_0)$. Since T is one-to-one, we have $\#(TI_{u_0v_0}(T_0)) = \#I_{u_0v_0}(T_0) \geq \#I_{uv_0}(TT_0)$ by the maximality (1). So by counting, we conclude that the subset is the whole thing, that is

$$I_{uv_0}(TT_0) = TI_{u_0v_0}(T_0). \tag{2}$$

Now let $x_0 = T_0(y_{v_0})$, $x_0 \in K_{u_0}$. Let

$$\varepsilon_1 = \min \left\{ |T'(y_{v_0}) - x_0| : T' \in I_{u_0v_0}(T_0), T'(y_{v_0}) \neq x_0 \right\}.$$

This is positive since $I_{u_0v_0}(T_0)$ is finite. Let $\varepsilon_2 = \rho(T_0)/2$ and $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2\}$. Now we claim: for any $u \in V$ and $R \in \mathcal{F}_{uu_0}(r)$, either R is the identity or $|R(x_0) - x_0| \geq \varepsilon_0$. Now $R = T^{-1} \circ S$ for some b and some $T, S \in F_{uu_0}(]rb, b])$. We may assume $\rho(S) \leq \rho(T)$, since in the other case we may apply the following to $S^{-1} \circ T$ and note that $|T^{-1}(S(x_0)) - x_0| > |x_0 - S^{-1}(T(x_0))| \geq \varepsilon_0$.

First consider the case $ST_0 \in I_{uv_0}(TT_0)$. By (2), $ST_0 = TT'$ for some $T' \in I_{u_0v_0}(T_0)$. Hence $|T^{-1}(S(x_0)) - x_0| = |T^{-1}(ST_0(y_{v_0})) - x_0| = |T'(y_{v_0}) - x_0|$. Either this is $\geq \varepsilon_1$ or it is 0 and $T^{-1} \circ S$ is the identity.

Next consider the case $ST_0 \notin I_{uv_0}(TT_0)$. This means $ST_0(y_{v_0}) \notin U_{TT_0}$, or $|ST_0(y_{v_0}) - TT_0(y_{v_0})| \geq \rho(TT_0)/2$. Thus $|T^{-1}(S(x_0)) - x_0| \geq \rho(T_0)/2 = \varepsilon_2$.

Finally, we must show that the same thing holds for any vertex v in place of u_0. Because G is strongly connected, there is $S' \in F_{vu_0}$. Let $x = S'(x_0)$, so $x \in K_v$. Let $\varepsilon = \rho(S')\varepsilon_0$. Now suppose $R \in \mathcal{F}_{uv}(r)$. Then $R = T^{-1} \circ S$ for $T, S \in F_{uv}(]rb, b])$. Then $TS', SS' \in F_{uu_0}(]b\rho(S')r, b\rho(S')])$. And

$$\begin{aligned} |T^{-1}(S(x)) - x| &= |T^{-1}(SS'(x_0)) - S'(x_0)| \\ &= \rho(S')|(TS')^{-1}((SS')(x_0)) - x_0| \geq \rho(S')\varepsilon_0 = \varepsilon, \end{aligned}$$

unless $(TS')^{-1} \circ (SS')$ is the identity, and then $TS' = SS'$ and $T = S$, so R is also the identity.

(1b) $\Longrightarrow$ (2a): Assume (1b). Let $u \in V$ be given. By (1b) we get $x \in X_u$, and $\varepsilon > 0$. Because G is strongly connected, K_u is contained in the closure of the set $A = \left\{ S_\tau(x) : \tau \in E_{uu}^{(*)} \right\}$. Since K_u is in general position, so is A. So there exist $\{x_0, \cdots, x_d\} \subseteq A$ in general position. Say $x_j = S_{\tau_j}(x)$ for $0 \leq j \leq d$. Define $\varepsilon' = \varepsilon \min_j \rho(\tau_j)$. Let $R \in \mathcal{F}_{uu}(r)$ and $j \in \{0, \cdots, d\}$. Then $R = T^{-1} \circ S$ with $T, S \in F_{uu}(]rb, b])$ for some b. So $T \circ S_{\tau_j}, S \circ S_{\tau_j} \in F_{uu}(]rb\rho(\tau_j), b\rho(\tau_j)])$

and $(T \circ S_{\tau_j})^{-1} \circ (S \circ S_{\tau_j}) \in \mathcal{F}_{uu}(r)$, so

$$|T^{-1}(S(x_j)) - x_j| = \rho(\tau_j)\,|(T \circ S_{\tau_j})^{-1} \circ (S \circ S_{\tau_j})(x) - x| \geq \varepsilon'$$

if it is not zero.

(2b) $\Longrightarrow$ (3a): Assume (2b). Let $u \in V$ be given. Note that $\mathcal{R}_{uu}(]r, r^{-1}[)$ is an open neighborhood of the identity in $\mathcal{R}_{uu}$ and $\mathcal{F}_{uu}(r) = \mathcal{F}_{uu} \cap \mathcal{R}_{uu}(]r, r^{-1}[)$. Let $\{x_0, \cdots, x_d\}$ and ε be as in (2b). The set

$$\{\, R \in \mathcal{F}_{uu}(r) : |R(x_j) - x_j| < \varepsilon \text{ for all } j \,\} = \{\mathrm{id}\}$$

is an open neighborhood of the identity. So the identity is an isolated point of $\mathcal{F}_{uu}$.

(3b) $\Longrightarrow$ (2b): Assume (3b). Let $u \in V$ be given. Then there exists a finite set $Y \subseteq X_u$ and $\varepsilon' > 0$ such that for all $R \in \mathcal{F}_{uu}(r) \setminus \{\mathrm{id}\}$, there is $y \in Y$ with $|R(y) - y| \geq \varepsilon'$. Let $\{x_0, \cdots, x_d\}$ be a set in general position. Then the map $R \mapsto (R(x_0), \cdots, R(x_d))$ is a homeomorphism since it is bijective and affine from one Euclidean space onto another [from the set of affine maps on $\mathbb{R}^d$ to $(\mathbb{R}^d)^{d+1}$]. So in particular for each y the value $R(y)$ is a continuous function of $(R(x_0), \cdots, R(x_d))$. Thus there exists $\varepsilon > 0$ so that for all $R \in \mathcal{F}_{uu}$, if $|R(x_j) - x_j| < \varepsilon$ for all j, then $|R(y) - y| < \varepsilon'$ for all $y \in Y$. $\square$

Definition. Let (S_e) be an IFS with G strongly connected and K_u in general position for all u. We say (S_e) satisfies the **weak separation property** (WSP) iff one of the equivalent conditions in the theorem holds.

Notes. For a graph that is not strongly connected, the conditions stated here need not all be equivalent. We intend to consider that case in a future paper.

The hypothesis of "general position" may be omitted in the following way. For each $u \in V$, let X_u be the smallest affine subspace that contains K_u. By strong connectivity, each K_u contains a similar copy of all the others, so all of these spaces X_u have the same dimension, and may therefore be identified with $\mathbb{R}^d$ for the same d. In practice, what this means is that for $e \in E_{uv}$, the maps S_e should be restricted to the subspace X_v.

3 The Open Set Condition

The IFS (S_e) satisfies the **open set condition** (OSC) iff there exist nonempty open sets $\Omega_u \subseteq X_u$ such that (i) for all $u, v \in V$ and $e \in E_{uv}$,

$$\Omega_u \supseteq S_e(\Omega_v)$$

and (ii) for all $u, v, v' \in V$, $e \in E_{uv}$, and $e' \in E_{uv'}$ with $e \neq e'$,

$$S_e(\Omega_v) \cap S_{e'}(\Omega_{v'}) = \varnothing.$$

We will say an IFS (S_e) **distinguishes paths** provided

$$\text{for all } u, v \in V \text{ and all } \sigma, \tau \in E_{uv}^{(*)}, \text{ if } \sigma \neq \tau, \text{ then } S_\sigma \neq S_\tau. \tag{3}$$

Proposition 3.1 *Let G be strongly connected, and K_u in general position for $u \in V$. Then OSC holds for (S_e) if and only if (S_e) has WSP and (S_e) distinguishes paths.*

PROOF. ($\Longrightarrow$) Suppose that the OSC holds, with open sets Ω_u. We claim (1b) holds. Let $u \in V$ and $r > 0$ be given. Choose any $x \in \Omega_u$. Then there is $\eta > 0$ so that $B(x, \eta) \subseteq \Omega_u$. Also $A = \left\{ \alpha \in E_{uu}^{(*)} : \rho(\alpha) \geq r \right\}$ is finite. Let

$$\eta' = \min \{ |S_\alpha(x) - x| : \alpha \in A, S_\alpha(x) \neq x \}.$$

Let $\varepsilon = \min\{\eta, \eta'\} > 0$. Now let $R \in \mathcal{F}_{uu}(r)$. We must show that either $R(x) = x$ or $|R(x) - x| \geq \varepsilon$. Now there is b so that $R = S_\tau^{-1} \circ S_\sigma$ for some $\tau, \sigma \in E_{uu}^{(*)}(]rb, b])$. Take three cases: (a) σ and τ are incomparable; (b) σ is a prefix of τ; (c) τ is a prefix of σ.

(a) Since σ and τ are incomparable, the two images $S_\sigma(\Omega_u), S_\tau(\Omega_u)$ are disjoint. So $S_\sigma(x)$ is not in the ball $S_\tau(B(x, \eta)) = B(S_\tau(x), \rho(\tau)\eta)$. Thus

$$|R(x) - x| = |S_\tau^{-1}(S_\sigma(x)) - x| = \rho(\tau)^{-1}|S_\sigma(x) - S_\tau(x)| \geq \eta.$$

(b) Say $\tau = \sigma\alpha$. Note $\rho(\alpha) = \rho(\tau)/\rho(\sigma) \geq r$, so $\alpha \in A$. Then

$$|R(x) - x| = |S_\tau^{-1}(S_\sigma(x)) - x| = |S_\alpha^{-1}(x) - x| = \rho(\alpha)^{-1}|x - S_\alpha(x)| \geq \eta'$$

if it is not zero.

(c) is similar to (b): $\sigma = \tau\alpha$, $\alpha \in A$,

$$|R(x) - x| = |S_\tau^{-1}(S_\sigma(x)) - x| = |S_\alpha(x) - x| \geq \eta'$$

if it is not zero.

Next suppose $\sigma, \tau \in E_{uv}^{(*)}$ and $\sigma \neq \tau$. Certainly if one is a prefix of the other then S_σ, S_τ have different contraction ratios, so $S_\sigma \neq S_\tau$. And if σ, τ are incomparable, then $S_\sigma(\Omega_v) \cap S_\tau(\Omega_v) = \varnothing$, so again $S_\sigma \neq S_\tau$.

Conversely, suppose that WSP holds and (S_e) distinguishes paths. Fix an $r \in]0,1]$ with $r \leq \rho_{\min}$. By (4a), for all $v \in V$ we have $\gamma_{uv}(]r,1], K_v) < \infty$. There are finitely many pairs u, v, so there is a single bound for all $\gamma_{uv}(]r,1], K_v)$. Now for $v \in V$ write

$$U_v = B(K_v, 1/2) = \left\{\, x \in X_v : \operatorname{dist}(x, K_v) < 1/2 \,\right\};$$

for $\sigma \in E_{uv}^{(*)}$ write $K_\sigma = S_\sigma(K_v)$ and $U_\sigma = S_\sigma(U_v) = B(K_\sigma, \rho(\sigma)/2)$. For $u, v \in V$ and $\sigma \in E_{uv}^{(*)}$, define

$$\begin{aligned} I_{uv}(\sigma) &= \left\{\, \tau \in E_{uv}^{(*)}(]r\rho(\sigma), \rho(\sigma)]) : K_\tau \cap U_\sigma \neq \varnothing \,\right\} \\ &= \left\{\, \tau : S_\tau \in F_{uv}(]r,1], U_\sigma, K_v) \,\right\}. \end{aligned}$$

By (3) we have $\#I_{uv}(\sigma) = \#F_{uv}(]r,1], U_\sigma, K_v)$, so $\sup_{u,v,\sigma} \#I_{uv}(\sigma) < \infty$.

Choose $u_0, v_0 \in V$, $\tau_0 \in E_{u_0v_0}^{(*)}$ so that

$$\#I_{u_0v_0}(\tau_0) = \sup\left\{\, \#I_{uv}(\tau) : u, v \in V, \tau \in E_{uv}^{(*)} \,\right\}. \tag{4}$$

For all $u \in V$ and all $\tau \in E_{uu_0}^{(*)}$ we claim $I_{uv_0}(\tau\tau_0) = \tau I_{u_0v_0}(\tau_0)$. We first prove $\supseteq$. Let $\sigma \in I_{u_0v_0}(\tau_0)$, so that $r\rho(\tau_0) < \rho(\sigma) \leq \rho(\tau_0)$ and $K_\sigma \cap U_{\tau_0} \neq \varnothing$. Then $\tau\sigma \in E_{uv_0}^{(*)}$, $r\rho(\tau\tau_0) < \rho(\tau\sigma) \leq \rho(\tau\tau_0)$, and $K_{\tau\sigma} \cap U_{\tau\tau_0} = S_\tau(S_\sigma(K_\sigma)) \cap S_\tau(U_{\tau_0}) \neq \varnothing$. Thus $\tau\sigma \in I_{uv_0}(\tau\tau_0)$. So we have $\tau I_{u_0v_0}(\tau_0) \subseteq I_{uv_0}(\tau\tau_0)$. So $\#(\tau I_{u_0v_0}(\tau_0)) = \#I_{u_0v_0}(\tau_0) \geq \#I_{uv_0}(\tau\tau_0)$ by the maximality (4). So by counting, we conclude that the subset is the whole thing, that is

$$I_{uv_0}(\tau\tau_0) = \tau I_{u_0v_0}(\tau_0). \tag{5}$$

Let $u \in V$, $e \in E_{uv}, e' \in E_{uv'}$, $e \neq e'$, $\tau \in E_{vu_0}^{(*)}$. We claim $\operatorname{dist}(K_{e'}, K_{e\tau\tau_0}) \geq \rho(e\tau\tau_0)/2$. Let $x \in K_{e'}$. Then, because $r \leq \rho_{\min}$, there is $w \in V$ and $\tau' \in E_{vw}^{(*)}$ so that $x \in K_{e'\tau'}$ and

$$r\rho(e\tau\tau_0) < \rho(e'\tau') \leq \rho(e\tau\tau_0).$$

Now by (5) we know that $e'\tau' \notin I_{uv_0}(e\tau\tau_0)$ since $e \neq e'$. So by the definition of $I_{uv_0}(e\tau\tau_0)$ we have $K_{e'\tau'} \cap U_{e\tau\tau_0} = \varnothing$. So $\operatorname{dist}(x, K_{e\tau\tau_0}) \geq \rho(e\tau\tau_0)/2$. This is true for all $x \in K_{e'}$, so

$$\operatorname{dist}\left(K_{e'}, K_{e\tau\tau_0}\right) \geq \frac{\rho(e\tau\tau_0)}{2}. \tag{6}$$

We are now ready to define the open sets for the OSC. Choose $x \in K_{\tau_0} \subseteq K_{u_0}$. For $v \in V$ and $\sigma \in E^{(*)}_{vu_0}$ write $G_\sigma = S_\sigma(B(x, \rho(\tau_0)/4)) = B(S_\sigma(x), \rho(\sigma\tau_0)/4)$. For all $u \in V$ define

$$\Omega_u = \bigcup_{\tau \in E^{(*)}_{uu_0}} G_\tau.$$

We claim that the OSC holds using these open sets.

Let $u, v \in V$ and $e \in E_{uv}$. We must show that $S_e(\Omega_v) \subseteq \Omega_u$. Let $y \in \Omega_v$. Then $y \in G_\tau$ for some $\tau \in E^{(*)}_{vu_0}$, so $e\tau \in E^{(*)}_{uu_0}$ and $S_e(y) \in S_e(G_\tau)) = G_{e\tau}$ so that $S_e(y) \in \Omega_u$.

Let $u, v, v' \in V$, $e \in E_{uv}$, $e' \in E_{uv'}$, $e \neq e'$. We must show that $S_e(\Omega_v) \cap S_{e'}(\Omega_{v'}) = \varnothing$. Suppose $y \in S_e(\Omega_v) \cap S_{e'}(\Omega_{v'})$. Then $y \in G_{e\tau}$ for some $\tau \in E^{(*)}_{vu_0}$ and $y \in G_{e'\tau'}$ for some $\tau' \in E^{(*)}_{v'u_0}$. Assume without loss of generality that $\rho(e'\tau') \leq \rho(e\tau)$. Then $z = S_{e\tau}(x) \in K_{e\tau\tau_0}$ with $|y - z| < \rho(e\tau\tau_0)/4$ and $z' = S_{e'\tau'} \in K_{e'\tau'\tau_0} \subseteq K_{e'}$ with $|y - z'| < \rho(e'\tau'\tau_0)/4$. So

$$|z - z'| < \frac{\rho(e\tau\tau_0)}{4} + \frac{\rho(e'\tau'\tau_0)}{4} \leq \frac{\rho(e\tau\tau_0)}{2},$$

and this contradicts (6). So, in fact, $S_e(\Omega_v) \cap S_{e'}(\Omega_{v'}) = \varnothing$. □

4 Similarity and Growth Dimensions

The **similarity dimension** α of the graph-directed IFS (S_e) is defined as follows [5]. For each $t \geq 0$ let $A(t)$ be a square matrix with rows and columns indexed by V, and the entry in row u column v is

$$\sum_{e \in E_{uv}} \rho_e^t.$$

Let $\Phi(t)$ be the spectral radius of $A(t)$. Then Φ is continuous, strictly decreasing, $\Phi(0) \geq 1$ and $\lim_{t\to\infty} \Phi(t) = 0$. So there is a unique $\alpha \in [0, \infty)$ with $\Phi(\alpha) = 1$. This α is called the **similarity dimension** of the IFS.

Suppose for each $u, v \in V$ we have a finite set L_{uv} of similitudes. Then we may consider this to be a new IFS with the same nodes V but new sets of edges. But still the above definition of similarity dimension makes sense. In particular, we will write α_b for the similarity dimension obtained from the sets $F_{uv}(]rb, b])$. That is, if matrix $A_b(t)$ has entry

$$\sum_{T \in F_{uv}(]rb,b])} \rho(T)^t$$

in row u column v, and its spectral radius is called $\Phi_b(t)$, then $\Phi_b(\alpha_b) = 1$.

The "growth dimension" β for the iterated function system (S_e) may be computed in several ways. Write

$$\begin{aligned}
F_{\bullet\bullet}(]a, b]) &= \bigcup_{u,v \in V} F_{uv}(]a, b]) \\
F^-_{uv}(b) &= \left\{\, S_\sigma : \sigma \in E^{(*)}_{uv}, \rho(\sigma) \le b < \rho(\sigma^-) \,\right\}. \\
F^-_{\bullet\bullet}(b) &= \bigcup_{u,v \in V} F^-_{uv}(b).
\end{aligned}$$

The following proof is adapted from the one-node case in [9].

Proposition 4.1 *Suppose G is strongly connected. Then there is a constant $r_0 > 0$ such that for all $r \in]0, r_0]$,*

$$\beta = \lim_{b \to 0} \frac{\log \# F_{\bullet\bullet}(]rb, b])}{-\log b}$$

exists and is independent of r. Also, for all $u, v \in V$ and all $r \in]0, r_0]$,

$$\beta = \lim_{b \to 0} \frac{\log \# F_{uv}(]rb, b])}{-\log b} = \lim_{b \to 0} \frac{\log \# F^-_{\bullet\bullet}(b)}{-\log b} = \lim_{b \to 0} \alpha_b.$$

PROOF. We will prove several claims.

(i) *Claim.* There is $u_0 \in V$ and $r_0 \in]0, \rho_{\min}]$ such that $\# F_{u_0u_0}(]r_0 b, b])$ increases as b decreases. For any $u \in V$ there exist $\sigma \in E^{(*)}_{uu}$ with $\rho(\sigma) \le \rho_{\min}$. Let

$$r_0 = \max \left\{\, \rho(\sigma) : \sigma \in E^{(*)}_{uu} \text{ for some } u \in V \text{ and } \rho(\sigma) \le \rho_{\min} \,\right\}.$$

Then let $u_0 \in V$ and $\sigma_0 \in E^{(*)}_{u_0u_0}$ be such that $r_0 = \rho(\sigma_0)$. Now if $T \in F_{u_0u_0}$ then $T \circ S_{\sigma_0} \in F_{u_0u_0}$ and $\rho(T \circ S_{\sigma_0}) = r_0 \rho(T)$. Also S_{σ_0} is bijective. So as b decreases, $\# F_{u_0u_0}(]r_0 b, b])$ increases.

(ii) *Claim.* For any $u, v, u', v' \in V$ there is $\gamma > 0$ such that for all $b > 0$ and all $r > 0$, we have $\#F_{uv}(]rb, b]) \leq \#F_{u'v'}(]rb\gamma, b\gamma])$. Since G is strongly connected, there exist $\sigma \in E^{(*)}_{u'u}$ and $\tau \in E^{(*)}_{vv'}$. Write $\gamma = \rho(\sigma)\rho(\tau)$. If $T \in F_{uv}(]rb, b])$, then $S_\sigma T S_\tau \in F_{u'v'}(]rb\gamma, b\gamma])$. Both S_σ and S_τ are bijective, so

$$\#F_{uv}(]rb, b]) \leq \#F_{u'v'}(]rb\gamma, b\gamma]).$$

(iii) *Claim.* For any $u, v \in V$ there is $c > 0$ such that if $cb_1 \geq b_2$ then $\#F_{uv}(]r_0b_1, b_1]) \leq \#F_{uv}(]r_0b_2, b_2])$. Choose γ so that

$$\#F_{uv}(]rb, b]) \leq \#F_{u_0u_0}(]rb\gamma, b\gamma]).$$

Choose γ' so that $\#F_{u_0u_0}(]rb, b]) \leq \#F_{uv}(]rb\gamma', b\gamma'])$. Let $c = \gamma\gamma'$.

$$\begin{aligned} \#F_{uv}(]r_0b_1, b_1]) &\leq \#F_{u_0u_0}(]r_0b_1\gamma, b_1\gamma]) \\ &\leq \#F_{u_0u_0}(]r_0b_2\gamma/c, b_2\gamma/c]) \leq \#F_{uv}(]r_0b_2, b_2]). \end{aligned}$$

(iv) *Claim.* There is $c > 0$ such that if $cb_1 \geq b_2$ then $\#F_{\bullet\bullet}(]r_0b_1, b_1]) \leq \#F_{\bullet\bullet}(]r_0b_2, b_2])$. Apply (iii) for each u, v, then take the minimum c.

(v) *Claim.* For $b_1, b_2 > 0$ we have

$$\#F_{\bullet\bullet}(]r_0b_1b_2, b_1b_2]) \leq 2\#F_{\bullet\bullet}(]r_0b_1, b_1])\#F_{\bullet\bullet}(]r_0b_2c, b_2c]).$$

If $\rho(\sigma) \in]r_0b_1b_2, b_1b_2]$, then write $\sigma = \sigma_1\sigma_2$ where $\rho(\sigma_1) \leq b_1 < \rho(\sigma_1^-)$. Then $\rho(\sigma_2) \in]r_0b_2, r_0^{-1}b_2]$. Now both b_2 and $r_0^{-1}b_2$ are $\geq b_2c$, so applying (iv) we get the inequality claimed.

(vi) *Claim.* The limit

$$\beta = \lim_{b \to 0} \frac{\log \#F_{\bullet\bullet}(]r_0b, b])}{-\log b}$$

exists. Write $H(b) = \#F_{\bullet\bullet}(]r_0b, b])$. So $H(b_1b_2) \leq 2H(b_1)H(b_2c)$. For $a, b \in]0, c[$, let $k = \lfloor \log b / \log(a/c) \rfloor + 1$, so that $b > (a/c)^k$. Then

$$H(b) \leq H\left(\frac{a^k}{c^{k-1}}\right) \leq 2H(a)H\left(\frac{a^{k-1}}{c^{k-2}}\right) \leq \cdots \leq 2^{k-1}H(a)^k \leq (2H(a))^k.$$

So

$$
\begin{aligned}
&H(b) \le (2H(a))^{\log(b)/\log(a/c)+1} \\
&H(b)^{-1/\log b} \le (2H(a))^{-1/\log(a/c)-1/\log b} \\
&\limsup_{b\to 0} H(b)^{-1/\log b} \le (2H(a))^{-1/\log(a/c)}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\limsup_{b\to 0} H(b)^{-1/\log b} &\le \inf_a \, (2H(a))^{-1/\log(a/c)} \\
&\le \liminf_{a\to 0} H(a)^{-1/\log(a/c)} = \liminf_{a\to 0} H(a)^{-1/\log a}.
\end{aligned}
$$

So $\lim_{b\to 0} H(b)^{-1/\log b}$ exists. Its logarithm is the limit claimed.

(vii) *Claim.* For all $r \in]0, r_0]$,

$$
\lim_{b\to 0} \frac{\log \#F_{\bullet\bullet}(]rb, b])}{-\log b} = \beta.
$$

Choose $k \in \mathbb{N}$ so that $r_0^k \le r$. Then for all b we have $]rb, b] \subseteq \bigcup_{1\le i\le k}]r_0^i b, r_0^{i-1} b]$. So

$$
\begin{aligned}
&F_{\bullet\bullet}(]r_0 b, b]) \subseteq F_{\bullet\bullet}(]rb, b]) \subseteq \bigcup_{1\le i\le k} F_{\bullet\bullet}(]r_0^i b, r_0^{i-1} b]) \\
&\#F_{\bullet\bullet}(]r_0 b, b]) \le \#F_{\bullet\bullet}(]rb, b]) \le \sum_{1\le i\le k} \#F_{\bullet\bullet}(]r_0^i b, r_0^{i-1} b]) \\
&\#F_{\bullet\bullet}(]r_0 b, b]) \le \#F_{\bullet\bullet}(]rb, b]) \le k \max_{1\le i\le k} \#F_{\bullet\bullet}(]r_0^i b, r_0^{i-1} b]) \\
&\log \#F_{\bullet\bullet}(]r_0 b, b]) \le \log \#F_{\bullet\bullet}(]rb, b]) \le \log k + \max_{1\le i\le k} \log \#F_{\bullet\bullet}(]r_0^i b, r_0^{i-1} b]) \\
&\frac{\log \#F_{\bullet\bullet}(]r_0 b, b])}{-\log b} \le \frac{\log \#F_{\bullet\bullet}(]rb, b])}{-\log b} \\
&\qquad\qquad \le \frac{\log k}{-\log b} + \max_{1\le i\le k} \frac{-\log(r_0^i b)}{-\log b} \, \frac{\log \#F_{\bullet\bullet}(]r_0^i b, r_0^{i-1} b])}{-\log(r_0^i b)}.
\end{aligned}
$$

Now as $b \to 0$, $\log k/(-\log b) \to 0$ and $\log b/\log(r_0^i b) \to 1$ for all i, so both extremes converge to

$$
\lim_{b\to 0} \frac{\log \#F_{\bullet\bullet}(]r_0 b, b])}{-\log b} = \beta
$$

and therefore the middle quantity also converges to β.

(viii) *Claim.* For any u, v,

$$\lim_{b\to 0} \frac{\log \#F_{uv}(]r_0 b, b])}{-\log b} = \beta.$$

By (ii) and (iii), there exists c so that for all $u, v, u'v'$ we have $\#F_{u'v'}(]r_0 b, b]) \leq \#F_{uv}(]r_0 bc, bc])$. So $\#F_{uv}(]r_0 b, b]) \leq \#F_{\bullet\bullet}(]r_0 b, b]) \leq (\#V)^2 \#F_{uv}(]r_0 bc, bc])$. Take logarithm, divide by $-\log b$ and let $b \to 0$ to get the result.

(ix) *Claim.* For any u, v, r,

$$\lim_{b\to 0} \frac{\log \#F_{uv}(]rb, b])}{-\log b} = \beta.$$

The same argument as (vii).

(x) *Claim.* For $r \in]0, \rho_{\min}]$ there is a constant C so that for all b_1, b_2, if $b_1 > b_2 \geq rb_1$, then $\#F^-_{\bullet\bullet}(b_2) \leq C\#F^-_{\bullet\bullet}(b_1)$. Let $C = \#\,\{\, S_\sigma : \rho(\sigma) \geq r\rho_{\min} \,\}$. It is enough to observe that every $T \in F^-_{\bullet\bullet}(b_2)$ can be written (perhaps not uniquely) as $T = T_1 T_2$ with $T_1 \in F^-_{\bullet\bullet}(b_1)$ and $\rho(T_2) \geq r\rho_{\min}$. If $T = S_\sigma$, write $\sigma = \sigma_1\sigma_2$ with $\rho(\sigma_1) \leq b_1 < \rho(\sigma_1^-)$, then $\rho(\sigma_2) = \rho(\sigma)/\rho(\sigma_1) \geq b_2\rho_{\min}/b_1 \geq r\rho_{\min}$.

(xi) *Claim.*

$$\lim_{b\to 0} \frac{\log \#F^-_{\bullet\bullet}(b)}{-\log b} = \beta.$$

Fix r and C as in (x), and s so that $1 > s \geq \rho_{\max} > 0$. Let $k \in \mathbb{N}$ be such that $s^k < r$. Then

$$F^-_{\bullet\bullet}(b) \subseteq F_{\bullet\bullet}(]rb, b]) \subseteq \bigcup_{i=0}^{k-1} F^-_{\bullet\bullet}(s^i b).$$

Now by (x), for $0 \leq i \leq k-1$ we have $\#F^-_{\bullet\bullet}(s^i b) \leq C\#F^-_{\bullet\bullet}(b)$, so

$$\#F^-_{\bullet\bullet}(b) \leq \#F_{\bullet\bullet}(]rb, b]) \leq kC\#F^-_{\bullet\bullet}(b).$$

Then as usual, take logarithm, divide by $-\log b$, and let $b \to 0$.

(xii) *Claim.* $\beta = \lim_{b\to 0} \alpha_b$. Let $t \in [0, \infty)$. The matrix $A(t)$ has entry

$$\sum_{T \in F_{uv}(]rb, r])} \rho(T)^t,$$

in row u column v. This lies between

$$(br)^t \# F_{uv}(]rb, r]) \quad \text{and} \quad b^t \# F_{uv}(]rb, r]).$$

Suppose $t < \beta$, so that there is $\delta > 0$ with $t + \delta < \beta$. Then for b close to 0 we have

$$t + \delta < \frac{\log \# F_{uv}(]rb, b])}{-\log b}$$

so $r^t b^{-\delta} < (br)^t \# F_{uv}(]rb, r])$. Now $r^t b^{-\delta} \to \infty$ as $b \to 0$, so all entries of the matrix $A(t)$ go to ∞. If b is close enough to 0 then all entries are > 1, so $\Phi(t) > 1$ and thus $t < \alpha_b$. This is true for all $t < \beta$, so we get $\beta \leq \liminf_b \alpha_b$.

Suppose $t > \beta$, so that there is $\delta > 0$ with $t - \delta > \beta$. Then for b close to 0 we have

$$t - \delta > \frac{\log \# F_{uv}(]rb, b])}{-\log b}$$

so $b^\delta > b^t \# F_{uv}(]rb, r])$. Now $b^\delta \to 0$ as $b \to 0$, so all entries of the matrix $A(t)$ go to 0. If b is close enough to 0 then all entries are $< 1/\# V$, so $\Phi(t) < 1$ and thus $t > \alpha_b$. This is true for all $t > \beta$, so we get $\beta \geq \limsup_b \alpha_b$.

Therefore $\lim_{b \to 0} \alpha_b = \beta$. $\square$

The growth dimension for the IFS provides an estimate for the dimension of the attractors K_u. If G is strongly connected, then each K_u contains a similar copy of all others, so they all have the same dimension. Here we will use "dim" for the upper box dimension. In fact, many types of dimension all coincide for the self-similar sets K_u, in particular the upper box dimension agrees with the lower box dimension, the packing dimension, the Hausdorff dimension.

The next three proofs are adapted from [9].

Theorem 4.2 *Suppose G is strongly connected. Let β be the growth dimension of the IFS (S_e). For any $u \in V$ we have* $\dim K_u \leq \beta$.

PROOF. Let $r \in]0, r_0]$, let $c = \max_{v \in V} \operatorname{diam} K_v$, and choose $x_v \in K_v$ for each v. Now $K_v \subseteq B(x_v, c)$, so

$$K_u = \bigcup_{v \in V} \bigcup_{T \in F_{uv}(]rb, b])} T(K_v) \subseteq \bigcup_{v \in V} \bigcup_{T \in F_{uv}(]rb, b])} B(T(x_v), cb).$$

Thus K_u is covered by at most $\#F_{\bullet\bullet}(]rb, b])$ sets of diameter $2cb$. This is true for all $b > 0$, so the upper box dimension $\dim K_u$ satisfies

$$\dim K_u \leq \limsup_{b\to 0} \frac{\log \#F_{\bullet\bullet}(]br, r])}{-\log(2cb)} = \beta. \qquad \square$$

Theorem 4.3 *Suppose G is strongly connected, K_u is in general position, and the IFS (S_e) has the WSP. Then for all $u \in V$, we have $\dim K_u = \beta$.*

PROOF. Let $v \in V$ and $r \in]0, \rho_{\min}]$. Choose $x \in K_v$ and $\varepsilon > 0$ as in (1a) and let $k = r\varepsilon/2$. So if $u \in V$ and $S, T \in F_{uv}(]rb, b])$ and $S \neq T$, then $|S(x) - T(x)| \geq rb\varepsilon = 2kb$. So in any cover of K_u by sets of diameter kb, the points $T(x)$ must lie in different sets. Thus such a cover must contain at least $\#F_{uv}(]rb, b])$ sets. So the upper box dimension satisfies

$$\dim K_u \geq \limsup_{b\to 0} \frac{\log \#F_{uv}(]rb, b])}{-\log b} = \beta. \qquad \square$$

The relation between the growth dimension β and the similarity dimension α is next. Recall that (S_e) **distinguishes paths** means

$$\text{for all } u, v \in V \text{ and all } \sigma, \tau \in E_{uv}^{(*)}, \text{ if } \sigma \neq \tau, \text{ then } S_\sigma \neq S_\tau.$$

Proposition 4.4 *Let G be strongly connected. In general $\beta \leq \alpha$. Equality holds if and only if (S_e) distinguishes paths.*

PROOF. Recall that the similarity dimention α is the exponent so that matrix $A(\alpha)$ has spectral radius 1. That is, by Perron-Frobenius, there exist $p_u > 0$ so that

$$\sum_{v\in V} \sum_{e\in E_{uv}} \rho_e^\alpha p_v = p_u$$

for all $u \in V$. Now for $b > 0$, write $E_{uv}^-(b) = \left\{ \sigma \in E_{uv}^{(*)} : \rho(\sigma) \leq b < \rho(\sigma^-) \right\}$. Then this forms a "cross-cut" of the forest of paths, so it follows that

$$\sum_{v\in V} \sum_{\sigma\in E_{uv}^-(b)} \rho(\sigma)^\alpha p_v = p_u$$

for all $u \in V$. Therefore 1 is the spectral radius for the matrix $A_b^-(\alpha)$ with entry

$$\sum_{\sigma\in E_{uv}^-(b)} \rho(\sigma)^\alpha$$

in row u column v. Now of course deleting repeated terms produces the matrix $A_b(\alpha)$ with entry in row u column v given by

$$\sum_{T \in F_{uv}^{-}(b)} \rho(T)^\alpha \leq \sum_{\sigma \in E_{uv}^{-}(b)} \rho(\sigma)^\alpha. \tag{7}$$

So the matrices are related $A_b(\alpha) \leq A_b^{-}(\alpha)$ entrywise, and therefore the spectral radius of $A_b(\alpha)$ is ≤ 1. So $\Phi_b(\alpha) \leq 1$ and thus $\alpha_b \leq \alpha$. Therefore $\beta = \lim_b \alpha_b \leq \alpha$.

In case (S_e) distinguishes paths, we have equality in (7), and therefore in the rest of the argument, so $\beta = \alpha$.

Conversely, suppose $S_\tau = S_\sigma$ for some $\sigma \neq \tau$. Let $b = \rho(\sigma) = \rho(S_\sigma)$ for such a pair. then for that b, the matrix with entry

$$\sum_{T \in F_{uv}^{-}(b)} \rho(T)^\alpha \tag{8}$$

in row u column v has at least one entry strictly smaller than matrix $A_b^{-}(\alpha)$. Because G is strongly connected, these matrices are irreducible, so we conclude the spectral radius of (8) is < 1. The IFS with maps $F_{uv}^{-}(b)$ then has similarity dimension strictly less than α. But the previous reasoning still shows $\beta \leq$ that dimension. So we have $\beta < \alpha$. $\square$

5 Finite Type

Another way has been proposed for computing the dimension for overlapping iterated function systems in certain cases, known as "finite type" in Ngai & Wang [6]. This has also been adapted to graph-directed IFSs by Das & Ngai [2]. For one-node IFSs, Nguyen [7] showed that finite type implies WSP. We will verify this for graph-directed IFSs here.

The actual definition for finite type will not be needed here. We mention just a few definitions and a consequence of the definition that we will use.

A new (infinite) graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined. Fix a value r with $0 < r \leq \rho_{\min}$. For $k \in \mathbb{N}$,

$$\begin{aligned} \mathcal{V}_k &= \left\{ \mathbf{v} = (S_\sigma, u, v, k) : u, v \in V, \sigma \in E_{uv}^{(*)}, \rho(\sigma) \leq r^k < \rho(\sigma^-) \right\}, \\ \mathcal{V} &= \bigcup_{k=0}^{\infty} \mathcal{V}_k. \end{aligned}$$

For notation: if $\mathbf{v} = (S_\sigma, u, v, k)$, write $S_{\mathbf{v}} = S_\sigma$. We will not need the definition of $\mathcal{E}$.

An **invariant system** of bounded open sets consists of a nonempty bounded open set $\Omega_u \subseteq X_u$, one for each node $u \in V$, such that $S_e(\Omega_v) \subseteq \Omega_u$ for all $e \in E_{uv}$ and all $u, v \in V$. Write $\mathbf{\Omega} = (\Omega_u)_{u \in V}$ for the system of open sets. For $\mathbf{u} = (S_\sigma, u, v, k)$, $\mathbf{u}' = (S_{\sigma'}, u', v', k) \in \mathcal{V}_k$, define $\mathbf{u}$ and $\mathbf{u}'$ are **neighbors** iff $u = u'$ and $S_\sigma(\Omega_v) \cap S_{\sigma'}(\Omega_{v'}) \neq \varnothing$. The neighborhood $\mathbf{\Omega}(\mathbf{u})$ is the set of all neighbors of $\mathbf{u}$.

From finite type we conclude: there is an invariant system $\mathbf{\Omega}$ and a bound $M < \infty$ so that $\#\mathbf{\Omega}(\mathbf{u}) \leq M$ for all $\mathbf{u}$ (see [2]). This is the only consequence of finite type we need in this proof. It is *not equivalent* to finite type (we will provide a counterexample elsewhere).

The proof for the following theorem is adapted from the one-node case in [7]. Alternatively, note that [2, Lemma 3.1] is a proof that finite type implies (4a).

Theorem 5.1 *Assume G is strongly connected and all K_v are in general position. Let $\mathbf{\Omega}$ be an invariant system of open sets, and let $r \in]0, \rho_{\min}]$. Assume (S_e) has finite type with respect to $\mathbf{\Omega}$ and r. Then (S_e) satisfies the weak separation property.*

Assume finite type with data $\mathbf{\Omega}$ and r. We will prove (5a). Fix $u, v, w \in V$, $z \in X_w$. Since any r is the same, use the one in the finite type. Let

$$\begin{aligned} M_1 &= \sup \left\{ \operatorname{diam} \Omega_{u'} : u' \in V \right\}, \\ M_2 &= \sup \left\{ |S_\tau(z) - x| : \tau \in E^{(*)}_{vw}, x \in \Omega_v \right\}. \end{aligned}$$

For future use, write $M_0 = 2M_1 + 2M_2 + 2$.

Lemma 5.2 *There exist $x_0 \in \Omega_v$ and $\delta > 0$ so that for all $b > 0$ and all $\sigma \in E^{(*)}_{uv}(]rb, b])$,*

$$S_\sigma(\Omega_v) \supseteq B(S_\sigma(x_0), \delta b).$$

PROOF. Take any $x_0 \in \Omega_v$, then choose $\delta > 0$ so that $B(x_0, r^{-1}\delta) \subseteq \Omega_v$. Now let $\sigma \in E^{(*)}_{uv}(]rb, b])$. We claim $B(S_\sigma(x_0), \delta b) \subseteq S_\sigma(B(x_0, r^{-1}\delta)) \subseteq S_\sigma(\Omega_v)$.

Note S_σ is a similitude with ratio $\rho(\sigma)$, so

$$B(S_\sigma(x_0), r^{-1}\delta\rho(\sigma)) = S_\sigma(B(x_0, r^{-1}\delta)).$$

Now σ satisfies $rb < \rho(\sigma) \leq b$, so

$$B(S_\sigma(x_0), \delta b) \subseteq B(S_\sigma(x_0), \delta r^{-1}\rho(\sigma)) \subseteq S_\sigma(\Omega_v). \qquad \square$$

For any $\tau \in E^{(*)}_{vw}$ and $b > 0$, consider a ball B of radius b in $X_u = \mathbb{R}^d$. Let

$$F = \left\{ S_{\sigma\tau}(z) : \sigma \in E^{(*)}_{uv}(]rb, b]) \right\} \cap B.$$

We are to show that there is an l, independent of b, B, τ, so that $\#F \leq l$. But it is enough to do it for b of the form $b = r^k$ since any interval of the type $]rb, b]$ is contained in at most two intervals of this form where b is a power of r. Say $b = r^k$.

Let

$$\widehat{F} = \{ \mathbf{v} = (S_\sigma, u, v, k) \in \mathcal{V} : S_{\sigma\tau}(z) \in F \}.$$

Then $\#\widehat{F} \geq \#F$. From finite type we get a bound M on the size of all neighborhoods.

Lemma 5.3 *There is $\widehat{G} \subseteq \widehat{F}$ such that $\#\widehat{G} \geq \#\widehat{F}/M$ and the family*

$$\left\{ S_{\mathbf{u}}(\Omega_v) : \mathbf{u} \in \widehat{G} \right\}$$

is pairwise disjoint.

PROOF. Take any $\mathbf{u}_1 \in \widehat{F}$ and consider

$$J(\mathbf{u}_1) = \left\{ \mathbf{u} \in \widehat{F} : S_{\mathbf{u}}(\Omega_v) \cap S_{\mathbf{u}_1}(\Omega_v) \neq \varnothing \right\}.$$

Then take $\mathbf{u}_2 \in \widehat{F} \setminus J(\mathbf{u}_1)$ and consider

$$J(\mathbf{u}_2) = \left\{ \mathbf{u} \in \widehat{F} : S_{\mathbf{u}}(\Omega_v) \cap S_{\mathbf{u}_2}(\Omega_v) \neq \varnothing \right\}.$$

Then take $\mathbf{u}_3 \in \widehat{F} \setminus (J(\mathbf{u}_1) \cup J(\mathbf{u}_2))$ and so on. Continuing until

$$\widehat{F} \setminus (J(\mathbf{u}_1) \cup \cdots \cup J(\mathbf{u}_m)) = \varnothing,$$

we obtain a set

$$\widehat{G} = \{\mathbf{u}_1, \cdots, \mathbf{u}_m\} \subseteq \widehat{F}.$$

By definition of *neighbor,* each $J(\mathbf{u}_i) \subseteq \mathbf{\Omega}(\mathbf{u}_i)$, and thus has at most M elements. So we get $\#\widehat{F} \leq mM$, or $\#\widehat{G} \geq \#\widehat{F}/M$. $\square$

Proposition 5.4 $\#F \leq M M_0^d \delta^{-d}$.

PROOF. Apply Lemma 5.2 to get x_0 and δ; then apply Lemma 5.3 to get $\widehat{G}$. For each $\mathbf{u} = (S_\sigma, u, v, k) \in \widehat{G}$, we have $B(S_\sigma(x_0), \delta b) \subseteq S_\sigma(\Omega_v)$, so these balls are disjoint. For any $\mathbf{u} = (S_\sigma, u, v, k) \in \widehat{G}$ and any $y \in S_\sigma(\Omega_v)$,

$$\begin{aligned} |y - S_{\sigma\tau}(z)| &\leq |y - S_\sigma(x_0)| + |S_\sigma(x_0) - S_\sigma(S_\tau(z))| \\ &\leq \rho(\sigma)\ M_1 + \rho(\sigma)\ M_2 \leq (M_1 + M_2)\rho(\sigma) \leq (M_1 + M_2)b. \end{aligned}$$

Now let

$$H = \bigcup_{\mathbf{u} \in \widehat{G}} B(S_{\mathbf{u}}(x_0), \delta b).$$

If $y_1, y_2 \in H$, then there exist $\mathbf{u}_1 = (S_{\sigma_1}, u, v, k), \mathbf{u}_2 = (S_{\sigma_2}, u, v, k) \in \widehat{G}$ with $|y_1 - S_{\sigma_1\tau}(z)| \leq (M_1 + M_2)b$, $|y_2 - S_{\sigma_2\tau}(z)| \leq (M_1 + M_2)b$. By the definition of F, both $S_{\sigma_1\tau}(z)$ and $S_{\sigma_2\tau}(z)$ are in the ball B, so their distance is at most $2b$. So the diameter of H is at most $(2M_1+2M_2+2)b = M_0 b$, so H is contained in a ball of radius $M_0 b$. So we have $\#\widehat{G}$ disjoint balls of radius δb contained inside one ball of radius $M_0 b$. Comparing volumes, we get $\#\widehat{G} \leq (M_0 b)^d/(\delta b)^d = M_0^d \delta^{-d}$. Therefore $\#F \leq \#\widehat{F} \leq M \#\widehat{G} \leq M M_0^d \delta^{-d}$. $\square$

This completes the proof of (5a).

Finite type. Finite type can be used as follows. Begin with an IFS consisting of similitudes, but failing the OSC. This means there are "overlaps" and it could happen that the attractors K_u have dimension strictly smaller than the similarity dimension α for the IFS. If the IFS has "finite type" then the construction provides a new (finite) "induced graph" $G_{\mathbf{\Omega}} = (V_{\mathbf{\Omega}}, E_{\mathbf{\Omega}})$. (This construction is in [6] for the one-node case, and [2] for the graph-directed case.) Even if G is a one-node graph, the result $G_{\mathbf{\Omega}}$ need not be. And we get a corresponding induced IFS. The attractors of the original IFS are finite unions of the attractors of the new IFS. We believe that the new IFS does satisfy the OSC (if it is interpreted properly in case $G_{\mathbf{\Omega}}$ is not strongly connected; we will deal with that case in a future paper). So the dimension of the original attractors may be computed as the similarity dimension for the induced IFS.

We had originally hoped to find cases where the finite type construction would yield new examples of IFSs with overlap that can be analyzed. But Theorem 5.1 shows that any dimension computed by the finite type construction also comes under the WSP. There could be cases where the finite type construction gives us a more explicit computation than the growth dimension, but it will not yield completely new cases.

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