FINE VARIATION AND FRACTAL MEASURES

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Abstract. Thomson noted that (in the line) the Hausdorff measures can be considered to be fine variations for appropriate choices of derivation basis and set function. We show that this point of view remains interesting in a general separable metric space. Use of the "centered ball" basis yields an alternate description of the covering measures of Saint Raymond and Tricot. Use of a "closed set" basis yields the Hausdorff measures. This paper may be considered a counterpart of [7], where the corresponding study of the packing measure may be found.

0. Introduction

The best-known fractal measures are the Hausdorff measures $\mathcal{H}^s$ for $s > 0$. Saint Raymond and Tricot [12] defined a variant, known as the (centered ball) covering measures $\mathcal{C}^s$. They differ by at most a constant factor from the Hausdorff measures, so they may be used in the computation of the Hausdorff dimension.

Thomson [13] studied "variation" measures defined on metric spaces. To a set-function, such as the s-th power of the diameter, he associated a "full" variation (or Method III measure) and a "fine" variation (or Method IV measure). [The term "method III" has been used for other constructions as well, so we do not use it here.] For the interval basis on the line, the full variation is related to the packing measure, and the fine variation is the Hausdorff measure. Meinershagen [10] showed that packing measure is exactly the full variation associated with the balanced-interval basis. This characterization was extended to general metric spaces in [7].

In this paper we show that the St.-Raymond-Tricot covering measure is a fine variation when we use the closed centered-ball basis. Corresponding integral and

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density results are proved for this setting. This paper may be considered a counterpart of [7], where the corresponding study of the packing measure may be found, but the two papers may be read independently.

The Hausdorff measures are treated in a similar way (with the same proofs in most cases) by changing to a different derivation basis in §6. Covering measure defined using the diameter of a ball (rather than the radius) is discussed in §4. The upper density defined with diameter may be (in some metric spaces) a nonmeasurable function (Example 5.3).

Background material on Carathéodory's outer measures and metric outer measures may be found in [4, § 1.3] or [5, Chapt. 5]; but an even better reference is Carathéodory's original paper [cara].

1. Covering measure

We begin with the definition of the covering measure (see [12]). It is a slight variant of the Hausdorff measure. Let (S,ρ) be a separable metric space. For $x \in S$ and $r>0$, the **open ball** is

$$B_r(x) = \{\, y \in S : \rho(y,x) < r \,\},$$

and the **closed ball** is

$$\overline{B}_r(x) = \{\, y \in S : \rho(y,x) \le r \,\}.$$

The **diameter** of a set $E \subseteq S$ is

$$\operatorname{diam} E = \sup \{\, \rho(x,y) : x,y \in E \,\}.$$

Let $A \subseteq S$ be a set. A **centered-ball cover** of A is a countable collection $\{\overline{B}_{r_1}(x_1), \overline{B}_{r_2}(x_2), \cdots\}$ of closed balls with center in A that covers A:

$$x_i \in A, \qquad A \subseteq \bigcup_i \overline{B}_{r_i}(x_i).$$

A **centered-ball packing** of A is a countable disjoint collection

$$\{\overline{B}_{r_1}(x_1), \overline{B}_{r_2}(x_2), \cdots\}$$

of closed balls with center in A:

$$x_i \in A, \qquad \overline{B}_{r_i}(x_i) \cap \overline{B}_{r_j}(x_j) = \varnothing \text{ for } i \ne j.$$

Let μ be a finite Borel measure on the metric space S. For any $x \in S$, the **upper s-density** of μ at x is

$$\overline{D}^s_\mu(x) = \limsup_{r\to 0} \frac{\mu\big(\overline{B}_r(x)\big)}{(2r)^s}.$$

If x_0 is an isolated point, then $\overline{B}_r(x_0) = \{x_0\}$ for r small enough. So of course we define $\overline{D}^s_\mu(x_0) = \infty$ if $\mu(\{x_0\}) > 0$. Let us say by convention that $\overline{D}^s_\mu(x_0) = 0$ if $\mu(\{x_0\}) = 0$.

(1.1) Proposition. *Let μ be a finite Borel measure and let $s > 0$. Then the function $\overline{D}^s_\mu$ is a Borel function.*

Proof. First, we note that for fixed r, the map $x \mapsto \mu(\overline{B}_r(x))$ is a Borel function. Indeed, for given $t > 0$, we claim that the set

$$V = \left\{\, x \in S : \mu\big(\overline{B}_r(x)\big) < t \,\right\}$$

is an open set. Let $x_0 \in V$. That is, $\mu(\overline{B}_r(x_0)) < t$. The balls $\overline{B}_{r+1/n}(x_0)$ decrease to $\overline{B}_r(x_0))$, so there is n with $\mu(\overline{B}_{r+1/n}(x_0)) < t$. Now for any x with $\rho(x, x_0) < 1/n$, the ball $\overline{B}_r(x)$ is contained in $\overline{B}_{r+1/n}(x_0)$ so $\mu(\overline{B}_r(x)) < t$. Thus V is an open set.

Next, for fixed x, the map $r \mapsto \mu(\overline{B}_r(x))$ is right-continuous. Indeed,

$$\overline{B}_{r_0}(x) = \bigcap_{r > r_0} \overline{B}_r(x),$$

so $\mu\big(\overline{B}_r(x)\big) \to \mu\big(\overline{B}_{r_0}(x)\big)$ as $r \downarrow r_0$.

The denominator $(2r)^s$ is also a right-continuous function of r. So in the definition of $\overline{D}^s_\mu$, the lim sup may be computed using only rational values:

$$\overline{D}^s_\mu(x) = \lim_{n\to\infty} \sup \left\{ \frac{\mu\big(\overline{B}_r(x)\big)}{(2r)^s} : r \in \mathbb{Q}, 0 < r < \frac{1}{n} \right\}.$$

Therefore it is also a Borel function. $\square$

(1.2) Definition. Let s be a positive number. For $\varepsilon > 0$, define

$$\mathcal{C}^s_\varepsilon(A) = \inf \sum_i (2r_i)^s,$$

where the infimum is over all countable covers $\big\{\overline{B}_{r_i}(x_i)\big\}$ of A by centered closed balls with $r_i \le \varepsilon$. [Of course, in Euclidean space diam $\overline{B}_r(x) = 2r$, but in a general separable metric space this need not be true.] The **s-dimensional covering pre-measure** of A is

$$\widetilde{\mathcal{C}}^s(A) = \lim_{\varepsilon\to 0} \mathcal{C}^s_\varepsilon(A).$$

The **s-dimensional covering outer measure** is the outer measure $\mathcal{C}^s$ defined by making $\widetilde{\mathcal{C}}^s$ increasing:

$$\mathcal{C}^s(A) = \sup \left\{ \widetilde{\mathcal{C}}^s(E) : E \subseteq A \right\}.$$

Then $\mathcal{C}^s$ is a metric outer measure on S. ([12, Lemma 3.1]; the result remains correct in any separable metric space S.) We will restrict attention here to separable metric spaces, since $\mathcal{C}^s(A) = \infty$ for any non-separable set.

The definition is a bit awkward to use, because of the "increasing" step added on the end. But the set function $\widetilde{\mathcal{C}}^s$ is not increasing so this final step is needed.

Here are a few observations about the definition. The proofs in [12] often apply unchanged for a general separable metric space.

(1.3) The covering measure $\mathcal{C}^s$ and the Hausdorff measure $\mathcal{H}^s$ differ at most by a constant factor:

$$2^{-s}\mathcal{C}^s(E) \le \mathcal{H}^s(E) \le \mathcal{C}^s(E).$$

In particular, the values of s for which $\mathcal{H}^s(E) = 0$ are the same as those for which $\mathcal{C}^s(E) = 0$; and similarly for $\mathcal{H}^s(E) = \infty$ and $\mathcal{C}^s(E) = \infty$. [12, Lemma 3.3]

(1.4) The value of $\mathcal{C}^s(E)$ is unchanged if we use covers by open balls rather than covers by closed balls.

(1.5) In Euclidean space, if μ is a finite Borel measure, and E is a Borel set with $\mathcal{C}^s(E) < \infty$, then

$$\mathcal{C}^s(E) \inf_{x\in E} \overline{D}^s_\mu(x) \le \mu(E) \le \mathcal{C}^s(E) \sup_{x\in E} \overline{D}^s_\mu(x).$$

[12, 1.1]; we will see a strengthened version of this below.

(1.6) Is $\mathcal{C}^s$ a regular outer measure? That is, for any set A, there is a measurable set $E \supseteq A$ with $\mathcal{C}^s(E) = \mathcal{C}^s(A)$. I do not know the answer.

2. Fine variation

We now consider (a special case of) Thomson's fine variation [13], [14]. The variations may be defined for a general "derivation basis". For the moment we will use only the "centered ball" basis, so reference to the basis will be suppressed from our notation and terminology.

Let (S, ρ) be a separable metric space. A **constituent** is a pair (r, x), where $x \in S$ is a point, and r is a positive real number. [The constituent (r, x) represents the closed ball $\overline{B}_r(x)$. In a general metric space, different centers x, x' and/or different radii r, r' may represent identical point sets:

$$\overline{B}_r(x) = \overline{B}_{r'}(x'),$$

so we emphasize a center and radius are given as the constituent.]

A **packing** is a disjoint collection π of constituents: that is, $\overline{B}_r(x) \cap \overline{B}_{r'}(x') = \varnothing$ if $(r, x), (r', x') \in \pi$, $(r, x) \ne (r', x')$. A packing of a set A is a packing π such that $x \in A$ for all $(r, x) \in \pi$. Note that since S is separable, any packing by balls of positive radius must be a countable packing. A **centered cover** of a set A is a collection β of constituents (r, x) with $x \in A$ and $A \subseteq \bigcup_{(r,x)\in\beta} \overline{B}_r(x)$. A **fine cover** (or **Vitali cover**) of a set A is a centered cover β of A such that, for every $x \in A$ and every $\varepsilon > 0$, there is $r < \varepsilon$ with $(r, x) \in \beta$.

We will need the following Vitali-type theorem. The usual (Banach) proof of Vitali's theorem proves this without trouble (for example [4, Theorem 6.2.1], [8, Theorem 1.10]). But note the use of *closed* balls.

(2.1) Theorem. *Let (S, ρ) be a separable metric space, let $E \subseteq S$ be a set, let β be a fine cover of E, and let $s > 0$. Then there is a (finite or infinite) packing $\pi = \{(r_i, x_i)\} \subseteq \beta$ such that:*

$$\text{either} \qquad \sum_i (2r_i)^s = \infty \qquad \text{or} \qquad \mathcal{C}^s\left(E \setminus \bigcup_i \overline{B}_{r_i}(x_i)\right) = 0.$$

(2.2) Definition. Let h be a finite nonnegative constituent function: that is, for each constituent (r, x), let $0 \le h(r, x) < \infty$. If β is a cover, write

$$V_\beta(h) = \sup \sum_{(r,x)\in\pi} h(r, x),$$

where the supremum is over all packings $\pi \subseteq \beta$. The **fine variation** of h is

$$V_*(h) = \inf V_\beta(h),$$

where the infimum is over all fine covers β.

The intersection of two fine covers may not be a fine cover at all, so this infimum is not a "limit" in the sense of Moore-Smith. If $h(r, x) = 0$ whenever x is outside a set A, then of course it is enough to use only fine covers β of A in the infimum.

When the constituent function h is of the special form $h(r, x) = f(x)(2r)^s$, for some nonnegative point-function $f\colon S \to \mathbb{R}$, we will write $V^s_\beta(f) = V_\beta(h)$ and $V^s_*(f) = V_*(h)$. We will call $V^s_*(f)$ the **fine s-variation** of f.

The following is a special case of Thomson's general results on fine variations [13]. We write $\mathbb{1}_A$ for the indicator function of a set A, and $h\,\mathbb{1}_A$ for the constituent function

$$\big(h\,\mathbb{1}_A\big)(r, x) = h(r, x)\,\mathbb{1}_A(x).$$

(2.3) Theorem. *Let h be a nonnegative constituent function. Then μ defined by $\mu(A) = V_*(h\,\mathbb{1}_A)$ for all $A \subseteq S$ is a metric outer measure on S.*

If S is a separable metric space, then the Cantor-Bendixson Theorem [9, p. 253] states that S may be written as a disjoint union

$$S = S_0 \cup Q,$$

where Q is countable, and S_0 has no isolated points. (Q consists of all points x_0 such that $\overline{B}_r(x_0)$ is countable, for some $r > 0$.) If a constituent function h is **continuous** in the sense that

$$\lim_{r\to 0} h(r, x) = 0$$

for every x, then the measure $V_*(h\,\mathbb{1}_A)$ vanishes on single points, so it vanishes on countable sets like Q. This fact will be used to reduce many of our assertions to the case where the metric space S has no isolated points.

3. Covering measure and fine variation

Next is the main result. The fine variation measure defined by the set function $h(r,x) = (2r)^s$ coincides with the covering measure. The one-dimensional version (for Hausdorff measure) occurs in [14, Theorem 6.4].

(3.1) Theorem. *Let (S,ρ) be a separable metric space, and let $s > 0$. Then for every set $E \subseteq S$, we have $V_*^s(\mathbb{1}_E) = \mathcal{C}^s(E)$.*

Proof. (a) We first prove $\mathcal{C}^s(E) \leq V_*^s(\mathbb{1}_E)$. If $V_*^s(\mathbb{1}_E) = \infty$, there is nothing to prove, so suppose $V_*^s(\mathbb{1}_E) < \infty$. Let β be a fine cover of E with $V_\beta^s(\mathbb{1}_E) < \infty$. Let $\varepsilon > 0$. By the Vitali theorem (2.1), there is a packing $\{(r_i, x_i)\} \subseteq \beta$ with $r_i < \varepsilon$ and either $\sum (2r_i)^s = \infty$ or $\mathcal{C}^s\big(E \setminus \bigcup \overline{B}_{r_i}(x_i)\big) = 0$. But $\sum (2r_i)^s \leq V_\beta^s(\mathbb{1}_E) < \infty$, so $\mathcal{C}^s\big(E \setminus \bigcup \overline{B}_{r_i}(x_i)\big) = 0$. Then $\mathcal{C}_\varepsilon^s\big(E \setminus \bigcup \overline{B}_{r_i}(x_i)\big) = 0$ for all $\varepsilon > 0$. Thus

$$\mathcal{C}_\varepsilon^s(E) \leq \mathcal{C}_\varepsilon^s\left(\bigcup_{i=1}^\infty \overline{B}_{r_i}(x_i)\right) \leq \sum_{i=1}^\infty (2r_i)^s \leq V_\beta^s(\mathbb{1}_E).$$

Now let $\varepsilon \to 0$ to obtain $\widetilde{\mathcal{C}}^s(E) \leq V_\beta^s(\mathbb{1}_E)$. Then take the infimum over β to obtain $\widetilde{\mathcal{C}}^s(E) \leq V_*^s(\mathbb{1}_E)$. Finally, take the supremum of this inequality over all subsets, to obtain $\mathcal{C}^s(E) \leq V_*^s(\mathbb{1}_E)$.

(b) Next we prove: if $\mathcal{C}^s(E) = 0$, then $V_*^s(\mathbb{1}_E) = 0$. Since S is separable, and $V_*^s\big(\mathbb{1}_{\{x\}}\big) = 0$ for any single point x, we may reduce to the case where E contains no isolated points.

Let $\varepsilon > 0$. For each integer n, since $\mathcal{C}_{1/n}^s(E) = 0$, there is a centered cover

$$\{\, (r_{in}, x_{in}) : i \in \mathbb{N} \,\}$$

of E with $r_{in} < 1/n$, $x_{in} \in E$, and

$$\sum_i \big(2r_{in}\big)^s < \frac{\varepsilon}{2^{n+1}}.$$

Now for each i and n let

$$\beta_{in} = \{\, (r_{in}, y) : y \in E, \rho(y, x_{in}) \leq r_{in} \,\}.$$

Then

$$\beta = \bigcup_{i,n} \beta_{in}$$

is a fine cover of E. Let $\pi \subseteq \beta$ be a packing. For each i, n, because all elements of β_{in} contain the point x_{in}, there is at most one element of β_{in} in π. Thus

$$\sum_{(r,x)\in\pi} (2r)^s \leq \sum_{i,n} (2r_{in})^s \leq \sum_n \frac{\varepsilon}{2^{n+1}} = \varepsilon.$$

Thus $V^s_\beta(1\!\!1_E) \le \varepsilon$. So $V^s_*(1\!\!1_E) \le \varepsilon$. But $\varepsilon > 0$ was arbitrary, so $V^s_*(1\!\!1_E) = 0$.

(c) Finally, we prove that $V^s_*(1\!\!1_E) \le \mathcal{C}^s(E)$. Again, we may reduce to the case where E contains no isolated points of S. If $\mathcal{C}^s(E) = \infty$, there is nothing to prove, so assume $\mathcal{C}^s(E) < \infty$. Let μ be the restriction of $\mathcal{C}^s$ to E: that is, $\mu(A) = \mathcal{C}^s(A \cap E)$ for all A. Then μ is a finite metric outer measure.

We will decompose E using the upper s-density $\overline{D}^s_\mu$. Fix a number $\alpha > 1$. Write

$$\begin{aligned} E_1 &= \left\{ x \in E : \overline{D}^s_\mu(x) \le \alpha^{-3} \right\} \\ E_2 &= \left\{ x \in E : \overline{D}^s_\mu(x) > \alpha^{-3} \right\}. \end{aligned}$$

Consider first E_1. For $n \in \mathbb{N}$, write

$$F_n = \left\{ x \in E_1 : \frac{\mu\big(\overline{B}_r(x)\big)}{(2r)^s} < \alpha^{-2} \text{ for all } r < \frac{1}{n} \right\}.$$

Then F_n increases to E_1 as $n \to \infty$, since $\alpha^{-2} > \alpha^{-3}$.

I claim that $\mathcal{C}^s(F_n) = 0$. If $\varepsilon < 1/n$, then when F_n is covered by $\big\{\overline{B}_{r_i}(x_i)\big\}$ with $r_i < \varepsilon$, we have

$$\sum (2r_i)^s \ge \alpha^2 \sum \mu\big(\overline{B}_{r_i}(x_i)\big) \ge \alpha^2 \mu\left(\bigcup \overline{B}_{r_i}(x_i)\right) \ge \alpha^2 \mu(F_n) = \alpha^2 \mathcal{C}^s(F_n).$$

Therefore $\mathcal{C}^s_\varepsilon(F_n) \ge \alpha^2 \mathcal{C}^s(F_n)$. Let $\varepsilon \to 0$ to obtain $\widetilde{\mathcal{C}}^s(F_n) \ge \alpha^2 \mathcal{C}^s(F_n)$. Therefore $\mathcal{C}^s(F_n) \ge \alpha^2 \mathcal{C}^s(F_n)$. Now $\mathcal{C}^s(F_n) < \infty$ and $\alpha^2 > 1$, so $\mathcal{C}^s(F_n) = 0$.

Thus $\mathcal{C}^s(F_n) = 0$ for all n. By countable subadditivity, we conclude $\mathcal{C}^s(E_1) = 0$. By part (b), $V^s_*(1\!\!1_{E_1}) = 0$ as well.

Next consider the set E_2. Since $\alpha^{-4} < \alpha^{-3}$, the set

$$\beta = \left\{ (r, x) \text{ constituent} : x \in E_2, \frac{\mu\big(\overline{B}_r(x)\big)}{(2r)^s} > \alpha^{-4} \right\}$$

is a fine cover of E_2. Now if $\pi \subseteq \beta$ is a packing, then

$$\sum_{(r,x)\in\pi} (2r)^s < \alpha^4 \sum_\pi \mathcal{C}^s\big(\overline{B}_r(x) \cap E\big) \le \alpha^4 \mathcal{C}^s(E).$$

This is true for all $\pi \subseteq \beta$, so $V^s_\beta(1\!\!1_{E_2}) \le \alpha^4 \mathcal{C}^s(E)$, and thus $V^s_*(1\!\!1_{E_2}) \le \alpha^4 \mathcal{C}^s(E)$.

Combining the two parts, we have

$$V^s_*(1\!\!1_E) \le V^s_*(1\!\!1_{E_1}) + V^s_*(1\!\!1_{E_2}) \le 0 + \alpha^4 \mathcal{C}^s(E).$$

Take the infimum over all $\alpha > 1$ to obtain $V^s_*(1\!\!1_E) \le \mathcal{C}^s(E)$. $\square$

For special h of the form $f(x)(2r)^s$, the fine variation $V^s_*(f)$ is the integral with respect to the covering measure $\mathcal{C}^s$. Note that f is not permitted to have the value ∞.

(3.2) Proposition. *Let S be a separable metric space, let $s > 0$, and let f be a nonnegative real-valued Borel function on S. Then*

$$V_*^s(f) = \int_S f(x)\, \mathcal{C}^s(dx).$$

Proof. First, by Theorem 3.1, we have $V_*^s(\mathbb{1}_A) = \mathcal{C}^s(A)$ for any set A. Clearly, if $a \geq 0$ is a constant and $f \geq 0$ is a finite non-negative function, then $V_*^s(af) = aV_*^s(f)$.

Now both $\mathcal{C}^s(A)$ and $V_*^s(f\, \mathbb{1}_A)$ are metric outer measures, so Borel sets are measurable. If A_n are disjoint Borel sets and $a_n \geq 0$ are constants, then the simple function $f = \sum a_n \mathbb{1}_{A_n}$ satisfies $V_*^s(f) = \int_S f(x)\, \mathcal{C}^s(dx)$.

Finally, if f is a nonnegative Borel function, then there is a sequence f_n of nonnegative Borel measurable simple functions that increases to f. If $c < 1$, then the sets

$$E_n = \{\, x : f_n(x) \geq cf(x) \,\}$$

increase to S. But $V_*^s(f_n) \geq cV_*^s(f\, \mathbb{1}_{E_n})$, so $\lim_n V_*^s(f_n) \geq cV_*^s(f)$. Let $c \to 1$ to conclude that $V_*^s(f_n) \to V_*^s(f)$. Therefore we have

$$V_*^s(f) = \lim_n V_*^s(f_n) = \lim_n \int_S f_n(x)\, \mathcal{C}^s(dx) = \int_S f(x)\, \mathcal{C}^s(dx),$$

as required. □

Next is a strengthened form of (1.5). It identifies $\overline{D}^s_\mu$ as the Radon-Nikodym derivative of μ with respect to $\mathcal{C}^s$. Because $\mathcal{C}^s$ is not σ-finite, the best we can hope for is equality for sets E on which $\mathcal{C}^s$ is finite (or σ-finite). We must also rule out $\overline{D}^s_\mu(x) = \infty$.

(3.3) Theorem. *(a) Let μ be a finite Borel measure on the separable metric space S and let $E \subseteq S$ be a Borel set. Then*

$$\mu(E) \geq \int_E \overline{D}^s_\mu(x)\, \mathcal{C}^s(dx).$$

(b) If, in addition, $\mathcal{C}^s(E) < \infty$, and $\overline{D}^s_\mu < \infty$ on E, then

$$\mu(E) = \int_E \overline{D}^s_\mu(x)\, \mathcal{C}^s(dx).$$

Proof. (a) Write $E = E_0 \cup Q$, where Q is countable, and E_0 has no isolated points. Then $\mathcal{C}^s(Q) = 0$, so $\int_Q \overline{D}^s_\mu(x)\, \mathcal{C}^s(dx) = 0$. Thus we may reduce to the case where E contains no isolated points.

Let $U \supseteq E$ be an open set. Let f be a finite Borel function, $0 \le f \le \overline{D}^s_\mu$ with strict inequality $f(x) < \overline{D}^s_\mu(x)$ whenever $\overline{D}^s_\mu(x) > 0$. Now

$$\beta = \left\{ (r,x) \text{ constituent} : x \in E, \overline{B}_r(x) \subseteq U, \frac{\mu\big(\overline{B}_r(x)\big)}{(2r)^s} \ge f(x) \right\}$$

is a fine cover of E. If $\pi \subseteq \beta$ is a packing, then

$$\sum_{(r,x)\in\pi} f(x)(2r)^s \le \sum_\pi \mu\big(\overline{B}_r(x)\big) = \mu\left(\bigcup_\pi \overline{B}_r(x)\right) \le \mu(U).$$

So $V^s_\beta(f\,1\!\!1_E) \le \mu(U)$, and therefore $V^s_*(f\,1\!\!1_E) \le \mu(U)$. Taking the infimum over U, we obtain $V^s_*(f\,1\!\!1_E) \le \mu(E)$. Now $f < \infty$ on E, so this means $\int_E f(x)\mathcal{C}^s(dx) \le \mu(E)$. But then by the choice of f we may conclude $\int_E \overline{D}^s_\mu(x)\mathcal{C}^s(dx) \le \mu(E)$.

(b) Now suppose $\mathcal{C}^s(E) < \infty$, and $\overline{D}^s_\mu < \infty$ on E. We claim that

$$\int_E \overline{D}^s_\mu(x)\mathcal{C}^s(dx) \ge \mu(E).$$

Since $\overline{D}^s_\mu < \infty$, we must have $\mu(\{x\}) = 0$ for all $x \in E$. So again we may reduce to the case where E contains no isolated points of S.

We claim first that $\mu \ll \mathcal{C}^s$ on E. Let $F \subseteq E$ with $\mathcal{C}^s(F) = 0$. We must show $\mu(F) = 0$. Since $\mathcal{C}^s(F) = 0$, we have $\mathcal{C}^s_\varepsilon(F) = 0$ for all $\varepsilon > 0$. Now for $n \in \mathbb{N}$, let

$$F_n = \left\{ x \in F : \frac{\mu\big(B_r(x)\big)}{(2r)^s} < n \text{ for all } r < \frac{1}{n} \right\}.$$

So F_n increases to F since $\overline{D}^s_\mu(x) < \infty$ on E. Now if $\varepsilon < 1/n$, then any centered ε-cover $\big\{\overline{B}_{r_i}(x_i)\big\}$ of F_n satisfies

$$\sum (2r_i)^s > \frac{1}{n}\sum \mu\big(\overline{B}_{r_i}(x_i)\big) \ge \frac{1}{n}\mu(F_n).$$

Thus, $\mathcal{C}^s_\varepsilon(F_n) \ge (1/n)\mu(F_n)$. Therefore $\mu(F_n) = 0$. By the countable subadditivity of μ, we conclude that $\mu(F) = 0$.

Now let β be a fine cover of E. Then

$$\beta' = \left\{ (r,x) \in \beta : \frac{\mu\big(\overline{B}_r(x)\big)}{(2r)^s} \le \overline{D}^s_\mu(x) + \varepsilon \right\}$$

is also a fine cover of E. But by the Vitali theorem (2.1) there is a packing $\pi \subseteq \beta'$ such that either $\sum_\pi (2r)^s = \infty$ or $\mathcal{C}^s(E\setminus\bigcup_\pi \overline{B}_r(x)) = 0$ so that $\mu(E\setminus\bigcup_\pi \overline{B}_r(x)) = 0$. In either case

$$\sum_{(r,x)\in\pi} \big(\overline{D}^s_\mu(x) + \varepsilon\big)(2r)^s \ge \mu(E) :$$

This is true in the first case since the left-hand side is ∞, and in the second case since $\sum(\overline{D}^s_\mu(x)+\varepsilon)(2r)^s \geq \sum \mu\big(\overline{B}_r(x)\big) \geq \mu(E)$. Thus

$$V^s_\beta\big((\overline{D}^s_\mu + \varepsilon)\,\mathbb{1}_E\big) \geq \mu(E).$$

The infimum over all β yields $V^s_*\big((\overline{D}^s_\mu+\varepsilon)\,\mathbb{1}_E\big) \geq \mu(E)$. The integrand is finite, so this means $\int_E \overline{D}^s_\mu(x)\mathcal{C}^s(dx) + \varepsilon\mathcal{C}^s(E) \geq \mu(E)$. Since $\mathcal{C}^s(E) < \infty$, we may let $\varepsilon \to 0$ to obtain $\int_E \overline{D}^s_\mu(x)\mathcal{C}^s(dx) \geq \mu(E)$. $\square$

(3.4) Corollary. *Let S be a separable metric space, let $s > 0$, and let $E \subseteq S$ be a Borel set with $\mathcal{C}^s(E) < \infty$.*

(a) *Let μ be a finite Borel measure. When the products are well-defined [i.e. not $0 \cdot \infty$], we have*

$$\mathcal{C}^s(E)\ \inf_{x\in E} \overline{D}^s_\mu(x) \leq \mu(E) \leq \mathcal{C}^s(E)\ \sup_{x\in E} \overline{D}^s_\mu(x).$$

(b) *The density theorem: Let μ be the restriction of $\mathcal{C}^s$ to E. Then $\overline{D}^s_\mu(x) = 1$ for $\mathcal{C}^s$-almost every $x \in E$ and $\overline{D}^s_\mu(x) = 0$ for $\mathcal{C}^s$-almost every $x \notin E$.*

Proof. (a) The constant $\inf_{x\in E} \overline{D}^s_\mu(x)$ is a Borel function $\leq \overline{D}^s_\mu$. By Theorem 3.3,

$$\mathcal{C}^s(E)\ \inf_{x\in E} \overline{D}^s_\mu(x) \leq \int_E \overline{D}^s_\mu(x)\mathcal{C}^s(dx) \leq \mu(E).$$

Similarly,

$$\mathcal{C}^s(E)\ \sup_{x\in E} \overline{D}^s_\mu(x) \geq \int_E \overline{D}^s_\mu(x)\mathcal{C}^s(dx) \geq \mu(E).$$

(b) Let μ be the restriction of $\mathcal{C}^s$ to E: that is, $\mu(A) = \mathcal{C}^s(A \cap E)$ for all A. So μ is a finite Borel measure.

For $c < 1$, let $E_c = \left\{ x \in E : \overline{D}^s_\mu(x) \leq c \right\}$. Then

$$\mathcal{C}^s(E_c) = \mu(E_c) \leq \mathcal{C}^s(E_c)\ \sup_{x\in E_c} \overline{D}^s_\mu(x) \leq c\mathcal{C}^s(E_c).$$

So $\mathcal{C}^s(E_c) = 0$. Thus $\overline{D}^s_\mu(x) \geq 1$ a.e. on E.

For $c > 1$, let $E^c = \left\{ x \in E : \overline{D}^s_\mu(x) \geq c \right\}$. Then

$$\mathcal{C}^s(E^c) = \mu(E^c) \geq \mathcal{C}^s(E^c)\ \inf_{x\in E^c} \overline{D}^s_\mu(x) \geq c\mathcal{C}^s(E^c).$$

So $\mathcal{C}^s(E^c) = 0$. Thus $\overline{D}^s_\mu(x) \geq 1$ a.e. on E.

For $c > 0$, let $F_c = \left\{ x \in S \setminus E : \overline{D}^s_\mu(x) \geq c \right\}$. Then

$$0 = \mu(F_c) \geq \mathcal{C}^s(F_c)\ \inf_{x\in F_c} \overline{D}^s_\mu(x) \geq c\mathcal{C}^s(F_c).$$

So $\mathcal{C}^s(F_c) = 0$. Thus $\overline{D}^s_\mu(x) = 0$ a.e. on $S \setminus E$. $\square$

4. Diameter vs. radius

Now let us consider a possible variant of the definitions used here. [In fact, these are the definitions I used at first.] Rather than using $(2r)^s$ for a closed ball $\overline{B}_r(x)$, let us use the actual diameter $\left(\operatorname{diam} \overline{B}_r(x)\right)^s$. In many metric spaces (such as Euclidean space) this makes no difference. But in other metric spaces (including certain subsets of Euclidean space) it does make a difference. In this section we will write the upper s-density of a measure μ as

$$\overline{\Delta}_\mu^s(x) = \limsup_{r\to 0} \frac{\mu\left(\overline{B}_r(x)\right)}{\left(\operatorname{diam} \overline{B}_r(x)\right)^s}.$$

In some metric spaces $\overline{\Delta}_\mu^s$ may fail to be a measurable function (Example 5.3). For fixed $r > 0$, the function $x \mapsto \mu\left(\overline{B}_r(x)\right)$ is a Borel function (in fact upper semi-continuous). But the function $x \mapsto \operatorname{diam} \overline{B}_r(x)$ need not be a measurable function (Example 5.2). If the metric space satisfies $\lim_{r\downarrow a} \operatorname{diam} \overline{B}_r(x) = \operatorname{diam} \overline{B}_a(x)$, then $x \mapsto \operatorname{diam} \overline{B}_r(x)$ is measurable, and then $\overline{\Delta}_\mu^s$ will be measurable. See [1, Lemma 3] and §7, below.

In this section we will temporarily use the notation $\mathcal{T}^s$ for the appropriate covering outer measure:

(4.1) Definition. Let s be a positive number. For $\varepsilon > 0$, define

$$\mathcal{T}_\varepsilon^s(A) = \inf \sum_i \left(\operatorname{diam} \overline{B}_{r_i}(x_i)\right)^s,$$

where the infimum is over all countable covers of A by centered closed balls with diameter $\le \varepsilon$. The **s-dimensional covering pre-measure** of A is

$$\widetilde{\mathcal{T}}^s(A) = \lim_{\varepsilon\to 0} \mathcal{T}_\varepsilon^s(A).$$

The **s-dimensional covering outer measure** is the outer measure $\mathcal{T}^s$ defined by making $\widetilde{\mathcal{T}}^s$ increasing:

$$\mathcal{T}^s(A) = \sup \left\{ \widetilde{\mathcal{T}}^s(E) : E \subseteq A \right\}.$$

Then $\mathcal{T}^s$ is a metric outer measure on S.

The fine variations to be used are, of course, the same as before, but we will write $W_*(h)$ and now we will use the notation $W_*^s(f)$ when the constituent function h is of the special form $h(r,x) = f(x)\left(\operatorname{diam} \overline{B}_r(x)\right)^s$.

With the new notation, most of the same results are proved as before:

(4.2) Theorem. *Let (S,ρ) be a separable metric space, and let $s > 0$. Then for every set $E \subseteq S$, we have $W_*^s(\mathbb{1}_E) = \mathcal{T}^s(E)$.*

Note that the sets F_n in the proof are not at first known to be measurable sets, but the argument shows they have outer measure zero, so they are, in fact, measurable.

(4.3) Proposition. *Let S be a separable metric space, let $s > 0$, and let f be a nonnegative real-valued Borel function on S. Then*

$$W_*^s(f) = \int_S f(x)\,\mathcal{T}^s(dx).$$

If g is a non-negative real function on S, possibly not $\mathcal{T}^s$-measurable, the integral $\int_S g(x)\mathcal{T}^s(dx)$ may not be defined. We will use the upper and lower integrals:

$$\overline{\int} g(x)\mathcal{T}^s(dx) = \inf\left\{ \int f(x)\mathcal{T}^s(dx) : f \geq g, f \text{ Borel measurable} \right\},$$
$$\underline{\int} g(x)\mathcal{T}^s(dx) = \sup\left\{ \int f(x)\mathcal{T}^s(dx) : f \leq g, f \text{ Borel measurable} \right\}.$$

(4.4) Theorem. *(a) Let μ be a finite Borel measure on the separable metric space S and let $E \subseteq S$ be a Borel set. Then*

$$\mu(E) \geq \underline{\int}_E \overline{\Delta}^s_\mu(x)\,\mathcal{T}^s(dx).$$

(b) If, in addition, $\mathcal{T}^s(E) < \infty$, and $\overline{\Delta}^s_\mu < \infty$ on E, then

$$\mu(E) \leq \overline{\int}_E \overline{\Delta}^s_\mu(x)\,\mathcal{T}^s(dx).$$

(c) If, in addition, $\overline{\Delta}^s_\mu$ is Borel measurable, then

$$\mu(E) = \int_E \overline{\Delta}^s_\mu(x)\,\mathcal{T}^s(dx).$$

(4.5) Corollary. *Let S be a separable metric space, let $s > 0$, and let $E \subseteq S$ be a Borel set with $\mathcal{T}^s(E) < \infty$.*

(a) *Let μ be a finite measure. When the products are well-defined [i.e. not $0\cdot\infty$], we have*

$$\mathcal{T}^s(E)\ \inf_{x\in E} \overline{\Delta}^s_\mu(x) \leq \mu(E) \leq \mathcal{T}^s(E)\ \sup_{x\in E} \overline{\Delta}^s_\mu(x).$$

(b) *The density theorem: Let μ be the restriction of $\mathcal{T}^s$ to E. Then $\overline{\Delta}^s_\mu(x) = 1$ for $\mathcal{T}^s$-almost every $x \in E$ and $\overline{\Delta}^s_\mu(x) = 0$ for $\mathcal{T}^s$-almost every $x \notin E$.*

5. Irregular examples

Here we display three examples showing that regularity properties (for diameter) may fail in certain metric spaces:

(a) A set $S \subseteq \mathbb{R}$ and a finite Borel measure μ on S so that

$$\limsup_{r\to 0} \frac{\mu\big(B_r(0)\big)}{\big(\operatorname{diam} B_r(0)\big)^s} \neq \limsup_{r\to 0} \frac{\mu\big(\overline{B}_r(0)\big)}{\big(\operatorname{diam} \overline{B}_r(0)\big)^s}.$$

(b) A set $S \subseteq \mathbb{R}^2$ so that the function $x \mapsto \operatorname{diam} \overline{B}_1(x)$ is a non-measurable function.

(c) A (separable) metric space S and a finite Borel measure μ on S so that $\overline{\Delta}^1_\mu$ is a non-measurable function.

(5.1) Example. Let $s = 1$ and $\alpha = 3/2$. The set is

$$S = \{0\} \cup \bigcup_{n=1}^{\infty} [2^{-n}, 2^{-n}\alpha] \cup \bigcup_{n=1}^{\infty} [-2^{-n}\alpha, -2^{-n}).$$

Let ρ be the restriction of the usual metric of $\mathbb{R}$ to the subset S. Let μ be a discrete measure with mass 2^{-n} at the point 2^{-n}, for $n = 1, 2, \cdots$. Certainly μ is a (regular) Borel measure on S with finite total mass.

Now in the metric space S, for $2^{-n} < r \le 2^{-n}\alpha$,

$$\frac{\mu\left(B_r(0)\right)}{\operatorname{diam} B_r(0)} = \frac{\mu\left(\overline{B}_r(0)\right)}{\operatorname{diam} \overline{B}_r(0)} = \frac{2^{-n+1}}{2r} < 1.$$

For $r = 2^{-n}$,

$$\frac{\mu\left(B_r(0)\right)}{\operatorname{diam} B_r(0)} = \frac{2^{-n}}{2^{-n}\alpha} = \frac{1}{\alpha} < 1,$$

$$\frac{\mu\left(\overline{B}_r(0)\right)}{\operatorname{diam} \overline{B}_r(0)} = \frac{2^{-n+1}}{2^{-n} + 2^{-n-1}\alpha} = \frac{4}{2+\alpha} = \frac{8}{7} > 1.$$

And for $2^{-n-1}\alpha < r < 2^{-n}$,

$$\frac{\mu\left(B_r(0)\right)}{\operatorname{diam} B_r(0)} = \frac{\mu\left(\overline{B}_r(0)\right)}{\operatorname{diam} \overline{B}_r(0)} = \frac{2^{-n}}{2^{-n}\alpha} = \frac{1}{\alpha} < 1.$$

So we have

$$\limsup \frac{\mu\left(\overline{B}_r(0)\right)}{\operatorname{diam} \overline{B}_r(0)} = \frac{8}{7} \qquad \text{but} \qquad \limsup \frac{\mu\left(B_r(0)\right)}{\operatorname{diam} B_r(0)} \le 1.$$

(5.2) Example. Let V be a non-measurable subset of the interval $[0,1/2]$. For example, let V be a Bernstein set: $K \cap V \neq \varnothing$ and $K \setminus V \neq \varnothing$ for every uncountable closed set $K \subseteq [0,1/2]$. Define $S \subseteq \mathbb{R}^2$ by:

$$S = \{\, (x,0) : 0 \leq x \leq 1/2 \,\} \cup \{\, (x,1) : x \in V \,\},$$

a subset of two horizontal intervals. Then in the metric space S, the diameter of a unit ball $\overline{B}_1(x,0)$ is either ≥ 1 or $\leq 1/2$, according as $x \in V$ or not. So $\operatorname{diam} \overline{B}_1(x,0)$ is a non-measurable function of the point $(x,0)$.

(5.3) Example. This example has non-measurable upper density $\overline{\Delta}^s_\mu$. It is a bit more involved than the others. It is not clear to me whether there is such an example where S is (isometric to) a subset of Euclidean space $\mathbb{R}^n$. (Of course, since the example below is a zero-dimensional separable metric space, it is *homeomorphic* to a subset of the line $\mathbb{R}$.)

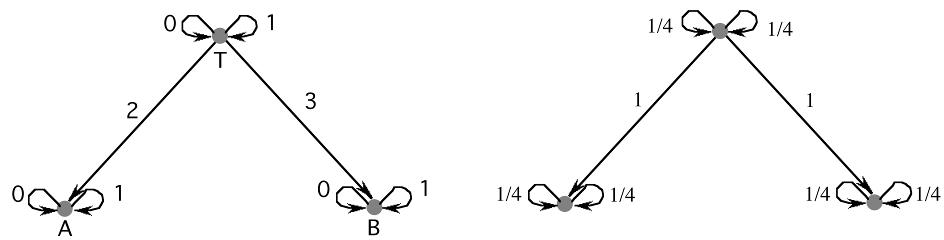

Figure 1. (a) labels; (b) ratios.

We will use the notation from my text [5, §4.3]. Begin with an alphabet $E = \{\mathbf{0},\mathbf{1},\mathbf{2},\mathbf{3}\}$. These letters label the edges of a directed multigraph (Figure 1). A compact space T of infinite strings is thus defined: Let T be the set of all infinite strings σ from the alphabet E where all letters (with possibly a single exception) are $\mathbf{0}$'s and $\mathbf{1}$'s.

Next we define a metric ρ on T. The recursive rules are:

$$\begin{aligned}
&\rho(\mathbf{0}\sigma,\mathbf{0}\tau) = \rho(\sigma,\tau)/4\\
&\rho(\mathbf{1}\sigma,\mathbf{1}\tau) = \rho(\sigma,\tau)/4\\
&\rho(\mathbf{2}\sigma,\mathbf{2}\tau) = \rho(\sigma,\tau)\\
&\rho(\mathbf{3}\sigma,\mathbf{3}\tau) = \rho(\sigma,\tau)\\
&\rho(\mathbf{0}\sigma,\mathbf{1}\tau) = 1\\
&\rho(\mathbf{0}\sigma,\mathbf{2}\tau) = 1 + \rho(\mathbf{0}\sigma,\tau)/4\\
&\rho(\mathbf{0}\sigma,\mathbf{3}\tau) = 1\\
&\rho(\mathbf{1}\sigma,\mathbf{2}\tau) = 1 + \rho(\mathbf{1}\sigma,\tau)/4\\
&\rho(\mathbf{1}\sigma,\mathbf{3}\tau) = 1\\
&\rho(\mathbf{2}\sigma,\mathbf{3}\tau) = 2.
\end{aligned}$$

Also, $\rho(\sigma,\tau)=\rho(\tau,\sigma)$ and $\rho(\sigma,\sigma)=0$. Verify by cases that ρ satisfies the triangle inequality. If $\alpha\in\{\mathbf{0},\mathbf{1}\}^{(*)}$ is a finite string of $\mathbf{0}$'s and $\mathbf{1}$'s, then the cylinder set $[\alpha]$ has diameter $4^{-|\alpha|}\cdot 2$. Indeed, if $\sigma,\tau\in[\alpha]$ do not involve $\mathbf{2}$ or $\mathbf{3}$, then $\rho(\sigma,\tau)\le 4^{-|\alpha|}$; except for $(\alpha\mathbf{2}\sigma',\alpha\mathbf{3}\tau')$ the maximum is $4^{-|\alpha|}+\rho(\sigma',\tau')/4$ for certain $\sigma',\tau'\in[\alpha]$, so it is $\le 4^{-|\alpha|}+4^{-|\alpha|}\cdot 2/4=4^{-|\alpha|}\cdot(3/2)$.

Now T is a "graph self-similar" set. It is made up of four parts: two parts $A=[\mathbf{2}]$, $B=[\mathbf{3}]$, which are $(1/4,1/4)$-Cantor sets; and two parts $[\mathbf{0}]$, $[\mathbf{1}]$ similar to all of T, shrunk by a factor $1/4$.

Next define a measure μ by defining it on the basic cylinder sets $[\alpha]$. If $\alpha,\beta\in\{\mathbf{0},\mathbf{1}\}^{(*)}$ are finite strings of $\mathbf{0}$'s and $\mathbf{1}$'s, let:

$$\mu[\alpha]=4^{-|\alpha|};\quad \mu[\alpha\mathbf{2}\beta]=0;\quad \mu[\alpha\mathbf{3}\beta]=4^{-|\alpha|}\cdot 2^{-1-|\beta|}.$$

The measure may be thought of as constructed by a Markov chain with probabilities shown in Figure 2. Note that $\mu\big(T\setminus\{\mathbf{0},\mathbf{1},\mathbf{3}\}^{(\omega)}\big)=0$: the set of strings with a letter $\mathbf{2}$ has measure 0.

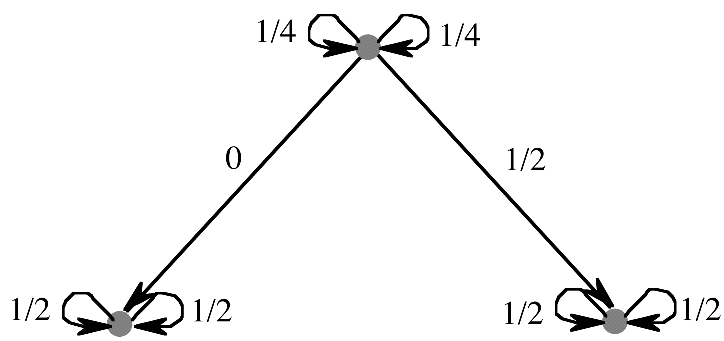

Figure 2. A Markov chain.

Now the subset $K=\{\mathbf{0},\mathbf{1}\}^{(\omega)}$ contained in T is a copy of the Cantor set (with metric $1/4,1/4$). Let V be a non-measurable subset of it, say a Bernstein set. Define a subset $S\subseteq T$ by removing all strings $\alpha\mathbf{2}\sigma'$ where $\alpha\sigma'\notin V$. Then S will be our (noncompact) metric space. We throw out only a subset of $T\setminus\{\mathbf{0},\mathbf{1},\mathbf{3}\}^{(\omega)}$, a closed set of measure 0, so μ is still a regular Borel measure on the remainder S.

Consider a ball $\overline{B}_r(\sigma)$ in S, where $\sigma\in K\subseteq S$. Choose the integer k so that $4^{-k}\le r<4^{-k+1}$. Write α for the first k letters of σ, and σ' for the rest of σ (so $\sigma=\alpha\sigma'$). If $4^{-k}<r<4^{-k+1}$, then

$$\begin{aligned}&\overline{B}_r(\sigma)\subseteq[\alpha]\\&\overline{B}_r(\sigma)\supseteq[\alpha\mathbf{0}]\cup[\alpha\mathbf{1}]\cup[\alpha\mathbf{3}]\cup[\alpha\mathbf{2}\beta]\end{aligned}$$

for a long enough prefix β of σ'. If $r=4^{-k}$, then

$$\begin{aligned}\overline{B}_r(\sigma)&=[\alpha\mathbf{0}]\cup[\alpha\mathbf{1}]\cup[\alpha\mathbf{3}]\cup\{\alpha\mathbf{2}\sigma'\} &&\text{if } \sigma\in V\\ \overline{B}_r(\sigma)&=[\alpha\mathbf{0}]\cup[\alpha\mathbf{1}]\cup[\alpha\mathbf{3}] &&\text{if } \sigma\notin V.\end{aligned}$$

So, for $4^{-k} < r < 4^{-k+1}$,

$$\mu\big(\overline{B}_r(\sigma)\big) = 4^{-k}, \qquad \operatorname{diam} \overline{B}_r(\sigma) = 4^{-k} \cdot 2, \qquad \frac{\mu\big(\overline{B}_r(\sigma)\big)}{\operatorname{diam} \overline{B}_r(\sigma)} = \frac{1}{2}.$$

And for $r = 4^{-k}$, $\mu\big(\overline{B}_r(\sigma)\big) = 4^{-k}$ and $\operatorname{diam} \overline{B}_r(\sigma) = 4^{-k} \cdot 2$ if $\sigma \in V$ but $\operatorname{diam} \overline{B}_r(\sigma) \le 4^{-k} \cdot (3/2)$ if $\sigma \notin V$ so that

$$\frac{\mu\big(\overline{B}_r(\sigma)\big)}{\operatorname{diam} \overline{B}_r(\sigma)} = \frac{1}{2} \text{ if } \sigma \in V \qquad \text{but} \qquad \frac{\mu\big(\overline{B}_r(\sigma)\big)}{\operatorname{diam} \overline{B}_r(\sigma)} \ge \frac{2}{3} \text{ if } \sigma \notin V.$$

Thus

$$\left\{ \sigma \in K : \overline{\Delta}^1_\mu(\sigma) < \frac{2}{3} \right\} = V$$

is a non-measurable set.

Note that $\mathcal{T}^1(S) = 0$ since $\dim T = 1/2$. But $\overline{\Delta}^1_\mu(\sigma) = \infty$ for σ that involve a letter **3**, so this measure μ does not contradict (4.4 b) or (4.5 a).

6. Hausdorff Measure

The Hausdorff measures $\mathcal{H}^s$ may be treated in a similar way. The derivation basis must be changed, however. The case for the metric space $\mathbb{R}$ was treated by Thomson [14].

Let (S, ρ) be a separable metric space. A **constituent** is a pair (B, x), where B is a closed bounded set with at least two points [or a single isolated point of S], and $x \in B$. A **packing** is a disjoint collection π of consitutents. A **closed cover** of a set A is a collection β of constituents (B, x) with $x \in A$ such that $A \subseteq \bigcup_{(B,x)\in\beta} B$. A **fine cover** of A is a collection β of constituents such that, for every $x \in A$, and every $\varepsilon > 0$, there is B such that $(B, x) \in \beta$, $x \in A$, and $\operatorname{diam} B < \varepsilon$.

The Banach argument in the Vitali theorem fits this setting [8, Theorem 1.10].

(6.1) Theorem. *Let (S, ρ) be a separable metric space, let $E \subseteq S$ be a set, let β be a fine cover of E, and let $s > 0$. Then there is a (finite or infinite) packing $\pi = \{(B_i, x_i)\} \subseteq \beta$ such that:*

$$\textit{either} \qquad \sum_i (\operatorname{diam} B_i)^s = \infty \qquad \textit{or} \qquad \mathcal{H}^s\left(E \setminus \bigcup_i B_i\right) = 0.$$

Let s be a positive number, and let $A \subseteq S$ be a set. For $\varepsilon > 0$, define

$$\mathcal{H}^s_\varepsilon(A) = \inf \sum_{(B,x)\in\beta} (\operatorname{diam} B)^s,$$

where the infimum is over all countable closed covers β of A. The **s-dimensional Hausdorff outer measure** of a set A is

$$\mathcal{H}^s(A) = \lim_{\varepsilon\to 0} \mathcal{H}^s_\varepsilon(A) = \sup_{\varepsilon>0} \mathcal{H}^s_\varepsilon(A).$$

The closure of a set has the same diameter as the set itself. Thus, in the definition of Hausdorff measure, we may consider only covers by closed sets.

Let μ be a finite Borel measure. The **upper s-density** required for this basis, defined for non-isolated points $x \in S$, is:

$$\overline{d}^s_\mu(x) = \lim_{r \to 0} \left(\sup \left\{ \frac{\mu(B)}{(\text{diam } B)^s} : B \text{ closed}, x \in B, 0 < \text{diam } B < r \right\} \right).$$

If S is Euclidean space, then since the convex hull of a closed set is a closed convex set with the same diameter, the same upper density is obtained if we restrict the sets B to convex sets; thus $\overline{d}^s_\mu$ is sometimes known as the **upper convex density** [8, p. 21].

(6.2) Proposition. *The upper density $\overline{d}^s_\mu(x)$ is a Borel measurable function of x.*

Proof. First note that open sets may be used in the definition:

$$\overline{d}^s_\mu(x) = \lim_{r \to 0} \left(\sup \left\{ \frac{\mu(U)}{(\text{diam } U)^s} : U \text{ open}, x \in U, 0 < \text{diam } U < r \right\} \right).$$

Indeed, for any closed set B, the $1/n$-neighborhood U_n of B decreases to B, and diam $U_n \to$ diam B; $\mu(U_n) \to \mu(B)$. And for any open set U, the set B_n of points with distance at least $1/n$ from the complement is closed, B_n increases to U, and diam $B_n \to$ diam U; $\mu(B_n) \to \mu(U)$.

Now for fixed $r > 0$, write

$$\theta_r(x) = \sup \left\{ \frac{\mu(U)}{(\text{diam } U)^s} : U \text{ open}, x \in U, 0 < \text{diam } U < r \right\}.$$

We claim that θ_r is a Borel function. In fact, for each t, the set $L_t = \{ x : \theta_r(x) > t \}$ is an open set. To see this, suppose $x_0 \in L_t$. Then there is an open set U with $x_0 \in U$, $0 <$ diam $U < r$, and

$$\frac{\mu(U)}{(\text{diam } U)^s} > t.$$

But then, for any $x \in U$, the same set U shows that $\theta_r(x) > t$. Thus L_t is open.

Finally, $\overline{d}^s_\mu(x) = \lim_n \theta_{1/n}(x)$ is the pointwise limit of a sequence of Borel functions, so it is Borel. □

If h is a non-negative real-valued constituent function, the fine variation is defined much as before: if β is a closed cover, then

$$v_\beta(h) = \sup \sum_{(B,x) \in \pi} h(B, x),$$

where the supremum is over all packings $\pi \subseteq \beta$; the **fine variation** of h is

$$v_*(h) = \inf v_\beta(h),$$

where the infimum is over all fine covers β. When $h(B, x) = f(x)(\text{diam } B)^s$, write $v^s_\beta(f) = v_\beta(h)$ and $v^s_*(f) = v_*(h)$.

The following results are proved in the same way as before. Replace $\mathcal{T}^s$ by $\mathcal{H}^s$, "centered cover" by "closed cover", V_β by v_β, $\overline{\Delta}^s_\mu$ by $\overline{d}^s_\mu$, and so on. The Vitali theorem (6.1) should be used.

(6.3) Theorem. *Let h be a nonnegative constituent function. Then μ defined by $\mu(A) = v_*(h\,\mathbb{1}_A)$ is a metric outer measure on S.*

(6.4) Theorem. *Let (S, ρ) be a separable metric space, and let $s > 0$. Then for every set $E \subseteq S$, we have $v_*^s(\mathbb{1}_E) = \mathcal{H}^s(E)$.*

In part (c), define

$$F_n = \left\{ x \in E_1 : \frac{\mu(B)}{(\text{diam } B)^s} < \alpha^{-2} \text{ for all } B \text{ with } x \in B, \text{diam } B < \frac{1}{n} \right\}.$$

(6.5) Proposition. *Let S be a separable metric space, let $s > 0$, and let f be a nonnegative real-valued Borel function on S. Then*

$$v_*^s(f) = \int_S f(x)\,\mathcal{H}^s(dx).$$

(6.6) Theorem. *(a) Let μ be a finite Borel measure on the separable metric space S and let $E \subseteq S$ be a Borel set. Then*

$$\mu(E) \geq \int_E \overline{d}_\mu^s(x)\,\mathcal{H}^s(dx).$$

(b) If, in addition, $\mathcal{H}^s(E) < \infty$, and $\overline{d}_\mu^s < \infty$ on E, then

$$\mu(E) = \int_E \overline{d}_\mu^s(x)\,\mathcal{H}^s(dx).$$

The following corollaries are known; see [1], [15], [8, Theorems 2.3, 2.4].

(6.7) Corollary. *Let S be a separable metric space, let $s > 0$, and let $E \subseteq S$ be a Borel set with $\mathcal{H}^s(E) < \infty$.*

(a) *Let μ be a finite Borel measure. When the products are well-defined [i.e. not $0 \cdot \infty$], we have*

$$\mathcal{H}^s(E) \inf_{x \in E} \overline{d}_\mu^s(x) \leq \mu(E) \leq \mathcal{H}^s(E) \sup_{x \in E} \overline{d}_\mu^s(x).$$

(b) *The density theorem: Let μ be the restriction of $\mathcal{H}^s$ to E. Then $\overline{d}_\mu^s(x) = 1$ for $\mathcal{H}^s$-almost every $x \in E$ and $\overline{d}_\mu^s(x) = 0$ for $\mathcal{H}^s$-almost every $x \notin E$.*

7. Open-ball diameter covering measure

Let us consider an alternative for the diameter covering measure. We are interested in using open balls in the centered covers and in the upper density. Since closed balls appear in the Vitali theorem (2.1), some care must be taken.

A simple variant of the lim sup and lim inf will be used. They are the limsup and liminf if we ignore countable sets.

Suppose $q(r) \in \mathbb{R}$ is defined for each $r > 0$. Then

(a) $\operatorname{u\,lim\,sup}_{r\to 0} q(r)$ is the infimum of all α such that, for some $\eta > 0$, we have $q(r) < \alpha$ for all but countably many r with $0 < r < \eta$.

(b) $\operatorname{u\,lim\,sup}_{r\to 0} q(r) \geq \alpha$ means: for all $\varepsilon > 0$ and all $\eta > 0$, there are uncountably many r with $0 < r < \eta$ and $q(r) > \alpha - \varepsilon$.

(c) $\operatorname{u\,lim\,inf}_{r\to 0} q(r)$ is the supremum of all α such that, for some $\eta > 0$, we have $q(r) > \alpha$ for all but countably many r with $0 < r < \eta$.

(d) $\operatorname{u\,lim\,inf}_{r\to 0} q(r) \leq \alpha$ means: for all $\varepsilon > 0$ and all $\eta > 0$, there are uncountably many r with $0 < r < \eta$ and $q(r) < \alpha + \varepsilon$.

Let us begin with the density. First, note that the *lower* density is the same for open and closed sets.

(7.1) Proposition. *Let (S, ρ) be a metric space, let $x \in S$ be a non-isolated point, let μ be a finite Borel measure, and let $s > 0$. Then the following four values are all equal:*

$$D_1 = \liminf_{r\to 0} \frac{\mu\big(B_r(x)\big)}{\big(\operatorname{diam} B_r(x)\big)^s}$$

$$D_2 = \operatorname{u}\liminf_{r\to 0} \frac{\mu\big(B_r(x)\big)}{\big(\operatorname{diam} B_r(x)\big)^s}$$

$$D_3 = \liminf_{r\to 0} \frac{\mu\big(\overline{B}_r(x)\big)}{\big(\operatorname{diam} \overline{B}_r(x)\big)^s}$$

$$D_4 = \operatorname{u}\liminf_{r\to 0} \frac{\mu\big(\overline{B}_r(x)\big)}{\big(\operatorname{diam} \overline{B}_r(x)\big)^s}.$$

Proof. The relevant facts are: $B_{r+\varepsilon}(x) \downarrow \overline{B}_r(x)$ as $\varepsilon \downarrow 0$, so

(i) $\mu\big(B_{r+\varepsilon}(x)\big) \to \mu\big(\overline{B}_r(x)\big)$;

and $\overline{B}_{r-\varepsilon}(x) \uparrow B_r(x)$, so

(ii) $\mu\big(\overline{B}_{r-\varepsilon}(x)\big) \to \mu\big(B_r(x)\big)$;

(iii) $\operatorname{diam} \overline{B}_{r-\varepsilon}(x) \to \operatorname{diam} B_r(x)$.

To begin, $D_1 \leq D_2$ and $D_3 \leq D_4$ are clear.

Next, suppose $\alpha > D_1$. For $\eta > 0$, there is r with $0 < r < \eta$ and

$$\frac{\mu\big(B_r(x)\big)}{\big(\operatorname{diam} B_r(x)\big)^s} < \alpha.$$

Use (ii) and (iii): if $\varepsilon > 0$ is small enough, then

$$\frac{\mu\big(\overline{B}_{r-\varepsilon}(x)\big)}{\big(\operatorname{diam} \overline{B}_{r-\varepsilon}(x)\big)^s} < \alpha.$$

Thus we have $D_4 \le \alpha$. This holds for any $\alpha > D_1$, so this means $D_4 \le D_1$.

Next suppose $\alpha > D_3$. For $\eta > 0$, there is r with $0 < r < \eta$ and

$$\frac{\mu\big(B_r(x)\big)}{\big(\text{diam } B_r(x)\big)^s} < \alpha.$$

If $\varepsilon > 0$ is small enough, then $\mu\big(B_{r+\varepsilon}(x)\big) \approx \mu\big(\overline{B}_r(x)\big)$ and $\text{diam } B_{r+\varepsilon}(x) \ge \text{diam } \overline{B}_r(x)$, so

$$\frac{\mu\big(\overline{B}_{r+\varepsilon}(x)\big)}{\big(\text{diam } \overline{B}_{r+\varepsilon}(x)\big)^s} < \alpha.$$

Thus we have $D_2 \le \alpha$. Therefore $D_2 \le D_3$.

To summarize: $D_1 \le D_2 \le D_3 \le D_4 \le D_1$, so they are all equal. $\square$

The lower density $\underline{\Delta}^s_\mu(x)$ is defined by any of the four expressions. It is a Borel measurable function: for a fixed r, the functions $x \mapsto \mu\big(B_r(x)\big)$ and $x \mapsto \text{diam } B_r(x)$ are both lower semicontinuous, hence measurable. For fixed x, the functions $r \mapsto \mu\big(B_r(x)\big)$ and $r \mapsto \text{diam } B_r(x)$ are continuous from the left, so the lim inf called D_1 above is the same when computed using only rational r. So $\underline{\Delta}^s_\mu$ is a measurable function of x.

An example (5.2) shows that things are not as nice for the upper density.

(7.2) Proposition. *Let (S,ρ) be a metric space, let $x \in S$ be a non-isolated point, let μ be a finite Borel measure, and let $s > 0$. Write:*

$$D_1 = \limsup_{r \to 0} \frac{\mu\big(B_r(x)\big)}{\big(\text{diam } B_r(x)\big)^s}$$

$$D_2 = \text{u}\limsup_{r \to 0} \frac{\mu\big(B_r(x)\big)}{\big(\text{diam } B_r(x)\big)^s}$$

$$D_3 = \limsup_{r \to 0} \frac{\mu\big(\overline{B}_r(x)\big)}{\big(\text{diam } \overline{B}_r(x)\big)^s}$$

$$D_4 = \text{u}\limsup_{r \to 0} \frac{\mu\big(\overline{B}_r(x)\big)}{\big(\text{diam } \overline{B}_r(x)\big)^s}.$$

Then $D_1 = D_2 = D_4 \le D_3$.

Proof. First, $D_1 \ge D_2$ and $D_3 \ge D_4$ are clear.

Next, suppose $\alpha < D_1$. For $\eta > 0$, there is r with $0 < r < \eta$ and

$$\frac{\mu\big(B_r(x)\big)}{\big(\text{diam } B_r(x)\big)^s} > \alpha.$$

For $\varepsilon > 0$ small enough,

$$\frac{\mu\big(\overline{B}_{r-\varepsilon}(x)\big)}{\big(\text{diam } \overline{B}_{r-\varepsilon}(x)\big)^s} > \alpha.$$

Thus $D_4 \geq \alpha$. This proves $D_4 \geq D_1$.

Now as ε decreases to 0, the open ball $B_{r+\varepsilon}(x)$ decreases to the closed ball $\overline{B}_r(x)$. But diam $B_{r+\varepsilon}(x)$ need not converge to diam $\overline{B}_r(x)$. However, diam $B_r(x)$ is a monotone function of r, so it is continuous at all but countably many values of r. Wherever it is continuous,

$$\operatorname{diam} B_r(x) = \operatorname{diam} \overline{B}_r(x) = \lim_{\varepsilon\to 0} \operatorname{diam} B_{r+\varepsilon}(x).$$

Next let $\alpha < D_4$. For $\eta > 0$ there are uncountably many r with $0 < r < \eta$ and

$$\frac{\mu\big(\overline{B}_r(x)\big)}{\big(\operatorname{diam} \overline{B}_r(x)\big)^s} > \alpha.$$

In particular, there is such an r where diam $\overline{B}_r(x) = \lim_{\varepsilon\to 0} \operatorname{diam} B_{r+\varepsilon}(x)$. So for $\varepsilon > 0$ small enough,

$$\frac{\mu\big(B_{r+\varepsilon}(x)\big)}{\big(\operatorname{diam} B_{r+\varepsilon}(x)\big)^s} > \alpha.$$

Thus, as usual, $D_2 \geq D_4$.

In summary, $D_2 \geq D_4 \geq D_1 \geq D_2$ so these three are all equal. Also, $D_3 \geq D_4$, but this inequality may be strict. □

Write $\overline{\mathcal{D}}^s_\mu(x)$ for the upper density defined by $D_1 = D_2 = D_4$. Clearly $\overline{\mathcal{D}}^s_\mu(x) \leq \overline{\Delta}^s_\mu(x)$. But (because it is defined by D_1) $\overline{\mathcal{D}}^s_\mu$ is a Borel measurable function of x.

The covering measure to use in this connection is defined with centered covers by open balls. This measure differs from $\mathcal{C}^s$ (and from $\mathcal{H}^s$ and $\mathcal{T}^s$) by at most a factor 2^s. In particular, measure 0 and finite measure are the same as for the other measures.

Now consider the Vitali theorem (6.1). A **constituent** will be a pair $(B_r(x), x)$; an open ball and its center. Let us say that a collection β of open constituents is a **very fine cover** of a set A iff for every $x \in A$ and every $\varepsilon > 0$, there exist uncountably many r with $0 < r < \varepsilon$ and $\big(B_r(x), x\big) \in \beta$. If β is a very fine cover of a set E, then

$$\beta' = \left\{ \big(\overline{B}_r(x), x\big) : \big(B_r(x), x\big) \in \beta, \operatorname{diam} B_r(x) = \operatorname{diam} \overline{B}_r(x) \right\}$$

is a fine cover of E by closed balls. Thus, (2.1) may be applied to β'. We get $\{B_1, B_2, \cdots\} \subseteq \beta$ with either $\sum(\operatorname{diam} B_i)^s = \infty$ [open or closed balls the same] or $\mathcal{T}^s(E \setminus \bigcup_i \overline{B}_i) = 0$; here we get only closed balls. But the result may still be used in the applications (3.1) and (3.3), provided we use very fine covers. In a very fine cover, we may delete all balls $B_r(x)$ with diam $B_r(x) \neq$ diam $\overline{B}_r(x)$, and if μ is a finite Borel measure, we may similarly delete all balls $B_r(x)$ with $\mu\big(B_r(x)\big) \neq \mu\big(\overline{B}_r(x)\big)$.

The very fine variation will be defined as

$$\inf V_\beta(h)$$

with infimum over all very fine covers β (by centered open balls).

The results of §4 should remain correct, using the density $\overline{\mathcal{D}}^s_\mu$, the very fine variation, and the covering measure defined with centered open ball covers. Because the upper density $\overline{\mathcal{D}}^s_\mu$ is Borel measurable, there is no need to use upper and lower integrals.

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