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REVIEW

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Fractals and Universal Spaces in Dimension Theory, by Stephen Leon Lipscomb. Springer Monographs in Mathematics, Springer, New York, 2009. xviii+241 pp., ISBN: 978-0-387-85493-9 (Print) 978-0-387-85494-6 (Online), US $79.95

"Dimension" in the fractal sense and "dimension" in the topological sense—although both born in the early 1900s from closely-related sources, and having much in common in the early years—developed more or less independently for fifty years. But recently they have been brought together again in connection with investigation of "universal spaces" as I will describe later. In the monograph under review, Stephen Lipscomb describes the mathematics involved in the study of these universal spaces. The account extends over most of the past century, and includes some work published as recently as 2008. This book will of course be an essential reference for those working on dimension theory in point-set topology. But I hope the next few pages will suggest that this book will also appeal to those interested in fractals—in particular in iterated function systems.

This review is not a historical account. Lipscomb devotes a full chapter to to describe who did what, when, and the interdependence of the results. So I do not attempt to sort that out here.

1. Iterated Function Systems

The *fractals* involved here are specifically those described by *iterated function systems.* Let A be a finite set; it will be called an "alphabet" and its elements will be called "letters." As usual, letters will be combined into *words* or *strings* denoted by juxtaposition: $a_1 a_2 \cdots a_n$.

Now write $N(A)$ for the set of all infinite strings from the alphabet A. An overbar will be used when a block repeats indefinitely from some point on: $01\overline{10} := 011010101010 10\cdots$. We consider $N(A)$ to be a topological space by taking A to be discrete and using the (infinite) product topology for $N(A)$.

Let X be a space: a topological space, usually a metric space. For each letter $a \in A$, let $w_a : X \to X$ be a function from X into itself. The collection $\{\, w_a : a \in A \,\}$ is an *iterated function system* or IFS. We iterate the functions w_a in all possible ways: if $a_1 a_2 \cdots a_n$ is a word, write

$$w_{a_1 a_2 \cdots a_n} = w_{a_1} \circ w_{a_2} \circ \cdots \circ w_{a_n}.$$

Under the right conditions, infinite iterates

$$w_{\mathbf{a}}(x) = w_{a_1 a_2 \cdots}(x) := \lim_{n\to\infty} w_{a_1 a_2 \cdots a_n}(x) \qquad (\text{for } x \in X), \tag{1}$$

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are defined for all $\mathbf{a} = a_1 a_2 \cdots \in N(A)$. If (X, ρ) is a complete metric space and

$$\exists c < 1, \quad \forall a \in A, \quad \forall x, y \in X, \quad \rho\bigl(w_a(x), w_a(y)\bigr) \le c\rho(x, y) \tag{2}$$

(so that each w_a is a contraction on X), then the the limits (1) exist for all $\mathbf{a} \in N(A)$, and the value is independent of x. So the iterated function system defines a function $p\colon N(A) \to X$ where $p(\mathbf{a}) = w_{\mathbf{a}}(x)$. And under condition (2) the function p is continuous. The set of values of p,

$$K = \{\, p(\mathbf{a}) : \mathbf{a} \in N(A) \,\} \tag{3a}$$

is called the *attractor* of the IFS. The string $\mathbf{a}$ is called an *address* of the point $p(\mathbf{a})$. Note that K is homeomorphically a quotient of $N(A)$ in the following sense: Define equivalence relation $\sim$ on $N(A)$ by: $\mathbf{a} \sim \mathbf{b} \iff p(\mathbf{a}) = p(\mathbf{b})$; then p induces a continuous bijection from the quotient topological space $N(A)/\!\sim$ onto K. Because of compactness, the bijection is a homeomorphism.

In the setting mentioned above, where X is a complete metric space and (2) holds, the attractor may also be constructed in another way. It it the unique nonempty compact set K satisfying

$$K = \bigcup_{a \in A} w_a(K). \tag{3b}$$

Write $\mathcal{C}_X$ for the collection of nonempty compact subsets of X. Under the Hausdorff metric, $\mathcal{C}_X$ is a complete metric space. With the definition

$$\mathcal{W}(E) := \bigcup_{a \in A} w_a(E), \qquad \text{for } E \in \mathcal{C}_X, \tag{4a}$$

we get a map $\mathcal{W}\colon \mathcal{C}_X \to \mathcal{C}_X$, the *Hutchinson operator*. Under hypothesis (2), the map $\mathcal{W}$ is a contraction on the complete metric space $\mathcal{C}_X$, so it has a unique fixed point K which is obtained as the limit of iterates

$$\mathcal{W} \circ \mathcal{W} \circ \cdots \circ \mathcal{W}(E),$$

starting with any nonempty compact set E.

Many examples (which are today known as fractals, and as attractors of IFSs) were first described about a century ago. Some of them will be seen again later, so I will mention them now.

Cantor set. Let $A = \{0, 1\}$, $X = \mathbb{R}$, $w_0(x) = x/3$, and $w_1(x) = (x + 2)/3$ to get the Cantor set C. The maps are homotheties, centered at 0 and 1, with contraction ratio 1/3. A few addresses are shown on the left side of Figure 1. "Identify adjacent endpoints" to get the interval $I = [0, 1]$ shown on the right side. The interval I is also an attractor, obtained using homotheties, centered at 0 and 1, but with contraction ratio 1/2. Imagine letting the ratio vary. When the ratio increases from 1/3, the gaps in the attractor decrease in size, until at ratio 1/2 adjacent endpoints merge. In I, the address of a point is its binary expansion.

Sierpinski gasket. Let $A = \{\mathrm{z}, \mathrm{a}, \mathrm{b}\}$, $X = \mathbb{R}^2$. Let $w_{\mathrm{z}}, w_{\mathrm{a}}, w_{\mathrm{b}}$ be homotheties with ratio 1/3 and centers at the vertices of an equilateral triangle. Then we get the attractor shown on the left of Figure 2. "Identify adjacent endpoints" to get the Sierpinski gasket G, shown on the right. (Or: G itself is the attractor if the ratio for the homotheties is 1/2 instead of 1/3. We can imagine the ratio increasing from 1/3 to see the adjacent endpoints approach each other, and at ratio 1/2 they merge.)

FIGURE 1. Cantor set C; interval I

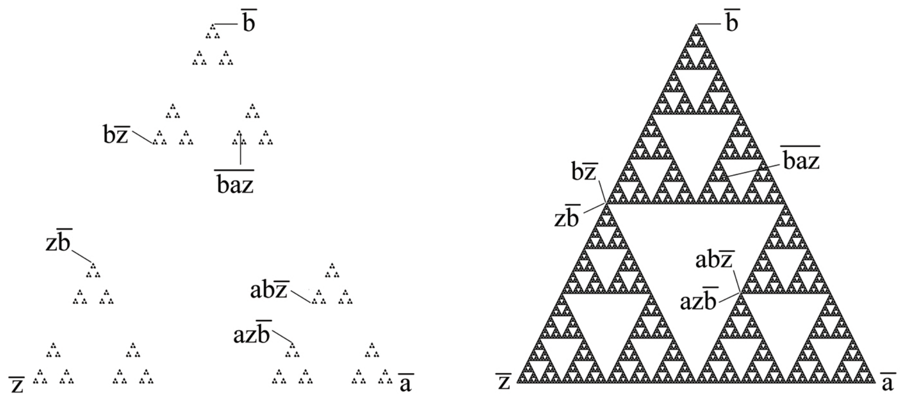

FIGURE 2. Space $N(\{\mathrm{z}, \mathrm{a}, \mathrm{b}\})$; Sierpinski gasket G

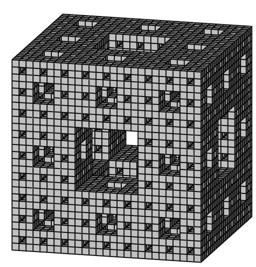

FIGURE 3. Menger sponge M

Note that G is the union of three parts $G_{\mathrm{z}} = w_{\mathrm{z}}(G)$, $G_{\mathrm{a}} = w_{\mathrm{a}}(G)$, $G_{\mathrm{b}} = w_{\mathrm{b}}(G)$, and that any two of these parts meet in exactly one point.

Menger sponge. This familiar fractal M is pictured in Figure 3. The alphabet has 20 letters, and the maps w_a are homotheties with ratio $1/3$. This time, however, some pairs $w_a(M), w_b(M)$ meet in more than one point, while other pairs do not meet at all.

2. Separable Metric Space

The *dimension theory* involved here is specifically the theory of topological dimension. I begin with a description of the *covering* dimension or *Lebesgue* dimension (from p. 200): A family $\mathcal{U}$ of subsets of a space X has *order* n iff some point of X belongs to n different sets of $\mathcal{U}$, but no point of X belongs to $n+1$ or more different sets of $\mathcal{U}$. The space X has covering dimension $\leq n$ iff for each finite open covering $\{V_1, \dots, V_k\}$ of X there is an open covering $\{U_1, \dots, U_k\}$ of order $\leq n+1$ such that each $U_i \subseteq V_i$. Then we may write $\dim X \leq n$. When $\dim X \leq n$ but not $\dim X \leq n-1$, then we write $\dim X = n$. Also, $\dim \varnothing = -1$ by convention; and $\dim X = \infty$ iff $\dim X \leq n$ fails for all natural numbers n. Of course $\dim \mathbb{R}^n = n$, although the usual proof of that relies on algebraic topology.

Another topological dimension is the (small or weak) *inductive dimension.* Begin with $\operatorname{ind} \varnothing = -1$. Then (inductively) define $\operatorname{ind} X \leq n$ if and only if there is a base $\mathcal{U}$ for the topology of X such that $\operatorname{ind} B(U) \leq n-1$ for all $U \in \mathcal{U}$. [Here we wrote $B(U)$ for the topological boundary of the set U because topologists may reserve the notation ∂U for another type of boundary.]

By the end of the 1930s there was an extensive theory of topological dimension for separable metric spaces. It can be found in the text by Hurewicz & Wallman [2]. The fact that $\dim X = \operatorname{ind} X$ (for separable metric space X) was an important element in some of the proofs. At that time there was no satisfactory dimension theory for more general spaces (not metrizable, or metrizable but not separable).

Here are a few of the basic results, taken from [1, pp. 257–260] (all spaces are metrizable):

Subspace Theorem: Let $S \subseteq X$. Then $\dim S \leq \dim X$.

Sum Theorem: Let $\{F_\gamma\}_{\gamma \in \Gamma}$ be a locally countable closed covering of X such that $\dim F_\gamma \leq n$ for each γ. Then $\dim X \leq n$.

Decomposition Theorem: Let $n \geq 0$. Then $\dim X \leq n$ if and only if there is a decomposition $X = \bigcup_{i=1}^{n+1} X_i$, where each $X_i \subseteq X$ satisfies $\dim X \leq 0$.

Product Theorem: $\dim(X \times Y) \leq \dim X + \dim Y$ [unless $X = Y = \varnothing$].

If "dim" is replaced by "ind," then these results are still true for separable metric spaces. But $\dim X = \operatorname{ind} X$ can fail for nonseparable metric spaces. And accordingly some of these basic results (with "dim" replaced by "ind") also fail for nonseparable metric spaces.

Universal Spaces. Let C be the Cantor set. Then C is a separable metric space and $\dim C = 0$. If X is any separable metric space with $\dim X = 0$, then X embeds in C, meaning X is homeomorphic to a subset of C. This is what it means to say that C is a *universal space* for the class of separable zero-dimensional metric spaces.

Menger [4] constructed the space M (now called the Menger sponge) specifically as a universal space for the class of separable one-dimensional metric spaces. Any separable metric space X with $\dim X = 1$ embeds in M and therefore in particular embeds in $\mathbb{R}^3$. [The existence of graphs that are not planar shows that not every one-dimensional separable metric space embeds in $\mathbb{R}^2$.] In the same paper, Menger described universal spaces for the higher-dimensional cases (complete proofs were supplied later by Lefschetz). These are fractals $M_n \subseteq \mathbb{R}^{2n+1}$ for $n \in \mathbb{N}$; and M_n is a universal space for separable metric spaces of dimension n. This sequence of spaces begins with $M_0 = C$, the Cantor set, and $M_1 = M$, the Menger sponge. The spaces M_n may be described as attractors for iterated function systems, although of course Menger did not use that language.

Another universal space was described in 1931 by Nöbeling [5]. Write $I = [0,1]$ for the interval, and $N_n := \{ x \in I^{2n+1} : x \text{ has at most } n \text{ rational coordinates} \}$. Then for all $n \in \mathbb{N}$, space N_n is universal for separable metric spaces of dimension

n. This sequence begins with $N_0 = [0,1] \setminus \mathbb{Q}$, the irrationals. The spaces N_n are no longer attractors for iterated function systems. After Menger, dimension theory abandoned IFSs for the most part.

Also notable is another universal space. The Hilbert cube I^∞, the product of countably many copies of the interval I, is a universal space for separable metric spaces.

3. General Metric Space

The theory of topological dimension developed in the 1950s into a theory for topological spaces, and in particular for metric spaces not necessarily separable. In order to keep the important "basic results" enumerated above, it is necessary to use the covering dimension and not the inductive dimension.

But what about universal spaces? Of course there can be no universal space for all zero-dimensional metric spaces simply on grounds of cardinality, since a discrete space of any cardinality is zero-dimensional. But what about spaces of at most a given weight? The *weight* of a space is the minimum cardinal of a base for the topology. A metric space is separable if and only if its weight is $\leq \aleph_0$.

A topological space J_A is a key construction. We think of J_A as an analog of the interval I. Let A be a set, considered as a discrete space; also considered as an alphabet. The set A could be finite, but we will also allow A to be infinite, even uncountable. The space $N(A)$ of infinite strings from A is zero-dimensional. If $|A| = \alpha$, then $N(A)$ has weight α. [We wrote $|A|$ for the cardinal of A.] To construct J_A we will "identify adjacent endpoints" in $N(A)$, as I now describe. We will say that two strings in $N(A)$ are *adjacent* iff they are of the form

$$a_1 \cdots a_n p\overline{q} \quad \text{and} \quad a_1 \cdots a_n q\overline{p},$$

where $n \geq 0$; $a_1, \cdots, a_n, p, q \in A$; and $p \neq q$. Some examples of adjacent strings have been seen above: In Figure 1, $0\overline{1}$ and $1\overline{0}$ are adjacent; $00\overline{1}$ and $01\overline{0}$ are adjacent. In Figure 2, $\mathrm{z}\overline{\mathrm{b}}$ and $\mathrm{b}\overline{\mathrm{z}}$ are adjacent; $\mathrm{az}\overline{\mathrm{b}}$ and $\mathrm{ab}\overline{\mathrm{z}}$ are adjacent.

On the topological space $N(A)$, define an equivalence relation $\sim$ of adjacency as follows: Every string is related to itself, and adjacent strings are related to each other. So the equivalence classes have either one or two elements only. Define J_A as the space $N(A)/{\sim}$ with the quotient topology. We say: J_A is obtained from $N(A)$ by *identifying adjacent endpoints.* The cardinal of A determines J_A up to homeomorphism, so we sometimes write J_α if the cardinal α is all that is of interest.

Theorem. *Let $|A| = \alpha \geq \aleph_0$. If X is a metric space with $\dim X = 1$ and weight α, then X embeds in J_A.*

We have already seen some of the spaces J_n in Figures 1 and 2. Think of $N(\{0,1\})$ as the Cantor set, and $J_{\{0,1\}}$ as the interval $[0,1]$. Similarly, $J_3 = J_{\{\mathrm{z,a,b}\}}$ is the Sierpinski gasket.

If n is finite, J_n may be realized as an attractor of an iterated function system in a natural way: Let A be an n-letter alphabet. Take the n vertices $\{v_a\}_{a \in A}$ of an $(n-1)$-simplex in Euclidean space of dimension $\geq n-1$. Let map w_a be the homothety with center v_a and ratio $1/2$. Then the attractor K satisfies

$$K = \bigcup_{a \in A} w_a(K).$$

This IFS is “just touching” in a very strong sense: For all $a, b \in A$, $a \neq b$, the two images $w_a(K), w_b(K)$ intersect in just the single point $(v_a + v_b)/2$.

Lipscomb described the spaces J_A in his thesis in 1973. In 1980 he attended a lecture by Michael Barnsley, which included an image of the Sierpinski gasket. It was then (50 years after the divorce) that dimension theory and iterated function systems got back together.

Infinite IFS. When A is infinite (even uncountable), we wish to similarly realize J_A using an IFS. Additionally, this will show how to place a natural metric on J_A. [When we say that J_A is a universal space for one-dimensional metric spaces of weight α, it should in particular be a metric space.]

The iterated function system discussion used above will need to be generalized for this purpose. First, we must allow an infinite alphabet A. Second, we must allow non-compact attractors: If A is uncountable, then space J_A is not separable, so it cannot even be embedded in a compact metric space.

Let X be a metric space. Write $\mathcal{B}_X$ for the set of all nonempty, closed, bounded subsets of X. The Hausdorff metric makes this a metric space. If X is complete, then so is $\mathcal{B}_X$.

Let A be our set, and for each $a \in A$ let w_a map X to itself. Then $\{\, w_a : a \in A \,\}$ will still be called an *iterated function system*. If, in addition, whenever $E \subseteq X$ is bounded, $\bigcup_{a\in A} w_a(E)$ is also bounded, then the IFS $\{\, w_a : a \in A \,\}$ is called *bounded*.

If we have a bounded IFS on a complete metric space and it satisfies (2), then the limits (1) exist for all $\mathbf{a} \in N(A)$ and are independent of x, so we get a continuous function $p\colon N(A) \to X$ defined by $p(\mathbf{a}) = w_{\mathbf{a}}(x)$. Thus we obtain a set of “points with addresses”:

$$K := \{\, p(\mathbf{a}) : \mathbf{a} \in A \,\}. \tag{3a}$$

However: since $N(A)$ is not compact, we no longer know that the set K in (3a) is closed. The map p still induces an equivalence relation $\sim$ on $N(A)$, and the map p still induces a continuous bijection from $N(A)/{\sim}$ onto K. But again we no longer know that this is a homeomorphism, that K has the quotient topology.

In this setting [a complete metric space X, a bounded IFS, and (2)], let us define a Hutchinson operator:

$$\mathcal{W}(E) := \overline{\bigcup_{a\in A} w_a(E)} \qquad \text{for } E \in \mathcal{B}_X. \tag{4b}$$

First note that, in general, if E is closed and bounded, the continuous images $w_a(E)$ need not be closed. But even if all images $w_a(E)$ are closed [as they are in Perry’s IFS, below], the infinite union $\bigcup_{a\in A} w_a(E)$ may still fail to be closed. These are the reasons for the closure in the definition (4b). Now (under our hypotheses), the Hutchinson operator $\mathcal{W}$ is a contraction on the complete metric space $\mathcal{B}_X$, so there is a unique nonempty closed bounded set K such that

$$K = \overline{\bigcup_{a\in A} w_a(K)}. \tag{3c}$$

However: we do not in general know that this set K satisfies the equation (3b) without the closure. We do not know that the set in (3a) is the same as the set in (3c); that is, we do not know that every point of the set in (3c) has an address.

The considerations above show that this new notion of IFS is more subtle than the old. By definition, the IFS has a *topological attractor* iff the set in (3a) is closed and has the quotient topology from $N(A)$. This implies that the set K in (3a) is the set in (3c) and so also satisfies (3b).

Aside. I include another note (not related to dimension theory) on a use for non-compact IFSs. For a prime p, the locally compact field $\mathbb{Q}_p$ of p-adic numbers can support a study in many ways much like real analysis. For an analog of complex analysis, we need the extension $\mathbb{C}_p$ of $\mathbb{Q}_p$, see [3]. It is both complete in the metric sense (Cauchy sequences must converge) and in the algebraic sense (algebraically closed). Now $\mathbb{C}_p$ is infinite-dimensional over $\mathbb{Q}_p$, is not locally compact, closed bounded sets are not necessarily compact. Iterated function systems are useful tools in conventional complex analysis, for example in the study of Julia sets. For the corresponding study in $\mathbb{C}_p$, in order to use an IFS we will have to allow attractors that are not compact (although closed and bounded).

IFS in Hilbert space. Now we return the question of realizing the universal one-dimensional space J_A as topological attractor of an IFS. In 1996 Perry proposed that such an IFS should be found in (nonseparable) Hilbert space. This was carried out by Miculescu and Mihail in 2008. Another realization of J_A in Hilbert space is due to Milotinović. The Perry IFS for this is easily described. The space $X = l^2_\alpha$ is a Hilbert space with orthogonal dimension $\alpha = |A|$. That is, there is an orthonormal set of cardinality α. The maps w_a will be homotheties with ratio $1/2$. For the centers v_a of the homotheties: choose one letter z and let $v_z = 0$; the other centers v_a are an orthonormal basis for (a subspace of) the Hilbert space. The topological attractor exists, and is homeomorphic to J_A. Carrying out this proof requires some effort: it is three chapters of the book. (Those chapters contain complete descriptions of the Hausdorff metric and of nonseparable Hilbert space, as well. Some of the analysis applies to infinite IFSs in general, not just to the particular IFS used here.)

Higher Dimension. Once the universal space J_A for one-dimensional weight α metric spaces is known, it can be used to construct other universal spaces.

Theorem. *Let $|A| = \alpha \geq \aleph_0$. The space J_A^∞, the product of countably many copies of J_A, is a universal space for the class of metric spaces of weight α.*

To provide an analog of the Nöbeling universal space, we need a notion of "rational" coordinate. A point of J_A is called *rational* iff it has two addresses, *irrational* iff it has one address. For example, in $J_2 = I$, the rational points of J_2 are the dyadic rationals in I.

Theorem. *Let $|A| = \alpha \geq \aleph_0$ and $n \in \mathbb{N}$. The space*

$$J_A^{n+1}(n) := \left\{ (x_1, \cdots, x_{n+1}) \in J_A^{n+1} : \textit{at most } n \textit{ of the } x_i \textit{ are rational} \right\}$$

is a universal space for the class of metric spaces with dimension n and weight α.

Visualization in 3-space. For finite n, the fractal J_n appears naturally in $\mathbb{R}^{n-1}$. But for visualization, sets embedded in $\mathbb{R}^k$ with $k \leq 3$ are best. Now since $\dim J_n = 1$ and J_n is a separable metric space, we know that J_n embeds homeomorphically into $\mathbb{R}^3$. But usually such an embedding will obscure the IFS structure. Is there an affine embedding into $\mathbb{R}^3$?

There is no problem with J_2 the interval, J_3 the Sierpinski gasket (Figure 2), or J_4 the "Sierpinski tetrahedron" or "Sierpinski cheese." It is a surprise [6] that J_5 does admit an affine embedding in $\mathbb{R}^3$. It may be realized as the attractor of an IFS of 5 maps, all homotheties with ratio 1/2.

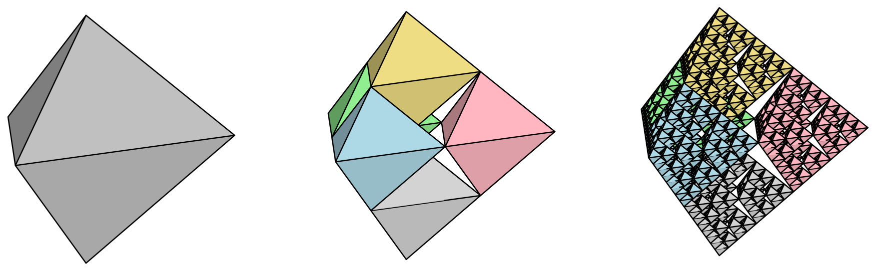

FIGURE 4. Hexahedron; five just touching; J_5

This is illustrated in Figure 4. Join two regular tetrahedra face-to-face to obtain a convex hexahedron with six triangular faces and five vertices. The vertices are the centers for the five homotheties used for the IFS. Consider the five images of the hexahedron: the important property is that any two of them intersect in exactly one point. See how the top and bottom hexahedra meet in the triangular gap not occupied by the other three hexahedra. (Links to the VRML files that will let you rotate these figures to help in visualizing them in three dimensions are available from `http://www.math.ohio-state.edu/~edgar/preprints/lipscomb/`.)

What about higher J_n? For a just-touching attractor of an IFS consisting of n homotheties with ratio 1/2, the Hausdorff dimension is $\log n / \log 2$. For $n \geq 8$, this Hausdorff dimension is ≥ 3, so our attractor J_n cannot embed in $\mathbb{R}^3$. But it seems the cases J_6 and J_7 are open. *Can one place six points in* 3*-space in such a way that the corresponding attractor* (*for the IFS of six homotheties with ratio* 1/2) *will consist of six parts, but that any two of the parts intersect exactly in one point*? Perhaps some pairs of the parts are interlocking, one passing through the spaces in the other. It is probably impossible, but still interesting to contemplate.

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