Eigenvalues of a self-differential operator

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Abstract

In this paper we study the asymptotic behavior of the eigenvalues of the compact operator $Tf(x) = \int_0^1 \varphi(x,s)f(s)ds$, where $f \in C[0,1]$ with $f(0) = 0$, and

$$\varphi(x,s) = \begin{cases} 0, & \text{if } 1 > s > 2x \\ 1/2, & \text{if } 2x > s > 2x-1 \\ 1, & \text{if } 2x-1 > s > 0. \end{cases} \tag{1}$$

We show that the eigenvalues $\{A_n\}_{n=-\infty}^{\infty}$ of T satisfy

$$\lim_{n\to\pm\infty} A_n = 0, \qquad \lim_{n\to\infty} \frac{A_n}{A_{-n}} = -1, \qquad \lim_{n\to\infty} \frac{A_n}{A_{n-1}} = \frac{1}{4}, \tag{2}$$

and that A_n^{-1} are zeros of the power series

$$\sum_{m=0}^{\infty} \frac{B_m x^m}{2^{(m^2-m)/2} m!}$$

where B_m are the Bernoulli numbers. An asymptotic estimate of a partial theta funcion is involved in the proof.

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1 Introduction

A “self-differential” function is a function satisfying an equation relating the value of the derivative at a point to values of the function itself at other points. So in a sense the notion is a mixture of the conventional differential equation and the self-similar (or self-affine) function.

In previous work [5] we considered the very simple self-differential equations

$$\begin{cases} f'(x) = af(2x) & \text{if } 0 \le x \le \frac{1}{2} \\ f'(x) = af(2-2x) & \text{if } \frac{1}{2} \le x \le 1 \\ f(0) = 2-a, f(1) = 2+a \end{cases} \tag{1.1}$$

and

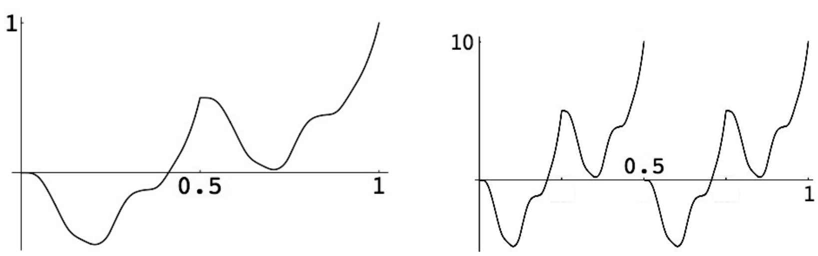
$$\begin{cases} f'(x) = af(2x) & \text{if } 0 \le x \le \frac{1}{2} \\ f'(x) = af(2-2x) & \text{if } \frac{1}{2} \le x \le 1 \\ f(0) = 0, f(1) = 0 \end{cases} \tag{1.2}$$

The main results were: Equation (1.2) has infinitely many solutions for values of $a \in S$ where $S = \left\{2^{2n+1} : n = 1, 2, ...\right\}$ while equation (1.1) has no solution for $a \in S$. When $a \notin S$ equation (1.1) has a unique solution and equation (1.2) does not have a solution.

In the present paper we will consider another simple functional differential equation:

$$\begin{cases} f'(x) = af(2x), & 0 \le x \le \frac{1}{2} \\ f'(x) = af(2x-1), & \frac{1}{2} \le x \le 1 \\ f(0) = 0, f(1) = d. \end{cases} \tag{$*$}$$

Without loss of generality we may assume $d = 1$. This was studied in [4] Chapter 3, Case 4. An example is in the figure: $a = 10.1952$, the graph of f is on the left, the graph of the derivative f' is on the right, discontinuous at $1/2$. The graph of f' consists of two affine images of the graph of f.



In [4] we showed that $(*)$ has a unique solution for values of a which are zeros of a function $G(x)$, and described $G(x)$ as a power series. The power series may be computed as $G(x) = \lim_{n\to\infty} G_n(x)$, where $G_n(x)$ is a polynomial of degree $n-1$ defined recursively: $G_1(x) = 1$ and

$$G_n(x) = G_{n-1}(x) - \sum_{k=1}^{n-1} \frac{x^k}{2^{nk-k/2-k^2/2+1}k!} G_{n-k}(x).$$

In Section 3 of this paper we will show that $G(x)$ is the power series

$$\sum_{M=0}^{\infty} \frac{B_M x^M}{2^{(M^2-M)/2} M!}$$

where B_M are the Bernoulli numbers.

The fact that the values of a for which $(*)$ has a solution are zeros of G is derived from the requirement that $f'(0) = af(0) = 0$. The smallest twelve zeros of G are:

$$\begin{array}{ll} a_1 = 10.1952 & a_{-1} = 2.53281 \\ a_2 = 37.9638 & a_{-2} = -52.1525 \\ a_3 = 145.083 & a_{-3} = -143.617 \\ a_4 = 571.596 & a_{-4} = -571.602 \\ a_5 = 2277.598 & a_{-5} = -2277.598 \\ a_6 = 9101.829 & a_{-6} = -9101.829 \end{array}$$

These values of a_n suggest that the zeros of $G(x)$ have the following pattern: $a_n \approx -a_{-n}$ and $a_{n+1} \approx 4a_n$ for large values of n. In Section 2 of this paper we will show that the solutions of $(*)$ are the eigenfunctions of the operator T, defined in the Abstract, with eigenvalues $A_n = 1/a_n$. The properties of a_n given above will follow from Theorem 2.1.

2 Eigenvalue Formulation

In this section we explain the relation between the eigenvalues of the operator T and the solutions of equation $(*)$.

Let f be a solution of $(*)$. The function f satisfies

$$f(x) = f(0) + \int_0^x f'(t)dt = \int_0^x af(2t)dt = \frac{a}{2}\int_0^{2x} f(s)ds \qquad (2.1)$$

for $0 \le x \le \frac{1}{2}$. From (2.1) with $x = \frac{1}{2}$ we obtain

$$f\left(\frac{1}{2}\right) = \frac{a}{2}\int_0^1 f(s)\,ds.$$

When $\frac{1}{2} \le x \le 1$ we have

$$\begin{aligned} f(x) &= f\left(\frac{1}{2}\right) + \int_{1/2}^{x} f'(t)\,dt = f\left(\frac{1}{2}\right) + \int_{1/2}^{x} af(2t-1)\,dt \\ &= \frac{a}{2}\int_0^1 f(s)\,ds + \frac{a}{2}\int_0^{2x-1} f(s)\,ds. \end{aligned} \tag{2.2}$$

From equations (2.1) and (2.2) the function f satisfies

$$f(x) = a\int_0^1 \varphi(x,s)f(s)\,ds$$

where $\varphi : [0,1]\times[0,1] \to \mathbb{R}$ is defined by (1) as in the Abstract. Define the operator T by

$$Tf(x) = \int_0^1 \varphi(x,s)f(s)\,ds.$$

The eigenfunctions of T are differentiable on the intervals $\left[0,\frac{1}{2}\right]$ and $\left[\frac{1}{2},1\right]$ and satisfy equations (2.1) and (2.2). Their derivative is discontinuous at $x = \frac{1}{2}$ where the derivative from the left is ad and the derivative from the right is 0. The solutions of equation $(*)$ satisfy $Tf(x) = \frac{1}{a}f(x)$ and the eigenfuncions of T are solutions of $(*)$.

Then by [4], the operator T has eigenvalues $\{A_n\}_{n=-\infty}^{\infty}$ where $A_n = 1/a_n$ and a_n are the zeros of G.

Theorem 2.1. *The eigenvalues of T satisfy*

(i) $\left|A_n - \dfrac{\sqrt{2}}{\pi 4^n}\right| < \dfrac{1}{8^{n+1}},$ *for n sufficiently large positive integer.*

(ii) $\left|A_{-n} + \dfrac{\sqrt{2}}{\pi 4^{-n}}\right| < \dfrac{1}{8^{1-n}},$ *for $n < 0$ and $|n|$ sufficiently large.*

The proof is in Section 5. Properties (2) of the eigenvalues of T follow from Theorem 2.1.

3 The Coefficients of G

We now evaluate the coefficients b_n of the power series $G(x) = \sum_{n=0}^{\infty} b_n x^n$. From [4], $G(x)$ is the limit of the sequence of polynomials $G_n(x)$ where

$$G_1(x) = 1 \quad \text{and} \quad G_n(x) = G_{n-1}(x) - \sum_{k=1}^{n-1} \frac{x^k}{2^{nk-k/2-k^2/2+1}k!}G_{n-k}(x).$$

Write this as

$$G_n(x) = G_{n-1}(x) - \sum_{k=1}^{n-1} b(n,k) x^k G_{n-k}(x) \tag{3.1}$$

where

$$b(n,k) = \frac{1}{2^{nk-k/2-k^2/2+1} k!}$$

for $1 \le k < n$. Writing $G_n(x) = c_{n,0} + c_{n,1}x + c_{n,2}x^2 + \cdots + c_{n,n-1}x^{n-1}$ and comparing the coefficients of x^M on both sides of (3.1) we obtain

$$c_{n,M} = c_{n-1,M} - \sum_{k=1}^{M} b(n,k) c_{n-k,M-k}. \tag{3.2}$$

Also $c_{n,0} = 1$ because the constant term of the polynomial G_n is 1. Now

$$\begin{aligned} c_{n,1} &= c_{n-1,1} - b(n,1)c_{n-1,0} = c_{n-1,1} - b(n,1) \\ &= c_{1,1} - \sum_{k=2}^{n} b(k,1) = -\sum_{k=2}^{n} \frac{1}{2^k} = -\frac{1}{2}\left(1 - \frac{1}{2^{n-1}}\right) \end{aligned}$$

because $c_{1,1} = 0$. Then $b_0 = \lim_{n\to\infty} c_{n,0} = 1$ and $b_1 = \lim_{n\to\infty} c_{n,1} = -\frac{1}{2}$. Using (3.2) and letting $n \to \infty$ we obtain for the coefficients b_M of $G(x)$:

$$b_M = \sum_{n_i,k_i,t} (-1)^t b(n_1,k_1) b(n_2,k_2) b(n_3,k_3) \cdots b(n_t,k_t), \tag{3.3}$$

where the sum is over all integers n_i, k_i and t satisfying

$$\begin{cases} 1 \le t \le M \\ k_1 + \cdots + k_t = M \\ k_i \ge 1 & \text{for} \quad i = 1, \ldots, t \\ n_t \ge k_t + 1 \\ n_i \ge n_{i+1} + k_i & \text{for} \quad i = 1, \ldots, t-1 \end{cases} \tag{3.4}$$

Let

$$h(t,M) = \sum_{n_i,k_i} b(n_1,k_1) b(n_2,k_2) b(n_3,k_3) \cdots b(n_t,k_t)$$

where n_i and k_i satisfy (3.4) with a fixed value of t. From equation (3.3)

$$b_M = \sum_{t=1}^{M} (-1)^t h(t,M). \tag{3.5}$$

Next we prove some formulas.

Claim 3.1. *Let* $s_m = \sum_{i=1}^m k_i$, $\sigma_m = \sum_{i=1}^m s_i$, *and*

$$u_m = \sigma_m + \frac{s_m}{2} - \frac{s_m^2}{2} - m.$$

Then

$$\sum_{n_m=n_{m+1}+k_m}^{\infty} \sum_{n_{m-1}=n_m+k_{m-1}}^{\infty} \cdots \sum_{n_2=n_3+k_2}^{\infty} \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^{m} b(n_i, k_i) \right) = \frac{N_m}{D_m} \tag{3.6}$$

where

$$N_m = 2^{u_m - n_{m+1} s_m}, \quad \text{and} \quad D_m = \prod_{i=1}^{m} (2^{s_i} - 1) k_i!$$

Proof. We use induction on m. When $m = 1$ we must show that

$$\sum_{n=n_2+k_1}^{\infty} b(n,k) = \frac{2^{u_1 - n_2 s_1}}{(2^{s_1} - 1)k_1!} = \frac{2^{(3/2 - n_2)k_1 - k_1^2/2 - 1}}{k_1!(2^{k_1} - 1)}$$

because $s_1 = \sigma_1 = k_1$ and $u_1 = (3/2)k_1 - k_1^2/2 - 1$. But

$$\begin{aligned} \sum_{n=n_2+k_1}^{\infty} b(n,k) &= \sum_{n=n_2+k_1}^{\infty} \frac{1}{2^{n_1 k_1 - k_1^2/2 - k_1/2 + 1} k_1!} \\ &= \frac{2^{k_1^2/2 + k_1/2 - 1}}{k_1!} \sum_{n=n_2+k_1}^{\infty} \frac{1}{2^{n_1 k_1}} \\ &= \frac{2^{k_1^2/2 + k_1/2 - 1}}{k_1!} \frac{1}{2^{k_1(n_2+k_1)}} \frac{2^{k_1}}{2^{k_1} - 1} \\ &= \frac{2^{(3/2 - n_2)k_1 - k_1^2/2 - 1}}{k_1!(2^{k_1} - 1)}, \end{aligned}$$

as required. Now suppose that (3.6) holds for $m - 1$:

$$\sum_{n_{m-1}=n_m+k_{m-1}}^{\infty} \cdots \sum_{n_2=n_3+k_2}^{\infty} \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^{m-1} b(n_i, k_i) \right) = \frac{N_{m-1}}{D_{m-1}}.$$

Then for m we have

$$\begin{aligned} &\sum_{n_m=n_{m+1}+k_m}^{\infty} \sum_{n_{m-1}=n_m+k_{m-1}}^{\infty} \cdots \sum_{n_2=n_3+k_2}^{\infty} \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^{m} b(n_i, k_i) \right) \\ &= \sum_{n_m=n_{m+1}+k_m}^{\infty} b(n_m, k_m) \left[\sum_{n_{m-1}=n_m+k_{m-1}}^{\infty} \cdots \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^{m} b(n_i, k_i) \right) \right] \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n_m=n_{m+1}+k_m}^{\infty} b(n_m,k_m)\frac{N_{m-1}}{D_{m-1}} \\
&= \sum_{n_m=n_{m+1}+k_m}^{\infty} \frac{2^{u_{m-1}-n_m s_{m-1}}}{2^{n_m k_m - k_m^2/2 - k_m/2+1} k_m! D_{m-1}} \\
&= \frac{2^{u_{m-1}} 2^{k_m^2/2+k_m/2-1}}{k_m! D_{m-1}} \sum_{n_m=n_{m+1}+k_m}^{\infty} \frac{1}{2^{n_m(k_m+s_{m-1})}} \\
&= \frac{2^{u_{m-1}+k_m^2/2+k_m/2-1}}{k_m! D_{m-1}} \sum_{n_m=n_{m+1}+k_m}^{\infty} \frac{1}{2^{n_m s_m}} \\
&= \frac{2^{\sigma_{m-1}+s_{m-1}/2-s_{m-1}^2/2-m+1+k_m^2/2+k_m/2-1}}{k_m! D_{m-1}} \frac{1}{2^{(n_{m+1}+k_m)s_m}} \frac{2^{s_m}}{2^{s_m}-1} \\
&= \frac{2^{\sigma_{m-1}+s_m/2-s_{m-1}^2/2-m+\frac{k_m^2}{2}+s_m-k_m s_m-n_{m+1}s_m}}{(2^{s_m}-1)k_m! D_{m-1}} \\
&= \frac{2^{\sigma_m+s_m/2-m-s_m^2/2-n_{m+1}s_m}}{D_m} = \frac{2^{u_m-n_{m+1}s_m}}{D_m} = \frac{N_m}{D_m},
\end{aligned}
$$

as required. This completes the induction. □

Claim 3.2.

$$
\sum_{n_m=k_m+1}^{\infty} \sum_{n_{m-1}=m_m+k_{m-1}}^{\infty} \cdots \sum_{n_2=m_3+k_2}^{\infty} \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^{m} b(n_i,k_i) \right)
= \frac{2^{\sigma_m-s_m/2-m-s_m^2/2}}{\prod_{i=1}^{m}(2^{s_i}-1)k_i!}
$$

Proof.

$$
\begin{aligned}
&\sum_{n_m=k_m+1}^{\infty} \sum_{n_{m-1}=m_m+k_{m-1}}^{\infty} \cdots \sum_{n_2=m_3+k_2}^{\infty} \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^{m} b(n_i,k_i) \right) \\
&\quad= \sum_{n_m=k_m+1}^{\infty} b(n_m,k_m) \left[\sum_{n_{m-1}=m_m+k_{m-1}}^{\infty} \cdots \sum_{n_1=n_2+k_1}^{\infty} \left(\prod_{i=1}^{m} b(n_i,k_i) \right) \right] \\
&\quad= \sum_{n_m=k_m+1}^{\infty} b(n_m,k_m) \frac{2^{u_{m-1}-n_m s_{m-1}}}{\prod_{i=1}^{m-1}(2^{s_i}-1)k_i!} \\
&\quad= \frac{2^{u_{m-1}}}{\prod_{i=1}^{m-1}(2^{s_i}-1)k_i!} \sum_{n_m=k_m+1}^{\infty} \frac{2^{-n_m s_{m-1}}}{k_m! 2^{n_m k_m - k_m^2/2 - k_m/2+1}} \\
&\quad= \frac{2^{\sigma_{m-1}+s_{m-1}/2-s_{m-1}^2/2-(m-1)+k_m^2/2+k_m/2-1}}{k_m! \prod_{i=1}^{m-1}(2^{s_i}-1)k_i!} \sum_{n_m=k_m+1}^{\infty} \frac{1}{2^{n_m s_m}}
\end{aligned}
$$

$$= \frac{2^{\sigma_{m-1}+s_m/2-m-s_{m-1}^2/2+k_m^2/2}}{k_m!\prod_{i=1}^{m-1}(2^{s_i}-1)k_i!}\frac{1}{2^{(k_m+1)s_m}}\frac{2^{s_m}}{2^{s_m}-1} = \frac{2^{\sigma_m-s_m/2-m-s_m^2/2}}{\prod_{i=1}^{m}(2^{s_i}-1)k_i!}$$

□

Claim 3.3.

$$h(t,M) = \frac{1}{P(M)}\sum_{k_1=1}^{M-t+1}\sum_{k_2=1}^{M-s_1-t+2}\cdots\sum_{k_{t-1}=1}^{M-s_{t-2}-1}\left[\binom{M}{k_1\ \ldots\ k_t}\prod_{i=1}^{t-1}\frac{2^{s_i-1}}{2^{s_i}-1}\right]$$

where $P(M) = M!(2^M-1)2^{M^2/2-M/2+1}$.

Proof. Recall the definition

$$h(t,M) = \sum_{n_i,k_i} b(n_1,k_1)b(n_2,k_2)b(n_3,k_3)\cdots b(n_t,k_t)$$

where the numbers n_i and k_i satisfy

$$\begin{cases} k_1+\cdots+k_t = M \\ k_i \geq 1 \qquad \text{for} \quad i=1,\ldots,t \end{cases} \tag{3.7}$$

and

$$\begin{cases} n_t \geq k_t+1 \\ n_i \geq n_{i+1}+k_i \qquad \text{for} \quad i=1,\ldots,t-1 \end{cases} \tag{3.8}$$

So

$$h(t,M) = \sum_{k_i}\left[\sum_{n_t=k_t+1}^{\infty}\sum_{n_{m-1}=n_t+k_{t-1}}^{\infty}\cdots\sum_{n_2=n_3+k_2}^{\infty}\sum_{n_1=n_2+k_1}^{\infty}\left(\prod_{i=1}^{t} b(n_i,k_i)\right)\right]$$

The first sum is over all $\{k_i\}_{i=1}^t$ which satisfy (3.7). From Claim 3.2:

$$h(t,M) = \sum_{k_i}\left[\frac{2^{\sigma_t-s_t/2-t-s_t^2/2}}{\prod_{i=1}^{t}(2^{s_i}-1)k_i!}\right] = \sum_{k_i}\left[\frac{1}{\prod_{i=1}^{t}k_i!}\frac{2^{\sigma_{t-1}+s_t/2-t-s_t^2/2}}{(2^{s_t}-1)\prod_{i=1}^{t-1}(2^{s_i}-1)}\right]$$

$$= \sum_{k_i}\left[\frac{M!}{\prod_{i=1}^{t}k_i!}\frac{2^{s_t/2-1-s_t^2/2}}{M!(2^{s_t}-1)}\frac{2^{\sigma_{t-1}-t+1}}{\prod_{i=1}^{t-1}(2^{s_i}-1)}\right]$$

We have that $s_t = M$. Then

$$h(t,M) = \sum_{k_i}\left[\binom{M}{k_1\ k_2\ \ldots\ k_t}\frac{1}{M!(2^M-1)\left(2^{M^2/2-M/2+1}\right)}\frac{\prod_{i=1}^{t-1}2^{s_i-1}}{\prod_{i=1}^{t-1}(2^{s_i}-1)}\right]$$

$$= \frac{1}{P(M)}\sum_{k_1=1}^{M-t+1}\sum_{k_2=1}^{M-s_1-t+2}\cdots\sum_{k_{t-1}=1}^{M-s_{t-2}-1}\left[\binom{M}{k_1\ \ldots\ k_t}\prod_{i=1}^{t-1}\frac{2^{s_i-1}}{2^{s_i}-1}\right].$$

□

Denote

$$c_M = M!(2^M - 1)2^{M^2/2-M/2+1}b_M$$
$$H(t,M) = M!(2^m - 1)2^{M^2/2-M/2+1}h(t,M).$$

Then $c_0 = 1$ and $c_1 = -1$. From Claim 3.3 we have

$$H(t,M) = \sum_{k_1=1}^{M-t+1} \sum_{k_2=1}^{M-s_1-t+2} \cdots \sum_{k_{t-1}=1}^{M-s_{t-2}-1} \left[\binom{M}{k_1\ k_2\ \dots\ k_t} \prod_{i=1}^{t-1} \frac{2^{s_i-1}}{2^{s_i}-1} \right]$$

so equation (3.5) becomes

$$c_M = \sum_{i=1}^{M} (-1)^i H(i,M) \tag{3.9}$$

Claim 3.4. *A recursive formula for $H(t,M)$:*

$$H(t,M) = \sum_{k=t-1}^{M-1} \binom{M}{k} \frac{2^{k-1}}{2^k - 1} H(t-1,k)$$

Proof. Substitute $k_1 = s_1$ and $k_i = s_i - s_{i-1}$ in the above formula for H.

$$\begin{aligned} H(t,M) &= \sum_{s_1=1}^{M-t+1} \sum_{s_2=s_1+1}^{M-t+2} \cdots \sum_{s_{t-1}=s_{t-2}+1}^{M-1} \left[\binom{M}{k_1\ k_2\ \dots\ k_t} \prod_{i=1}^{t-1} \frac{2^{s_i-1}}{2^{s_i}-1} \right] \\ &= \sum_{s_1=1}^{M-t+1} \sum_{s_2=2}^{M-t+2} \sum_{s_3=3}^{M-t+3} \cdots \sum_{s_{t-1}=t-1}^{M-1} \left[\binom{M}{k_1\ k_2\ \dots\ k_t} \prod_{i=1}^{t-1} \frac{2^{s_i-1}}{2^{s_i}-1} \right] \end{aligned}$$

where the numbers s_i satisfy $1 \le s_1 < s_2 < s_3 < \cdots < s_{t-1} < M$. Then $H(t,M) =$

$$\sum_{s_1=1}^{M-t+1} \sum_{s_2=2}^{M-t+2} \cdots \sum_{s_{t-2}=t-2}^{M-2} \left(\sum_{s_{t-1}=t-1}^{M-1} \left[\frac{M!}{s_{t-1}!k_t!} \frac{s_{t-1}!}{k_1!k_2!\cdots k_{t-1}!} \prod_{i=1}^{t-1} \frac{2^{s_i-1}}{2^{s_i}-1} \right] \right)$$

We have that $k_t + s_{t-1} = s_t = M$. So $H(t,M) =$

$$\sum_{s_{t-1}=t-1}^{M-1} \binom{M}{s_{t-1}} \frac{2^{s_{t-1}-1}}{2^{s_{t-1}}-1} \left(\sum_{s_1=1}^{M-t+1} \cdots \sum_{s_{t-2}=t-2}^{M-2} \binom{s_{t-1}}{k_1\ k_2 \cdots k_{t-1}} \prod_{i=1}^{t-2} \frac{2^{s_i-1}}{2^{s_i}-1} \right)$$

where $1 \le s_1 < s_2 < s_3 < \cdots < s_{t-2} < s_{t-1}$. Hence

$$H(t,M) = \sum_{s_{t-1}=t-1}^{M-1} \binom{M}{s_{t-1}} \frac{2^{s_{t-1}-1}}{2^{s_{t-1}}-1} H(t-1, s_{t-1})$$

By substituting $k = s_{t-1}$ we obtain

$$H(t, M) = \sum_{k=t-1}^{M-1} \binom{M}{k} \frac{2^{k-1}}{2^k - 1} H(t-1, k) \tag{3.10}$$

as claimed. □

Claim 3.5. *A recursive formula for* c_n*:*

$$c_n = -1 - \sum_{k=1}^{n-1} \frac{2^{k-1}}{2^k - 1} \binom{n}{k} c_k$$

Proof. We have that $H(1, n) = P(n)h(1, n) = n!(2^n - 1)2^{n^2/2 - n/2 + 1} h(1, n)$. Then

$$H(1, n) = P(n) \sum_{i=n+1}^{\infty} b(i, n) = n!(2^n - 1)2^{n^2/2 - n/2 + 1} \sum_{i=n+1}^{\infty} \frac{1}{2^{in - n^2/2 - n/2 + 1} n!}$$

$$= (2^n - 1)2^{n^2/2} \sum_{i=n+1}^{\infty} \frac{1}{2^{in}} = (2^n - 1)2^{n^2/2} \frac{1}{2^{n(n+1)}} \frac{2^n}{2^n - 1} = 1$$

From (3.9)

$$c_n = -1 + \sum_{t=2}^{n} (-1)^t H(t, n) = -1 + \sum_{t=2}^{n} \sum_{k=t-1}^{n-1} (-1)^t \binom{n}{k} \frac{2^{k-1}}{2^k - 1} H(t-1, k)$$

By substituting $s = t - 1$ then changing the order of summation we obtain

$$\begin{aligned} c_n &= -1 + \sum_{s=1}^{n} \sum_{k=s}^{n-1} (-1)^{s-1} \binom{n}{k} \frac{2^{k-1}}{2^k - 1} H(s, k) \\ &= -1 + \sum_{s=1}^{n} (-1) \binom{n}{k} \frac{2^{k-1}}{2^k - 1} \sum_{k=s}^{n-1} (-1)^s H(s, k) \\ &= -1 - \sum_{k=1}^{n-1} \binom{n}{k} \frac{2^{k-1}}{2^k - 1} \sum_{s=1}^{k} (-1)^s H(s, k) = -1 - \sum_{k=1}^{n-1} \frac{2^{k-1}}{2^k - 1} \binom{n}{k} c_k \end{aligned}$$

□

Bernoulli Numbers

The sequence c_n satisfies:

$$c_0 = 1; \qquad c_1 = -1; \qquad c_n = -1 - \sum_{k=1}^{n-1} \frac{2^{k-1}}{2^k - 1} \binom{n}{k} c_k \tag{3.11}$$

The first few terms:

$$c_2 = 1, \qquad c_3 = 0, \qquad c_4 = -1, \qquad c_5 = 0, \qquad c_6 = 3, \qquad c_7 = 0, \qquad \cdots$$

Recall the Bernoulli numbers B_n satisfy $B_0 = 1$,

$$\sum_{k=0}^{n} \binom{n}{k} B_k = B_n, \tag{3.12}$$

$$\sum_{k=0}^{n} \binom{n}{k} (2^k - 1) B_k = -(2^n - 1) B_n, \quad n \geq 2. \tag{3.13}$$

Proposition 3.6. *Let B_n be the Bernoulli numbers. Then $c_0 = 1$,*

$$c_n = 2(2^n - 1) B_n, \qquad n \geq 1 \tag{3.14}$$

is the sequence that satisfies (3.11).

Proof. First note that $c_1 = -1$ in (3.14). In (3.12), cancel B_n and use $B_0 = 1$ to get

$$\sum_{k=1}^{n-1} \binom{n}{k} B_k = -1$$

for $n \geq 2$. Then substitute (3.14) to get

$$\sum_{k=1}^{n-1} \frac{1}{2^k - 1} \binom{n}{k} c_k = -2.$$

Similarly from (3.13) we get

$$\sum_{k=1}^{n-1} \binom{n}{k} (2^k - 1) B_k = -2(2^n - 1) B_n, \qquad \sum_{k=1}^{n-1} \binom{n}{k} c_k = -2c_n.$$

Therefore

$$\begin{aligned} \sum_{k=1}^{n-1} \frac{2^{k-1}}{2^k - 1} \binom{n}{k} c_k &= \sum_{k=1}^{n-1} \frac{1}{2} \left(1 + \frac{1}{2^k - 1}\right) \binom{n}{k} c_k \\ &= \frac{1}{2} \sum_{k=1}^{n-1} \binom{n}{k} c_k + \frac{1}{2} \sum_{k=1}^{n-1} \frac{1}{2^k - 1} \binom{n}{k} c_k = -c_n - 1 \end{aligned}$$

Therefore c_n defined by (3.14) is the solution to (3.11). □

Thus we see our problem is related to the Bernoulli numbers in this way:

$$
\begin{aligned}
c_n &= 2(2^n-1)B_n, \\
b_n &= \frac{c_n}{n!(2^n-1)2^{n^2/2-n/2+1}} = \frac{B_n}{n!2^{n^2/2-n/2}}, \\
G(x) &= \sum_{m=0}^{\infty} b_m x^m = \sum_{m=0}^{\infty} \frac{B_m x^m}{2^{(m^2-m)/2} m!}
\end{aligned}
$$

4 Asymptotic Estimates for the Zeros of G

In Section 3 we determined the coefficients of the power series G:

$$G(x) = \sum_{M=0}^{\infty} \frac{B_M x^M}{2^{M^2/2-M/2} M!}$$

The Bernoulli numbers B_M satisfy: $B_0 = 1, B_1 = -\frac{1}{2}$ and

$$
\begin{aligned}
B_{2n+1} &= 0 \qquad \text{for } n \geq 1 \\
B_{2n} &= \frac{2(2n)!}{\pi^{2n}2^{2n}}(-1)^n \zeta(2n)
\end{aligned}
$$

where $\zeta(2n) = \sum_{k=1}^{\infty} 1/k^{2n}$ is the Riemann zeta function. Then

$$
\begin{aligned}
G(x) &= 1 - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{B_{2n} x^{2n}}{2^{2n^2-n}(2n)!} = 1 - \frac{x}{2} + \sum_{n=1}^{\infty} \frac{2(2n)!(-1)^n \zeta(2n) x^{2n}}{\pi^{2n} 2^{2n} 2^{2n^2-n}(2n)!} \\
&= 1 - \frac{x}{2} + 2\sum_{n=1}^{\infty} \frac{(-1)^n \zeta(2n) x^{2n}}{\pi^{2n} 2^{2n^2+n}} = 1 - \frac{x}{2} + 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{4^{n^2} 2^n \pi^{2n} k^{2n}}
\end{aligned}
$$

Define f, g and F by

$$
\begin{aligned}
f(x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{4^{n^2}} \\
g(x) &= \sum_{k=1}^{\infty} f\left(\frac{x}{k^2}\right) \qquad (4.1) \\
F(x) &= \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{x^n 4^{n^2}}
\end{aligned}
$$

Then writing $y = y(x) = x^2/(2\pi^2)$ we get

$$
\begin{aligned}
G(x) &= 1 - \frac{x}{2} + 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^n}{4^{n^2}} \left(\frac{y}{k^2}\right)^n \\
&= 1 - \frac{x}{2} - 2\sum_{k=1}^{\infty} f\left(\frac{y}{k^2}\right) = 1 - \frac{x}{2} - 2g(y)
\end{aligned}
$$

We have $y > 0$ and $x = \pm\pi\sqrt{2y}$. Define the functions G_1 and G_2 by

$$G_i(y) = 1 + (-1)^i \pi \sqrt{\frac{y}{2}} - 2g(y) \tag{4.2}$$

for $i = 1, 2$, so that $G(x) = G_1(y(x))$ when $x > 0$ and $G(x) = G_2(y(x))$ when $x < 0$.

Equation $(*)$ has a solution if $a = a_n$ where $G(a_n) = 0$. We number them so that $a_n > 0$ for $n > 0$ and $a_n < 0$ for $n < 0$ (except a_{-1}). Then

$$G_1\left(\frac{a_n^2}{2\pi^2}\right) = 0 \qquad \text{if } n > 0 \text{ and}$$
$$G_2\left(\frac{a_n^2}{2\pi^2}\right) = 0 \qquad \text{if } n < -1.$$

Technical Estimates

Claim 4.1. *The functions $f(x)$ and $F(x)$ satisfy*

(i) $0 < f(x) < \dfrac{x}{4}$ *for* $0 < x < 1$.

(ii) $0 < F(x) < \dfrac{1}{4x}$ *for* $x > 1$.

Proof. (i) First,

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{4^{n^2}} = \sum_{n=1}^{\infty} \frac{x^{2n-1}}{4^{(2n-1)^2}} \left(1 - \frac{x}{4^{4n-1}}\right)$$

Note $1 - x/4^{4n-1} > 0$ and so $f(x) > 0$. Next

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{4^{n^2}} = \frac{x}{4} + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{x^n}{4^{n^2}} = \frac{x}{4} - \sum_{n=1}^{\infty} \frac{x^{2n}}{4^{4n^2}} \left(1 - \frac{x}{4^{4n+1}}\right)$$

Then $f(x) < x/4$ because $1 - x/4^{4n+1} > 0$.

(ii) follows from (i) since $F(x) = f(1/x)$. □

Claim 4.2. *The function $f(x)$ satisfies the functional equations*

(i) $f(16x) = 4x - 4xf(x)$

(ii) $f'(16x) = \dfrac{1}{4} - \dfrac{f(x)}{4} - \dfrac{xf'(x)}{4}$

Proof. (i) $f(x) = \sum_{n=1}^{\infty}(-1)^{n-1}x^n/4^{n^2}$ so

$$\begin{aligned} f(16x) &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{16^n x^n}{4^{n^2}} = x\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{n-1}}{4^{n^2-2n}} \\ &= 4x - 4x\sum_{n=2}^{\infty}(-1)^{n-2}\frac{x^{n-1}}{4^{n^2-2n+1}} = 4x - 4x\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{4^{n^2}} \\ &= 4x - 4xf(x) \end{aligned}$$

(ii) By differentiating (i) we obtain: $16f'(16x) = 4 - 4f(x) - 4xf'(x)$. □

Define intervals I_n and J_n:

$$I_n = [4\cdot 16^n - 8^n, 4\cdot 16^n + 8^n], \quad J_n = [4\cdot 16^n + 8^n, 4\cdot 16^{n+1} - 8^{n+1}]$$

Claim 4.3. **(i)** $1 - x > e^{-2x}$ *for* $0 < x < 1/3$

(ii) $\frac{x^s}{4^{s^2}} < \frac{x^t}{4^{t^2}}$ *for* $x \in J_n$ *and* $1 \le s < t \le n+1$

Proof. (i) Let $h(x) = 1 - x - e^{-2x}$. Then $h'(x) = 2e^{-2x} - 1 = 0$ for $x = \frac{\ln 2}{2}$. The function $h(x)$ is increasing because $\frac{\ln 2}{2} > \frac{1}{3}$ so $h(x) > h(0) = 0$.

(ii) $x^{t-s} > (4\cdot 16^n)^{t-s} = (4^{2n+1})^{t-s} \ge 4^{(t+s)(t-s)} = 4^{t^2-s^2}$ □

Theta Function

We will use a theta function and the corresponding partial theta function with the notation:

$$\tau(p,q) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/2}p^n$$

$$\rho(p,q) = \sum_{n=0}^{\infty}(-1)^n q^{n(n-1)/2}p^n$$

The theta function satisfies the Jacobi triple product identity [8, Theorem 352]:

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n(n-1)/2}p^n = (q)_\infty (p)_\infty \left(\frac{q}{p}\right)_\infty$$

where

$$(q)_\infty = \prod_{n=1}^{\infty}(1-q^n),\ (p)_\infty = \prod_{n=1}^{\infty}(1-pq^{n-1}),\ \text{and}\ \left(\frac{q}{p}\right)_\infty = \prod_{n=1}^{\infty}\left(1-\frac{q^n}{p}\right)$$

When $q = 1/16$ and $p = x/4$ the theta function becomes

$$\tau\left(\frac{x}{4}, \frac{1}{16}\right) = \sum_{n=-\infty}^{\infty} (-1)^n \frac{x^n}{16^{n(n-1)/2}4^n} = \sum_{n=-\infty}^{\infty} (-1)^n \frac{x^n}{4^{n^2}}$$

and

$$(q)_\infty = \prod_{n=1}^{\infty}(1-q^n) = \prod_{n=1}^{\infty}\left(1-\frac{1}{16^n}\right)$$
$$(p)_\infty = \prod_{n=1}^{\infty}(1-pq^{n-1}) = \prod_{n=1}^{\infty}\left(1-\frac{x}{4\cdot 16^{n-1}}\right)$$
$$\left(\frac{q}{p}\right)_\infty = \prod_{n=1}^{\infty}\left(1-\frac{q^n}{p}\right) = \prod_{n=1}^{\infty}\left(1-\frac{4}{x\cdot 16^{n-1}}\right)$$

So the Jacobi triple product identity for $\tau(x/4, 1/16)$ yields

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{x^n}{4^{n^2}} = \prod_{n=1}^{\infty}\left(1-\frac{1}{16^n}\right)\prod_{n=1}^{\infty}\left(1-\frac{x}{4\cdot 16^{n-1}}\right)\prod_{n=1}^{\infty}\left(1-\frac{4}{x\cdot 16^{n-1}}\right)$$

Now we find a convenient expression for f. Note that $f(x) = \rho(x/4, 1/16) - 1$ where ρ is the partial theta function. Using our previous notation,

$$\sum_{n=-\infty}^{\infty} (-1)^n \frac{x^n}{4^{n^2}} = \sum_{n=-\infty}^{-1} (-1)^n \frac{x^n}{4^{n^2}} + 1 + \sum_{n=1}^{\infty}(-1)^n \frac{x^n}{4^{n^2}} = F(x) + 1 + f(x)$$

$$f(x) = \prod_{n=1}^{\infty}\left(1-\frac{1}{16^n}\right)\prod_{n=1}^{\infty}\left(1-\frac{x}{4\cdot 16^{n-1}}\right)\prod_{n=1}^{\infty}\left(1-\frac{4}{x\cdot 16^{n-1}}\right) - 1 - F(x)$$

From the triangle inequality we have the following estimates for $|f|$.

$$|f(x)| < \prod_{n=1}^{\infty}\left(1-\frac{1}{16^n}\right)\prod_{n=1}^{\infty}\left(1-\frac{x}{4\cdot 16^{n-1}}\right)\prod_{n=1}^{\infty}\left(1-\frac{4}{x\cdot 16^{n-1}}\right) + |F(x)| + 1$$

$$|f(x)| > \prod_{n=1}^{\infty}\left(1-\frac{1}{16^n}\right)\prod_{n=1}^{\infty}\left(1-\frac{x}{4\cdot 16^{n-1}}\right)\prod_{n=1}^{\infty}\left(1-\frac{4}{x\cdot 16^{n-1}}\right) - |F(x)| - 1$$

From Claim 4.1: $|F(x) < 1/(4x) < 1$ for $x > 1$. Then

$$|f(x)| < \prod_{n=1}^{\infty}\left(1-\frac{1}{16^n}\right)\prod_{n=1}^{\infty}\left(1-\frac{x}{4\cdot 16^{n-1}}\right)\prod_{n=1}^{\infty}\left(1-\frac{4}{x\cdot 16^{n-1}}\right) + 2 \quad (4.3)$$

$$|f(x)| > \prod_{n=1}^{\infty}\left(1-\frac{1}{16^n}\right)\prod_{n=1}^{\infty}\left(1-\frac{x}{4\cdot 16^{n-1}}\right)\prod_{n=1}^{\infty}\left(1-\frac{4}{x\cdot 16^{n-1}}\right) - 2 \quad (4.4)$$

We will also use the following inequalities

$$|f(16x)| \geq 4|x|(|f(x)|-1), \quad |f'(16x)| \geq \frac{1}{4}\,|x|\,|f'(x)| - \frac{1}{4}\,|f(x)| - \frac{1}{4} \tag{4.5}$$

$$|g(x)| \geq |f(x)| - \sum_{k=2}^{\infty}\left|f\left(\frac{x}{k^2}\right)\right|, \quad |g'(x)| \geq |f'(x)| - \sum_{k=2}^{\infty}\frac{1}{k^2}\left|f'\left(\frac{x}{k^2}\right)\right| \tag{4.6}$$

$$|G_i(x)| \geq 2|g(x)| - \pi\sqrt{\frac{x}{2}} - 1, \quad |G_i'(x)| \geq 2|g'(x)| - \frac{\pi}{2\sqrt{2x}} \tag{4.7}$$

Inequalities (4.5) follow from Claim 4.2 and the triangle inequality and (4.6) and (4.7) follow from the definitions (4.1), (4.2) of g and G_i. The main result of Section 4 is the following theorem.

Theorem 4.4. *The functions $G_i(x)$ have a unique zero in each interval I_n and have no zero on the intervals J_n, for $i = 1, 2$ and n sufficiently large.*

We prove Theorem 4.4 in Sections 4.1 and 4.2.

4.1 The Intervals $\mathbf{J_n}$

In Section 4.1 we show that the functions $G_1(x)$ and $G_2(x)$ have no zeros on the intervals J_n, for sufficiently large values of n.

Lemma 4.5. $|f(x)| > \dfrac{1}{100}\dfrac{x^n}{2^n 4^{n^2}}$ *for $x \in J_n$ and $n \geq 2$.*

Proof. First,

$$|f(x)| > \prod_{n=1}^{\infty}\left|1 - \frac{1}{16^n}\right| \prod_{n=1}^{\infty}\left|1 - \frac{x}{4\cdot 16^{n-1}}\right| \prod_{n=1}^{\infty}\left|1 - \frac{4}{x\cdot 16^{n-1}}\right| - 2$$

From Claim 4.3(i) we have $1-1/16^n > e^{-2/16^n}$ and $1-4/(x\cdot 16^n) > e^{-8/(x\cdot 16^n)}$. So

$$\prod_{n=1}^{\infty}\left(1 - \frac{1}{16^n}\right) > \prod_{n=1}^{\infty} e^{-2/16^n} = \exp\left(-\frac{2}{16}\sum_{n=0}^{\infty}\frac{1}{16^n}\right) = e^{-2/15} \tag{4.8}$$

$$\prod_{n=1}^{\infty}\left(1 - \frac{4}{x\cdot 16^n}\right) > \prod_{n=1}^{\infty} e^{-8/(x\cdot 16^n)} = \exp\left(-\frac{8}{16x}\sum_{n=0}^{\infty}\frac{1}{16^n}\right) = e^{-8/(15x)} \tag{4.9}$$

Now we estimate

$$\prod_{k=0}^{\infty}\left|1 - \frac{x}{4\cdot 16^k}\right| = \left[\prod_{k=0}^{n-1}\left|1 - \frac{x}{4\cdot 16^k}\right|\right] |h(x)| \left[\prod_{k=n+2}^{\infty}\left|1 - \frac{x}{4\cdot 16^k}\right|\right]$$

where

$$h(x) = \left(1 - \frac{x}{4\cdot 16^n}\right)\left(1 - \frac{x}{4\cdot 16^{n+1}}\right)$$

The function $|h(x)|$ has minimum on the interval J_n at $x = 4\cdot 16^n + 8^n$:

$$h(4\cdot 16^n + 8^n) = \left(1 - \frac{4\cdot 16^n + 8^n}{4\cdot 16^n}\right)\left(1 - \frac{4\cdot 16^n + 8^n}{4\cdot 16^{n+1}}\right) = \frac{7}{32\cdot 2^n}$$

Hence $|h(x)| \geq 7/(32\cdot 2^n)$ for $x \in J_n$.

$$\prod_{k=0}^{\infty}\left|1 - \frac{x}{4\cdot 16^k}\right| = \left[\prod_{k=0}^{n-1}\frac{x}{4\cdot 16^k}\prod_{k=0}^{n-1}\left|\frac{4\cdot 16^k}{x} - 1\right|\right]|h(x)|\prod_{k=n+2}^{\infty}\left|1 - \frac{x}{4\cdot 16^k}\right|$$

and

$$\prod_{k=0}^{n-1}\frac{x}{4\cdot 16^k} = \frac{x^n}{4^n 16^{(n-1)n/2}} = \frac{x^n}{4^n 4^{n(n-1)}} = \frac{x^n}{4^{n^2}}$$

Then

$$\prod_{k=0}^{\infty}\left|1 - \frac{x}{4\cdot 16^k}\right| > \frac{x^n}{4^{n^2}}\frac{7}{32\cdot 2^n}\prod_{k=0}^{n-1}\left|1 - \frac{4\cdot 16^k}{x}\right|\prod_{k=n+2}^{\infty}\left|1 - \frac{x}{4\cdot 16^k}\right| \tag{4.10}$$

We have

$$\frac{4\cdot 16^k}{x} < \frac{4\cdot 16^{n-1}}{4\cdot 16^n + 8^n} < \frac{1}{16} < \frac{1}{3}$$

for $0 \leq k \leq n-1$, and

$$\frac{x}{4\cdot 16^k} < \frac{4\cdot 16^{n+1} - 8^{n+1}}{4\cdot 16^{n+2}} < \frac{1}{16} < \frac{1}{3}$$

for $k \geq n+2$. From Claim 4.3(i) and (4.10)

$$\prod_{k=0}^{\infty}\left|1 - \frac{x}{4\cdot 16^k}\right| > \frac{7}{32}\frac{x^n}{2^n 4^{n^2}}\prod_{k=0}^{n-1}e^{-8\cdot 16^k/x}\prod_{k=n+2}^{\infty}e^{-2x/(4\cdot 16^k)}$$

$$\prod_{k=0}^{\infty}\left|1 - \frac{x}{4\cdot 16^k}\right| > \frac{7}{32}\frac{x^n}{2^n 4^{n^2}}\exp\left(-\frac{8(16^n-1)}{15x} - \frac{x}{30\cdot 16^{n+1}}\right)$$

Next: for $x \in J_n$,

$$\begin{aligned}\frac{8(16^n-1)}{15x} + \frac{x}{30\cdot 16^{n+1}} &< \frac{8}{15}\frac{16^n-1}{4\cdot 16^n + 8^n} + \frac{1}{30}\frac{4\cdot 16^{n+1} - 8^{n+1}}{16^{n+1}} \\ &< \frac{8}{15}\frac{16^n}{4\cdot 16^n} + \frac{1}{30}\frac{4\cdot 16^{n+1}}{16^{n+1}} < \frac{8}{15}\frac{1}{4} + \frac{4}{30} = \frac{4}{15}\end{aligned}$$

Then

$$\prod_{k=0}^{\infty}\left|1-\frac{x}{4\cdot 16^k}\right| > \frac{7e^{-4/15}}{32}\frac{x^n}{2^n 4^{n^2}} \tag{4.11}$$

From (4.8), (4.9), and (4.11):

$$\begin{aligned}|f(x)| &> \prod_{n=1}^{\infty}\left|1-\frac{1}{16^n}\right|\prod_{n=1}^{\infty}\left|1-\frac{x}{4\cdot 16^{n-1}}\right|\prod_{n=1}^{\infty}\left|1-\frac{4}{x\cdot 16^{n-1}}\right|-2\\ &> \frac{7e^{-\frac{4}{15}}}{32}\frac{x^n}{2^n 4^{n^2}}e^{-2/15}e^{-8/(15x)}-2 > \frac{7e^{-14/15}}{32}\frac{x^n}{2^n 4^{n^2}}-2\end{aligned}$$

So $|f(x)| - \frac{1}{100}\frac{x^n}{2^n 4^{n^2}} > \left(\frac{7e^{-14/15}}{32}-\frac{1}{100}\right)\frac{x^n}{2^n 4^{n^2}} - 2 > \frac{19}{250}2^2 4^4 - 2 > 0.$ □

Lemma 4.6. *$|f(x)| < \frac{4x^n}{4^{n^2}}$ for $x \in [4\cdot 16^n, 4\cdot 16^{n+1}]$ and $n \geq 1$.*

Proof. Since

$$\prod_{n=1}^{\infty}\left(1-\frac{1}{16^n}\right) < 1 \qquad \text{and} \qquad \prod_{n=1}^{\infty}\left(1-\frac{4}{x\cdot 16^n}\right) < 1,$$

we have

$$\prod_{k=0}^{\infty}\left|1-\frac{x}{4\cdot 16^k}\right| = \left[\prod_{k=0}^{n-1}\left|1-\frac{x}{4\cdot 16^k}\right|\right]|h(x)|\prod_{k=n+2}^{\infty}\left|1-\frac{x}{4\cdot 16^k}\right|$$

where

$$h(x) = \left(1-\frac{x}{4\cdot 16^n}\right)\left(1-\frac{x}{4\cdot 16^{n+1}}\right).$$

The maximum of $|h(x)|$ on J_n is at $x = 34\cdot 16^n$:

$$|h(x)| \leq \frac{1}{16^{2n+1}}(30\cdot 16^n)^2 = \frac{900\cdot 16^{2n}}{16^{2n+2}} = \frac{225}{64}$$

We have $\prod_{k=n+2}^{\infty}\left(1-x/(4\cdot 16^k)\right) < 1$ because $x < 4\cdot 16^{n+1}$. So

$$\left|\prod_{k=0}^{n-1}\left(1-\frac{x}{4\cdot 16^k}\right)\right| < \prod_{k=0}^{n-1}\frac{x}{4\cdot 16^k} = \frac{x^n}{4^n 16^{n(n-1)/2}} = \frac{x^n}{4^{n^2}}$$

Therefore

$$\prod_{k=0}^{\infty}\left|1-\frac{x}{4\cdot 16^k}\right| < \left[\prod_{k=0}^{n-1}\left|1-\frac{x}{4\cdot 16^k}\right|\right]|h(x)| < \frac{225}{64}\frac{x^n}{4^{n^2}}$$

and

$$|f(x)| < \prod_{n=1}^{\infty}\left(1-\frac{1}{16^n}\right)\prod_{n=1}^{\infty}\left(1-\frac{x}{4\cdot 16^{n-1}}\right)\prod_{n=1}^{\infty}\left(1-\frac{4}{x\cdot 16^{n-1}}\right)+2$$
$$< \frac{225}{64}\frac{x^n}{4^{n^2}}+2$$

Then $\frac{4\cdot x^n}{4^{n^2}} - |f(x)| > \frac{31}{64}\frac{x^n}{4^{n^2}} - 2 > \frac{31}{64}\frac{(4\cdot 16^n)^n}{4^{n^2}} - 2 > \frac{31\cdot 4^2}{64} - 2 = 5 > 0$ □

Claim 4.7. *The following inequalities hold for n sufficiently large.*

(i) $\left|f\left(\frac{x}{k^2}\right)\right| \le \frac{4x^n}{k^{2m}4^{n^2}}$ *for* $4^{n-m-1} < k \le 4^{n-m}$

(ii) $\sum_{k=4^{n-m-1}+1}^{4^{n-m}} \left|f\left(\frac{x}{k^2}\right)\right| < \frac{192x^n}{4^n 4^{n^2}}$ *for* $1 \le m \le n-2$

(iii) $\sum_{k=5}^{4^{n-1}} \left|f\left(\frac{x}{k^2}\right)\right| < \frac{192(n-2)x^n}{4^n 4^{n^2}}$

Proof. (i) Let $4\cdot 16^l \le x/k^2 \le 4\cdot 16^{l+1}$. We have that

$$\frac{x}{k^2} \ge \frac{4\cdot 16^n + 8^n}{(4^{n-m})^2} \ge \frac{4\cdot 16^n}{16^{n-m}} = 4\cdot 16^m$$

Then $l \ge m$. From Lemma 4.6

$$\left|f\left(\frac{x}{k^2}\right)\right| \le \frac{4\cdot x^l}{k^{2l}4^{l^2}}.$$

But $x^l/4^{l^2} < x^n/4^{n^2}$ and $1/k^{2l} < 1/k^{2m}$ so

$$\left|f\left(\frac{x}{k^2}\right)\right| \le \frac{4\cdot x^n}{k^{2m}4^{n^2}}$$

(ii)
$$\sum_{k=4^{n-m-1}+1}^{4^{n-m}} \left|f\left(\frac{x}{k^2}\right)\right| \le \sum_{k=4^{n-m-1}+1}^{4^{n-m}} \frac{4x^n}{4^{n^2}k^{2m}}$$
$$< \frac{4x^n}{4^{n^2}}\sum_{k=4^{n-m-1}+1}^{4^{n-m}} \frac{1}{4^{2m(n-m-1)}} \le \frac{4x^n}{4^{n^2}}\frac{4^{n-m}-4^{n-m-1}}{4^{2m(n-m-1)}}$$
$$= \frac{4x^n}{4^{n^2}}\frac{3\cdot 4^{n-m-1}}{4^{2m(n-m-1)}} = \frac{12x^n}{4^{n^2}}\frac{1}{4^{(2m-1)(n-m-1)}}$$

The minimum value of $h(m) = (2m-1)(n-m-1)$ is $h(1) = n-2$. So

$$\sum_{k=4^{n-m-1}+1}^{4^{n-m}} \left| f\left(\frac{x}{k^2}\right) \right| < \frac{12x^n}{4^{n^2}} \frac{1}{4^{n-2}} = \frac{192x^n}{4^n 4^{n^2}}$$

(iii) $$\sum_{k=5}^{4^{n-1}} \left| f\left(\frac{x}{k^2}\right) \right| = \sum_{m=1}^{n-2} \sum_{k=4^{n-m-1}+1}^{4^{n-m}} \left| f\left(\frac{x}{k^2}\right) \right| < \frac{192(n-2)x^n}{4^n 4^{n^2}}$$ □

Write $\widehat{f} = \max\{8\,|f(x)| : 0 \le x \le 1024\}$ and $\widehat{f}_2 = \max\{|f'(x)| : 0 \le x \le 2\}$.

Claim 4.8. *The following inequalities hold for n sufficiently large.*

(i) $$\sum_{k=2}^{4} \left| f\left(\frac{x}{k^2}\right) \right| \le \frac{48x^n}{4^n 4^{n^2}}$$

(ii) $$\sum_{k=4^{n-1}+1}^{8\cdot 4^n - 1} \left| f\left(\frac{x}{k^2}\right) \right| < 4^n \widehat{f}$$

(iii) $$\sum_{k=8\cdot 4^n}^{\infty} \left| f\left(\frac{x}{k^2}\right) \right| \le x$$

Proof. (i) We have $x/k^2 \ge x/16 \ge (4\cdot 16^n + 8^n)/16 \ge 4 \cdot 16^{n-1}$ and

$$\frac{x}{k^2} \le \frac{x}{4} \le \frac{4\cdot 16^{n+1} - 8^{n+1}}{4} \le 16^{n+1} < 4\cdot 16^{n+1}$$

Therefore x/k^2 satisfies the conditions of Lemma 4.6 with $n-1$ or n and so

$$f\left(\frac{x}{k^2}\right) < \frac{4x^{n-1}}{k^{2(n-1)} 4^{(n-1)^2}} \quad \text{or} \quad f\left(\frac{x}{k^2}\right) < \frac{4x^n}{k^{2n} 4^{n^2}}$$

In both cases

$$f\left(\frac{x}{k^2}\right) < \frac{4x^n}{k^{2(n-1)} 4^{n^2}}$$

because $1/k^{2n} < 1/k^{n-1}$ and $x^{n-1}/4^{(n-1)^2} < x^n/4^{n^2}$ for $x \in J_n$. Hence

$$\sum_{k=2}^{4} \left| f\left(\frac{x}{k^2}\right) \right| \le \sum_{k=2}^{4} \frac{4\cdot x^n}{k^{2(n-1)} 4^{n^2}} \le \frac{12\cdot x^n}{2^{2(n-1)} 4^{n^2}} = \frac{48x^n}{4^n 4^{n^2}}$$

(ii) We have $0 < x/k^2 < 4\cdot 16^{n+1}/16^{n-1} = 1024$. So

$$\sum_{k=4^{n-1}+1}^{8\cdot 4^n - 1} \left| f\left(\frac{x}{k^2}\right) \right| \le (8\cdot 4^n - 4^{n-1} - 1)\frac{\widehat{f}}{8} < \frac{8\cdot 4^n \widehat{f}}{8} = 4^n \widehat{f}$$

(iii) We have $0 < x/k^2 < (4\cdot 16^{n+1} - 8^{n+1})/(64\cdot 16^n) < 4\cdot 16^{n+1}/(64\cdot 16^n) = 1$, so by Claim 4.1(i) we have $\left|f\left(x/k^2\right)\right| \leq x/(4k^2)$. Therefore

$$\sum_{k=8\cdot 4^n}^{\infty} \left|f\left(\frac{x}{k^2}\right)\right| \leq \sum_{k=8\cdot 4^n}^{\infty} \frac{x}{4k^2} < \frac{x}{4}\sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2 x}{4\cdot 6} < x \qquad \square$$

Lemma 4.9. $|g(x)| > \dfrac{1}{200}\dfrac{x^n}{2^n 4^{n^2}}$ *for $x \in J_n$ and n sufficiently large.*

Proof. By Lemma 4.5 we have $|f(x)| > (1/100)x^n/(2^n 4^{n^2})$. By Claims 4.7(iii) and 4.8

$$\sum_{k=2}^{\infty}\left|f\left(\frac{x}{k^2}\right)\right| < \frac{192(n-2)x^n}{4^n 4^{n^2}} + \frac{48x^n}{4^n 4^{n^2}} + 4^n \widehat{f} + x$$

Then

$$\begin{aligned} |g(x)| &\geq |f(x)| - \sum_{k=2}^{\infty}\left|f\left(\frac{x}{k^2}\right)\right| \\ &> \frac{1}{100}\frac{x^n}{2^n 4^{n^2}} - \frac{192(n-2)x^n}{4^n 4^{n^2}} - \frac{48x^n}{4^n 4^{n^2}} - 4^n\widehat{f} - x \end{aligned}$$

Therefore for large enough n we have $|g(x)| > \frac{1}{200}\frac{x^n}{2^n 4^{n^2}}$ and therefore

$$\frac{1}{200}\frac{x^n}{2^n 4^{n^2}} - \frac{48x^n}{4^n 4^{n^2}} - \frac{192(n-2)x^n}{4^n 4^{n^2}} - 4^n\widehat{f} - x > 0 \qquad \square$$

Corollary 4.10. $|G_i(x)| > \dfrac{1}{200}\dfrac{x^n}{2^n 4^{n^2}}$ *for $y \in J_n$ and $i = 1, 2$.*

Proof. $|G_i(x)| \geq 2|g(x)| - \pi\sqrt{x/2} - 1 > (1/100)x^n/(2^n 4^{n^2}) - \pi\sqrt{x/2} - 1$. Therefore $|G_i(x)| > (1/200)x^n/(2^n 4^{n^2})$ for $i = 1, 2$ and n sufficiently large. $\square$

Corollary 4.11. *For sufficiently large n, the functions $G_1(x)$ and $G_2(x)$ have no zeros in the interval J_n.*

Proof. If $x \in J_n$ we have

$$|G_i(x)| > \frac{1}{200}\frac{x^n}{2^n 4^{n^2}} > \frac{1}{200}\frac{(4\cdot 16^n)^n}{2^n 4^{n^2}} = \frac{1}{200}2^n 4^{n^2}.$$

Therefore $G_i(x) \neq 0$. $\square$

4.2 The Intervals $\mathbf{I_n}$

In this section we show that the functions G_1 and G_2 have unique zeros on the intervals I_n. Let $\text{sgn}[x] = 1$ if $x > 0$ and $\text{sgn}[x] = -1$ if $x < 0$.

Lemma 4.12. *$\mathrm{sgn}[G_i(4 \cdot 16^n - 8^n)] = (-1)^n$ and $\mathrm{sgn}[G_i(4 \cdot 16^n + 8^n)] = (-1)^{n+1}$ for $i = 1, 2$ and n sufficiently large.*

Proof. The numbers $4 \cdot 16^n - 8^n$ and $4 \cdot 16^n + 8^n$ are endpoints of the intervals I_n and J_n. By the proofs of Lemma 4.9 and Corollary 4.10, the signs of $g(x)$ and $G_i(x)$ are the same as the sign of $f(x)$ for sufficiently large values of n. By Lemma 4.5,

$$\mathrm{sgn}[f(x)] = \mathrm{sgn}\left[\prod_{n=1}^{\infty}\left(1 - \frac{1}{16^n}\right)\prod_{n=1}^{\infty}\left(1 - \frac{x}{4 \cdot 16^{n-1}}\right)\prod_{n=1}^{\infty}\left(1 - \frac{4}{x \cdot 16^{n-1}}\right)\right]$$

But

$$\prod_{n=1}^{\infty}\left(1 - \frac{1}{16^n}\right) > 0, \qquad \prod_{n=1}^{\infty}\left(1 - \frac{4}{x \cdot 16^n}\right) > 0,$$

$$1 - \frac{x}{4 \cdot 16^{k-1}} > 0 \quad \text{for } k \geq n+2, \text{ and} \quad 1 - \frac{x}{4 \cdot 16^{k-1}} < 0 \quad \text{for } 1 \leq k \leq n.$$

So

$$\begin{aligned}\mathrm{sgn}[f(x)] &= \mathrm{sgn}\left[\prod_{k=1}^{n}\left(1 - \frac{x}{4 \cdot 16^{k-1}}\right)\right]\mathrm{sgn}\left[1 - \frac{x}{4 \cdot 16^n}\right] \\ &= (-1)^n \mathrm{sgn}\left[1 - \frac{x}{4 \cdot 16^n}\right]\end{aligned}$$

for $x \in J_n$. Hence

$$\mathrm{sgn}[f(4 \cdot 16^n + 8^n)] = (-1)^n \mathrm{sgn}\left[1 - \frac{4 \cdot 16^n + 8^n}{4 \cdot 16^n}\right] = (-1)^{n+1}$$

$$\mathrm{sgn}[f(4 \cdot 16^n - 8^n)] = (-1)^n \mathrm{sgn}\left[1 - \frac{4 \cdot 16^n - 8^n}{4 \cdot 16^n}\right] = (-1)^n$$

□

Lemma 4.12 establishes the existence of zeros of the functions G_1 and G_2 on the intervals I_n. It remains to show that these zeros are unique.

Claim 4.13. *$|f(x)| < 2 \cdot 4^{n^2}$ for $x \in [1, 16^n + 8^n]$ and n sufficiently large.*

Proof. Let n be large enough such that $2 \cdot 4^{n^2} > \max\{f(x) \,|\, 1 \leq x \leq 16^3 + 8^3\}$. Let $16^{k-1} + 8^{k-1} \leq x \leq 16^k + 8^k$ where $4 \leq k \leq n$. Then

$$4 \cdot 16^{k-2} < 16^{k-1} + 8^{k-1} < x < 16^k + 8^k < 4 \cdot 16^k$$

The conditions of Lemma 4.6 are satisfied with $n = k-2$ or $n = k-1$. Now $4 \cdot x^{k-2}/4^{(k-2)^2} < 4 \cdot x^{k-1}/4^{(k-1)^2}$ for $x > 16^{k-1}$. Therefore

$$\begin{aligned}|f(x)| &< \frac{4 \cdot x^{k-1}}{4^{(k-1)^2}} < \frac{4 \cdot x^{k-1}}{4^{(k-1)^2}} < \frac{4 \cdot (16^k + 8^k)^{k-1}}{4^{k^2-2k+1}} \\ &= \frac{16^{k(k-1)}\left(1+1/2^k\right)^{k-1}}{4^{k^2-2k}} < 4^{k^2}\left(1+\frac{1}{2^k}\right)^{k-1} \\ &< 4^{k^2}\left(1+\frac{1}{2(k-1)}\right)^{k-1} < e^{1/2}4^{k^2} < 2 \cdot 4^{k^2}\end{aligned}$$

because $2^{k-1} > k-1$. Since $k \leq n$, the claim is established. □

Lemma 4.14. *$|f'(x)| < 4^{(n-1)^2+2(n-1)/3-1}$ for $x \in [1, 16^n+8^n]$ and $n \geq 2$.*

Proof. We use induction on n. When $n = 2$ we must show that $|f'(x)| < 4^{2/3}$ for $x \in [1, 320]$.

$$\begin{aligned}f'(x) &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{nx^{n-1}}{4^{n^2}} = \frac{1}{4} - \frac{x}{128} + \frac{3x^2}{4^9} - \frac{4x^3}{4^{16}} + \sum_{n=4}^{\infty}(-1)^{n-1}\frac{nx^{n-1}}{4^{n^2}} \\ &\leq \left|\frac{1}{4} - \frac{x}{128} + \frac{3x^2}{4^9} - \frac{4x^3}{4^{16}}\right| + \sum_{n=5}^{\infty}\frac{nx^{n-1}}{4^{n^2}}\end{aligned}$$

The maximum value of

$$h(x) = \left|\frac{1}{4} - \frac{x}{128} + \frac{3x^2}{4^9} - \frac{4x^3}{4^{16}}\right|$$

for $x \in [1, 320]$ is $h(320) = 4541/4096 < 1.11$. We have $(n+1)x^n/4^{(n+1)^2} < (1/16)nx^{n-1}/4^{n^2}$ for $n \geq 5$. Then

$$\sum_{n=5}^{\infty}\frac{nx^{n-1}}{4^{n^2}} = \sum_{n=0}^{\infty}\frac{(n+5)x^{n+4}}{4^{(n+5)^2}} < \frac{5x^4}{4^{25}}\sum_{n=0}^{\infty}\frac{1}{16^n} < \frac{5 \cdot 320^4}{4^{25}}\frac{16}{15} < 0.0001$$

because $(n+5)x^{n+4}/4^{(n+5)^2} < 5x^4/(16^n 4^{25})$. Therefore

$$|f'(x)| < 1.11 + 0.0001 = 1.1101 < 4^{2/3}$$

Now suppose $n \geq 2$ and that $|f'(x)| < 4^{(n-1)^2+2(n-1)/3-1}$ for $x \in [1, 16^n + 8^n]$.

Now using Claim 4.13,

$$\begin{aligned}
|f'(16x)| &\le \frac{1}{4} + \frac{|f(x)|}{4} + \frac{|xf'(x)|}{4} \\
&< \frac{1}{4} + \frac{1}{2}4^{n^2} + \frac{1}{4}(16^n + 8^n)\frac{1}{4}4^{n^2-2n+1+2n/3-2/3} \\
&< \frac{1}{4} + \frac{1}{2}4^{n^2} + \frac{16^n+8^n}{16^n}\frac{4^{1/3}}{4}\frac{1}{4}4^{n^2+2n/3} \\
&= \frac{1}{4} + \frac{1}{2}4^{n^2} + \left(1+\frac{1}{2^n}\right)\frac{4^{1/3}}{4}\frac{1}{4}4^{n^2+2n/3}
\end{aligned}$$

But $1 + 1/2^n < 5/4$ for $n \ge 2$, so

$$|f'(16x)| < \frac{1}{4} + \frac{1}{2}4^{n^2} + \frac{5}{4}\frac{4^{1/3}}{4}\frac{1}{4}4^{n^2+2n/3}$$

and therefore

$$\begin{aligned}
\frac{1}{4}4^{n^2+2n/3} - |f'(16x)| &> \left(1 - \frac{5\cdot 4^{1/3}}{16}\right)\frac{1}{4}4^{n^2+2n/3} - \frac{1}{2}4^{n^2} - \frac{1}{4} \\
&> \left(1 - \frac{5\cdot 4^{1/3}}{16}\right)\frac{1}{4}4^{2^2+2\cdot 2/3} - \frac{1}{2}4^{2^2} - \frac{1}{4} > 76 > 0
\end{aligned}$$

Therefore, $|f'(16x)| < 4^{n^2+2n/3-1}$ for $x \in I_n$. Also

$$\begin{aligned}
16x &\ge 16(4\cdot 16^n - 8^n) = 4\cdot 16^{n+1} - 2\cdot 8^{n+1} \\
16x &\le 16(4\cdot 16^n + 8^n) = 4\cdot 16^{n+1} + 2\cdot 8^{n+1}
\end{aligned}$$

Then

$$|f'(x)| < 4^{n^2+2n/3}$$

for $x \in I_{n+1} \subset [4\cdot 16^{n+1} - 2\cdot 8^{n+1}, 4\cdot 16^{n+1} + 2\cdot 8^{n+1}]$. □

Claim 4.15. *$|f(x)| < \frac{1}{2}2^n 4^{n^2}$ for $x \in I_n$ and $n \ge 2$.*

Proof. We have $\prod_{n=1}^{\infty}(1 - 1/16^n) < 1$, $\prod_{n=1}^{\infty}\left(1 - 4/(x\cdot 16^{n-1})\right) < 1$ and $\left|1 - x/(4\cdot 16^{k-1})\right| < 1$ for $k \ge n+2$. Now we estimate:

$$\begin{aligned}
\prod_{k=1}^{\infty}\left|1 - \frac{x}{4\cdot 16^{k-1}}\right| &= \prod_{k=1}^{n+1}\left|1 - \frac{x}{4\cdot 16^{k-1}}\right|\prod_{k=n+2}^{\infty}\left|1 - \frac{x}{4\cdot 16^{k-1}}\right| \\
&< \left(\prod_{k=1}^{n}\left|1 - \frac{x}{4\cdot 16^{k-1}}\right|\right)\left|1 - \frac{x}{4\cdot 16^{n}}\right|
\end{aligned}$$

We have $x < 4 \cdot 16^n + 8^n$, so

$$\prod_{k=1}^{\infty}\left|1-\frac{x}{4\cdot 16^{k-1}}\right| < \left(\prod_{k=0}^{n-1}\left|1-\frac{4\cdot 16^n+8^n}{4\cdot 16^k}\right|\right)\left|1-\frac{4\cdot 16^n+8^n}{4\cdot 16^n}\right|$$
$$< \frac{1}{4\cdot 2^n}\prod_{k=0}^{n-1}\left|1-16^{n-k}+\frac{8^n}{4\cdot 16^k}\right|$$
$$= \frac{1}{4\cdot 2^n}\prod_{k=0}^{n-1}16^{n-k}\left(1-\frac{1}{16^{n-k}}-\frac{1}{4\cdot 2^n}\right)$$
$$< \frac{1}{4\cdot 2^n}\prod_{k=0}^{n-1}16^{n-k} = 16^{n(n+1)/2}2^{-n-2} = \frac{1}{4}2^n 4^{n^2}$$

Then

$$|f(x)| < \prod_{n=1}^{\infty}\left|1-\frac{1}{16^n}\right|\prod_{n=1}^{\infty}\left|1-\frac{4}{x\cdot 16^{n-1}}\right|\prod_{n=1}^{\infty}\left|1-\frac{x}{4\cdot 16^{n-1}}\right|+2$$
$$< \frac{1}{4}2^n 4^{n^2}+2 < \frac{1}{2}2^n 4^{n^2}$$

for $n \geq 2$. □

Claim 4.16. *$|f'(x)| > \frac{1}{5}4^{(n-1)^2+2(n-1)/3}$ for $x \in I_n$ and n sufficiently large.*

Proof. We use induction on n. When $n = 2$ we must show that $|f'(x)| > 4^{5/3}/5$ for $x \in [960, 1088]$. Compute:

$$f'(x) = \frac{1}{4} - \frac{x}{128} + \frac{3x^2}{4^9} - \frac{4x^3}{4^{16}} + \sum_{n=2}^{\infty}\left[\frac{(2n+1)x^{2n}}{4^{(2n+1)^2}} - \frac{(2n+2)x^{2n+1}}{4^{(2n+2)^2}}\right]$$

We have

$$\frac{(2n+1)x^{2n}}{4^{(2n+1)^2}} - \frac{(2n+2)x^{2n+1}}{4^{(2n+2)^2}} > 0$$

for $x \in [960, 1088]$ and the function

$$h(x) = \frac{1}{4} - \frac{x}{128} + \frac{3x^2}{4^9} - \frac{4x^3}{4^{16}}$$

has minimum $h(960) = 10129/4096$. Therefore

$$f'(x) > h(x) \geq h(960) > \frac{4^{5/3}}{5}$$

Now let $n \geq 2$ and suppose that $|f'(x)| > 4^{(n-1)^2+2(n-1)/3}$ for $x \in I_n$. Then

$$|f'(16x)| > \frac{1}{4}|x|\,|f'(x)| - \frac{1}{4}|f(x)| - \frac{1}{4}$$

By Claim 4.15, $|f(x)| < (1/2)2^n 4^{n^2}$, so

$$\begin{aligned}|f'(16x)| &> \frac{1}{4}(4\cdot 16^n - 8^n)\frac{1}{5}4^{(n-1)^2+2(n-1)/3} - \frac{1}{8}2^n4^{n^2} - \frac{1}{4}\\ &> \frac{1}{4}4\cdot 16^n\left(1-\frac{1}{4\cdot 2^n}\right)\frac{1}{5}4^{n^2-2n+1+2n/3-2/3} - \frac{1}{8}4^{n^2}2^n - \frac{1}{4}\end{aligned}$$

Now $1 - 1/(4\cdot 2^n) > 15/16$ because $n \geq 2$. Then

$$\begin{aligned}|f'(16x)| &> \frac{3\cdot 4^{1/3}}{16}4^{n^2+2n/3} - \frac{1}{8}4^{n^2+n/2} - \frac{1}{4}\\ |f'(16x)| - \frac{1}{5}4^{n^2+2n/3} &> \frac{4^{1/3}\cdot 15 - 16}{80}\cdot 4^{n^2+2n/3} - \frac{1}{8}4^{n^2+n/2} - \frac{1}{4}\end{aligned}$$

The right-hand side is increasing and has minimum for $n = 2$, so

$$|f'(16x)| - \frac{1}{5}4^{n^2+2n/3} > \frac{4^{1/3}\cdot 15 - 16}{80}\cdot 4^{16/3} - \frac{1}{8}4^5 - \frac{1}{4} > 30 > 0$$

Therefore, $|f'(x)| > (1/5)4^{n^2+2n/3}$ for $x \in I_{n+1}$, since $x/16 \in I_n$. □

Lemma 4.17. *$|g'(x)| > \frac{1}{40}4^{(n-1)^2+2(n-1)/3}$ for $x \in I_n$ and n sufficiently large.*

Proof. We have $(4\cdot 16^n + 8^n)/k^2 < 16^{n-m+1} + 8^{n-m+1}$ for $2\cdot 4^{m-1} \leq k \leq 2\cdot 4^m - 1$. By Lemma 4.14, $\left|f\left(x/k^2\right)\right| < 4^{(n-m)^2+2(n-m)/3-1}$. Then

$$\begin{aligned}\sum_{k=8}^{2\cdot 4^n-1}\frac{1}{k^2}\left|f'\left(\frac{x}{k^2}\right)\right| &= \sum_{m=2}^{n}\sum_{k=2\cdot 4^{m-1}}^{2\cdot 4^m-1}\frac{1}{k^2}\left|f'\left(\frac{x}{k^2}\right)\right|\\ &< \sum_{m=2}^{n}\sum_{k=2\cdot 4^{m-1}}^{2\cdot 4^m-1}\frac{1}{4}4^{(n-m)^2+2(n-m)/3}\\ &< \sum_{m=2}^{n}6\cdot 4^{m-1}\frac{1}{4}4^{(n-m)^2+2(n-m)/3}\\ &< \frac{3\cdot 4^{1/3}}{2}(n-1)4^{(n-2)^2+2(n-1)/3}\end{aligned}$$

because the maximal value of $h(m) = m - 1 + (n-m)^2 + 2(n-m)/3$ on the interval $[2, n]$ is $h(2) = (n-2)^2 + 2(n-1)/3 + 1/3$. So

$$\sum_{k=8}^{2\cdot 4^n-1}\frac{1}{k^2}\left|f'\left(\frac{x}{k^2}\right)\right| < \frac{6\cdot 4^{\frac{1}{3}}(n-1)}{16^{n-1}}4^{(n-1)^2+2(n-1)/3} < \frac{1}{50}4^{(n-1)^2+2(n-1)/3}$$

because $6 \cdot 4^{1/3}(n-1)/16^{n-1} < 1/50$ for $n \geq 4$. Recall $\widehat{f}_2 = \max\{|f'(x)| : 0 \leq x \leq 2\}$. We have

$$\sum_{k=2\cdot 4^n}^{\infty} \frac{1}{k^2} \left| f'\left(\frac{x}{k^2}\right) \right| < \sum_{k=2\cdot 4^n}^{\infty} \frac{\widehat{f}_2}{k^2} < \sum_{k=1}^{\infty} \frac{\widehat{f}_2}{k^2} = \frac{\pi^2}{6}\widehat{f}_2 < 2\widehat{f}_2$$

because $x/k^2 < 2$ and $\sum_{k=1}^{\infty} 1/k^2 = \pi^2/6$. Also

$$\sum_{k=2}^{7} \frac{1}{k^2} \left| f'\left(\frac{x}{k^2}\right) \right| < \frac{1}{4} 4^{(n-1)^2+2(n-1)/3} \sum_{k=2}^{7} \frac{1}{k^2} < \frac{3}{20} 4^{(n-1)^2+2(n-1)/3}$$

using $x/k^2 \leq x/4 < (4 \cdot 16^n + 8^4)/4 < 16^n + 8^n$ and Lemma 4.14. Then

$$\begin{aligned}
&\sum_{k=2}^{\infty} \frac{1}{k^2} \left| f'\left(\frac{x}{k^2}\right) \right| \\
&\quad = \sum_{k=2}^{7} \frac{1}{k^2} \left| f'\left(\frac{x}{k^2}\right) \right| + \sum_{k=8}^{2\cdot 4^n - 1} \frac{1}{k^2} \left| f'\left(\frac{x}{k^2}\right) \right| + \sum_{k=2\cdot 4^n}^{\infty} \frac{1}{k^2} \left| f'\left(\frac{x}{k^2}\right) \right| \\
&\quad < \left(\frac{3}{40} + \frac{1}{50} \right) 4^{(n-1)^2+2(n-1)/3} + 2\widehat{f}_2 < \frac{7}{40} 4^{(n-1)^2+2(n-1)/3}
\end{aligned}$$

By Claim 4.16, $|f'(x)| > (1/5)4^{(n-1)^2+2(n-1)/3}$ for $x \in I_n$. Therefore

$$\begin{aligned}
|g'(x)| &> |f'(x)| - \sum_{k=2}^{\infty} \frac{1}{k^2} \left| f'\left(\frac{x}{k^2}\right) \right| \\
&> \frac{1}{5} 4^{(n-1)^2+2(n-1)/3} - \frac{7}{40} 4^{(n-1)^2+2(n-1)/3} \\
&= \frac{1}{40} 4^{(n-1)^2+2(n-1)/3}
\end{aligned}$$

as claimed. □

Corollary 4.18. *$|G_i'(x)| > \dfrac{1}{100} 4^{(n-1)^2+2(n-1)/3}$ for $x \in I_n$ and n sufficiently large.*

Proof.

$$\begin{aligned}
|G_i'(x)| > 2|g'(x)| - \left| \frac{\pi}{2\sqrt{2x}} \right| &> \frac{1}{40} 4^{(n-1)^2+2(n-1)/3} - \frac{\pi}{2\sqrt{2(4 \cdot 16 - 8)}} \\
&> \frac{1}{40} 4^{(n-1)^2+2(n-1)/3} - \frac{\pi}{64\sqrt{2}} > \frac{1}{100} 4^{(n-1)^2+2(n-1)/3}
\end{aligned}$$

□

4.3 Conclusion

We have shown that the functions G_1 and G_2 have zeros on the intervals I_n while they have no zeros on the remaining intervals J_n when n is large enough. From Corollary 4.18 the derivatives $G_i'(x)$ have no zeros on the intervals I_n and the functions $G_i(x)$ are either increasing or decreasing on I_n. Therefore the functions G_1 and G_2 have unique zeros on the intervals I_n for sufficiently large values of n.

5 The Eigenvalues of T

In this section we prove (as announced in Theorem 2.1) that the eigenvalues of T satisfy

(i) $\left|A_n - \frac{\sqrt{2}}{\pi 4^n}\right| < \frac{1}{8^{n+1}},$ for n sufficiently large positive integer.

(ii) $\left|A_{-n} + \frac{\sqrt{2}}{\pi 4^{-n}}\right| < \frac{1}{8^{1-n}},$ for $n < 0$ and $|n|$ sufficiently large.

For (i), let $A_n = 1/a_n$ be an eigenvalue of T where $n > 0$ is large enough that the results in Section 4 hold. The function G_1 has unique zeros on the intervals I_n and $G_1\left(a_n^2/(2\pi^2)\right) = 0$ where $a_n^2/(2\pi^2) = 1/(2\pi^2 A_n^2) \in I_{n-1}$. So

$$4 \cdot 16^{n-1} - 8^{n-1} < \frac{1}{2\pi^2 A_n^2} < 4 \cdot 16^{n-1} + 8^{n-1}$$

$$\frac{1}{\sqrt{2}\pi\sqrt{4 \cdot 16^{n-1} + 8^{n-1}}} < A_n < \frac{1}{\sqrt{2}\pi\sqrt{4 \cdot 16^{n-1} - 8^{n-1}}}$$

The Mean Value Theorem states $h(x) = h(x_0) + h'(\theta)(x - x_0)$ where θ is a number between x and x_0. Applying this for $h(x) = 1/\sqrt{x}$, we get

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{x_0}} - \frac{1}{2\theta^{3/2}}(x - x_0)$$

Using $x = 4 \cdot 16^{n-1} - 8^{n-1}$ and $x_0 = 4 \cdot 16^{n-1}$, we get

$$\frac{1}{\sqrt{4 \cdot 16^{n-1} - 8^{n-1}}} = \frac{1}{\sqrt{4 \cdot 16^{n-1}}} + \frac{8^{n-1}}{2}\frac{1}{\theta^{3/2}}$$

for some θ with $4 \cdot 16^{n-1} - 8^{n-1} < \theta < 4 \cdot 16^{n-1}$, so $\theta > 4 \cdot 16^{n-1} - 8^{n-1} > 3.9 \cdot 16^{n-1}$ for $n \geq 5$. Thus

$$\frac{1}{\sqrt{4 \cdot 16^{n-1} - 8^{n-1}}} < \frac{1}{2 \cdot 4^{n-1}} + \frac{8^{n-1}}{2}\frac{1}{(3.9 \cdot 16^{n-1})^{3/2}} < \frac{2}{4^n} + \frac{13}{25 \cdot 8^n} \quad (5.1)$$

Using $x = 4 \cdot 16^{n-1} + 8^{n-1}$ and $x_0 = 4 \cdot 16^{n-1}$, we get

$$\frac{1}{\sqrt{4 \cdot 16^{n-1} + 8^{n-1}}} = \frac{1}{\sqrt{4 \cdot 16^{n-1}}} - \frac{8^{n-1}}{2} \frac{1}{\theta^{3/2}}$$

for some θ with $4 \cdot 16^{n-1} < \theta < 4 \cdot 16^{n-1} + 8^{n-1}$, Thus

$$\frac{1}{\sqrt{4 \cdot 16^{n-1} + 8^{n-1}}} > \frac{1}{2 \cdot 4^{n-1}} - \frac{8^{n-1}}{2} \frac{1}{(4 \cdot 16^{n-1})^{3/2}} > \frac{2}{4^n} - \frac{13}{25 \cdot 8^n} \tag{5.2}$$

From (5.1) and (5.2),

$$\frac{1}{\sqrt{2}\pi}\left(\frac{2}{4^n} - \frac{13}{25 \cdot 8^n}\right) < A_n < \frac{1}{\sqrt{2}\pi}\left(\frac{2}{4^n} + \frac{13}{25 \cdot 8^n}\right)$$
$$-\frac{13}{\sqrt{2}\pi 25 \cdot 8^n} < A_n - \frac{2}{\sqrt{2}\pi 4^n} < \frac{13}{\sqrt{2}\pi 25 \cdot 8^n}$$
$$\left| A_n - \frac{2}{\sqrt{2}\pi 4^n} \right| < \frac{13}{\sqrt{2}\pi 25 \cdot 8^n} < \frac{1}{8^{n+1}}$$

This proves (i).

(ii) If $A_n = 1/a_n$ is an eigenvalue of T with $n < -1$, then $G_2\left(a_n^2/(2\pi^2)\right) = 0$. The proof of (ii) is similar to the proof of (i).

Remarks

Theorem 2.1 establishes upper and lower bounds for the asymptotic behavior of the eigenvalues of T. These bounds allow us to prove properties (2) for these eigenvalues.

We showed that Theorem 2.1 holds for values of n large enough so that all statements in Section 4 hold. Numerical calculations suggest that in fact Theorem 2.1 holds for all integer values of n except $n = -1$ and $n = -2$.

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