ICMI-East Asia Regional Conference on Mathematics Education

Multiplicity of Algebraic Methods Using Graphing Calculator Technology

A Variety of Ways of Doing and Teaching Algebra

ABSTRACT:

The use of graphing-calculator technology has promoted the use of a variety of new algorithms in the algebra curriculum. At the same time, new teaching methods are also possible because of this technology. This paper will address a limited number of examples from each area.

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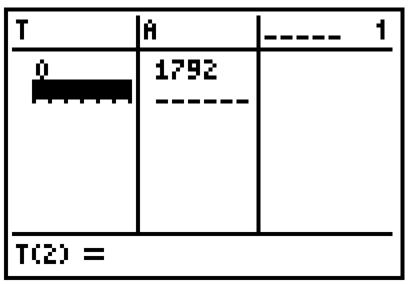
THE TEACHING OF ALGEBRA

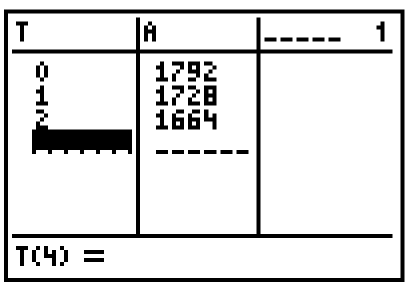
All of the following examples assume that mathematical topics are being developed using a function approach. That is, all mathematics is derived from using functions.

Connecting Different Representations of Functions:

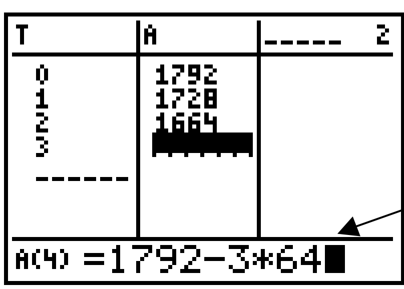
House Painter Measurements on a potential customer’s house shows a surface area of 1792 square feet. Let *t* be time in hours and the area *A* that remains to be painted. Since you paint at a rate of 64 square feet per hour, and the initial conditions are known, develop the algebraic expression.



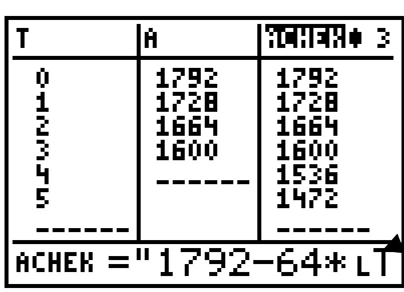
The initial conditions are known.



Since the rate is known, we also know that each hour worked is another 64 square feet less to paint. The mathematical process is subtraction of 64. Thus, we can get a numeric representation from the verbal description.



But repeated subtraction of 64 can best be accomplished by subtracting the product of 64 and the number of times it has been subtracted – as shown in the edit line.



From a pattern of arithmetic operations, we can generalize to algebraic symbols such as 1792 – 64*T*.

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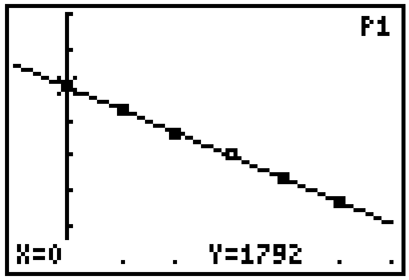


Confirmation that the symbols $1792 - 64T$ represent a linear relationship is shown along with the data points calculated.

Conclusion: This kind of activity works especially well with beginning algebra students in the United States. It is reasonable to assume that any teacher can do this kind of teaching with the use of a calculator like the TI-83.

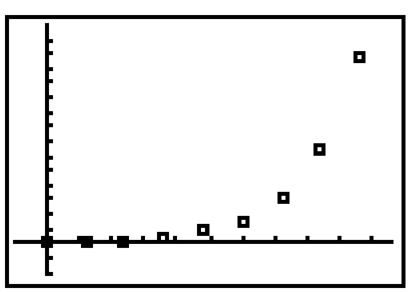
Use of Data to Teach Representations of Functions

TB Bacteria Below is the data relationship between time and the population of tuberculosis bacteria growing under unrestricted conditions. This is something like what might happen in a petri dish in a medical laboratory. Time is in hours and the bacteria population is in thousands.

Tuberculosis Data

t	0	6	12	18	24	30	36	42	48
B	5	10	20	40	80	160	320	640	1280

The initial conditions are shown and the bacteria population is doubling every 6 hours. If it doesn't follow that the symbolic representation is exponential, look at the graphical representation.



The data "looks" like symbolic form will be exponential.

The feature of "doubling" is represented symbolically as 2 raised to an exponent. What exponent? The second guess at symbolic form is shown in Figure 1 as 5×2^0. The first guess was 2^0, but it produced a 1.

Thus 1 times 5 gives the correct check for the initial conditions. The most common next attempt at symbolic form is shown in Figure 2.

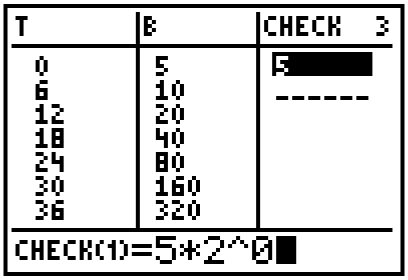


Figure 1

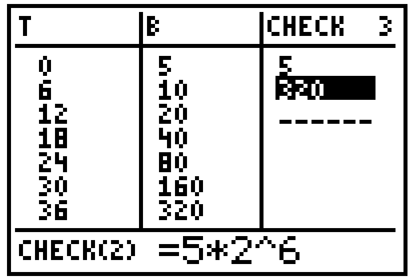


Figure 2

The guess of 5×2^6 didn’t work. The idea is OK but the exponent is too large. Likewise, 5×2^{12} doesn’t work for the next data point. The *T* values are rising by 6, but the exponent needs to rise by 1. What arithmetic operation will take numbers rising by 6 and make them rise by 1? Division by 6. Figure 3 shows a correct guess.

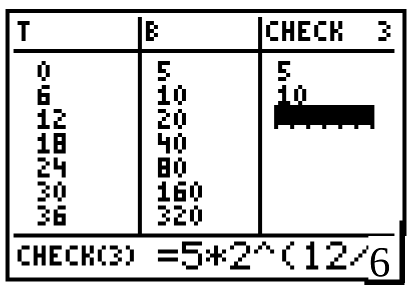


Figure 3

Finally, the generalized symbolic form is $5 \times 2^{\frac{T}{6}}$. This can be confirmed by graphing it and the data points.

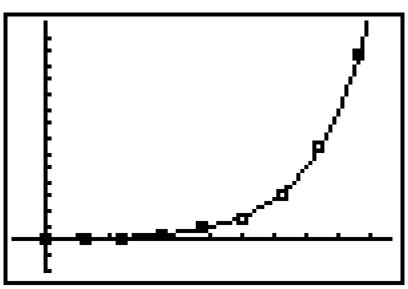


Figure 4

The graphical representation of $5 \times 2^{\frac{T}{6}}$ matches the data given in the problem.

Conclusion: This problem uses data that is based on a pattern that students would not necessarily recognize. Knowing the general shape of exponential growth is the clue to being able to "discover" the symbolic representation of the data.

Using Calculator Technology to Discover Parameter-Behavior Connection

Below is a graphing calculator exploration assigned to students before the topic is discussed in class.
Directions: For each of the following square root functions, identify the ***first number*** in the domain, the maximum or minimum value of the function, and state whether the function is increasing or decreasing.

	First # in Domain	Max/Min	Inc/Dec
1. $2\sqrt{x-1}+3$	______________	______________	______________
2. $3\sqrt{x+4}-5$	______________	______________	______________
3. $1\sqrt{x-2}+4$	______________	______________	______________
4. $-\sqrt{x+1}-2$	______________	______________	______________
5. $-2\sqrt{x+3}-6$	______________	______________	______________
6 $-3\sqrt{x-2}+4$	______________	______________	______________

7. Based on the above examples, make a conjecture about each statement below if *d*, *e*, and *f* are real numbers.

What is the first number in the domain of $d\sqrt{x+e}+f$?____________________________

What is the maximum or minimum of $d\sqrt{x+e}+f$?_______________________________

How do you know if $d\sqrt{x+e}+f$ is increasing or decreasing?_______________________

Conclusion: This exploration encourages students to learn the topic while working with their peer group. Mathematics is sometimes best taught by the self-learner.

Using Calculator Technology to Promote an Understanding of the Factor-Zero Connection

Below is a graphing calculator exploration that can be assigned to students before the topic is discussed in class, or it can form the basis of a lesson plan on factoring.
Directions: For each of the functions, find the zeros.

1. $(x-2)(x+3)$ Zeros ______________________
2. $(x+1)(x-3)$ Zeros ______________________

3. $(x - 4)(x - 1)$ Zeros ______________________

4. $(x + 5)(x + 2)$ Zeros ______________________

5. $(x + 3)(x + 1)$ Zeros ______________________

6. $(x - 6)(x + 4)$ Zeros ______________________

a. $x^2 + x - 6$ Zeros ______________________ Make a conjecture on another way of writing $x^2 + x - 6$ ____

b. $x^2 - 2x - 3$ Zeros ______________________ Make a conjecture on another way of writing $x^2 - x - 3$ ____

c. $x^2 - 5x + 4$ Zeros ______________________ Make a conjecture on another way of writing $x^2 - 5x + 4$ ____

d. Etc.

Conclusion: This is a teaching approach that works well with factoring any polynomial with linear factors of the form $dx + e$, not just in the $x + e$ form. Not only can the students learn to factor using this method, they also learn the connection between linear factors and zeros. An added benefit when the graph is used to find zeros is that students no longer think something like $x^2 + 1$ is factorable.

Teaching Mathematical Concepts in the Context of an Application – Like Rate of Change

Consider what happens as a person walks away from the Texas Instruments Calculator-Based Ranger (CBR) as it is collecting time-distance data. The CBR measures and records the distance from the CBR in feet and the time at that distance in seconds. Figure 5 shows the time-distance relationship as the person walk slowly away from the CBR at a constant rate.

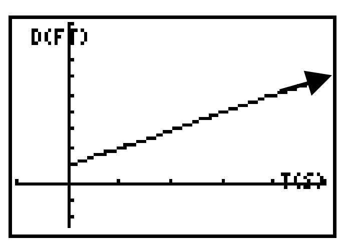


Figure 5

- How can we find the average speed of the walker? Typical student responses:

1. The speedometer in my car measures speed as miles per hour $\left(\frac{\text{miles}}{\text{hour}}\right)$.
2. The CBR is measuring distance in feet and time in seconds. So the speed of the walker is measured in $\left(\frac{\text{feet}}{\text{second}}\right)$.
3. Divide the distance (feet) by time (seconds).
4. Divide the distance traveled by the time walking.

If we use the idea in response 4, let's see how we can find the speed, but let's make a subtle change in the idea. The distance traveled can be thought of as the change in position from one data point to another. Likewise, the time to travel this distance can be thought of as the change in the time from one data point to another. Consider the numeric representation of a typical walker shown below.

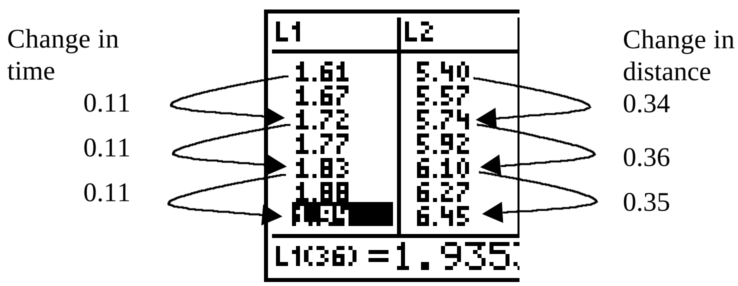


Figure 6

The rates of the walker from the three sample data points are shown in Figure 7. The person is averaging at about 3.2 feet per second.

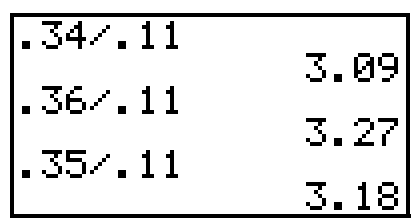


Figure 7

- Is the person walking at a constant rate?

  Typical student responses:

1. No.
2. Almost.
3. If the average is 3.2, the rates above only differ by a tenth or a few hundredth of a foot per second.

4. Yes, the person is nearly walking at a constant rate.
5. I suppose – for all practical purposes.

Looking at the speed from the graphical representation, note that from 1.02 seconds to 1.24 seconds the change in time is 0.22 second. The walker has changed position by 0.73 foot from the first to second data point. From this data, the average speed, or rate, is 3.32 feet per second. That is, the walker has averaged a change in position by 3.32 feet in one second. $\left(\dfrac{\text{Change in position}}{\text{Change in time}} = \dfrac{0.73}{0.22} = 3.32 \right)$

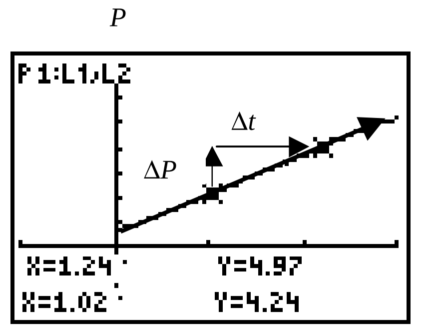


Figure 8

What would happen if the walker walked at obviously different speeds, would the graph be straight? Figure 9 shows data collected from a walker who started quickly, slowed to almost no motion, then started forward again.

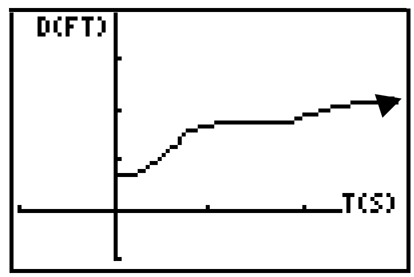


Figure 9

Suppose that in the experiment described above, the walker now walks slowly toward the CBR as shown in Figure 10. Then the walker walks a little faster as shown in Figure 11. After seeing this relationship, must we conclude that the faster we walk the faster the graph of the time-distance relationship rises? No! As the walker moves closer to the CBR, the graph doesn't rise. It falls because the distance is decreasing – not increasing. Note the decreasing graphs in Figures 10 and 11. Figure 12 shows that the average rate is decreasing – not increasing.

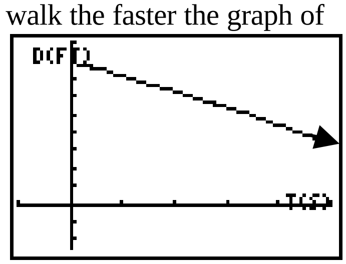
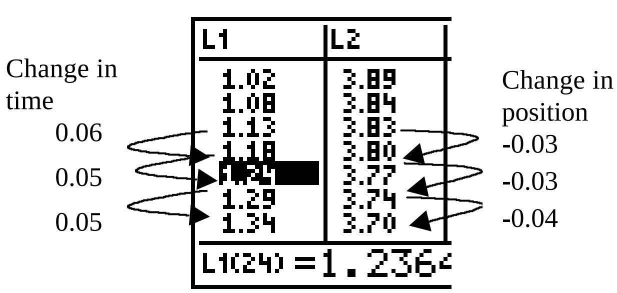


Figure 10



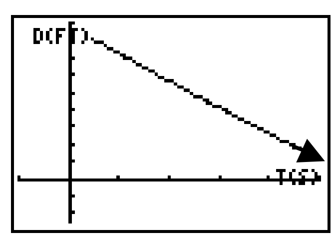


Figure 11

The use of negative numbers indicates a decreasing distance. The average rate of change is about –0.6 foot per second. Is the walker walking at a faster rate in Figure 10 or 11?

Conclusion: We can conclude from this demonstration that the walker’s average speed is a quotient of two numbers; it is positive if the distance is increasing (the graph is rising); it is negative if the distance is decreasing (the graph is decreasing), and it measures how fast a person is changing position. The speed is quite often referred to as the rate. Since the walker has changed position from one data point to another, we can refer to speed as a rate of change. Since we are not finding the walker’s rate of change at just one data point, it is called an average rate of change (average speed) – just like when you take an auto trip, you usually find an average speed for the trip. How is average rate of change related to the steepness of a graph?

THE DOING OF ALGEBRA

In addition to using the graphing calculator as a teaching tool, it can also be used as a tool for doing mathematics. Below are a few examples demonstrating the calculator doing mathematics. Again, please note that these examples imply that mathematics is taught using a function approach.

Solving Equations and Inequalities

The Numeric Method

Suppose that you raise red raspberry plants and you notice that on June 28 there are 17 beetles on your small set of plants. The next day your see 4 more and the day after 4 more, etc. Students must be able to explain that the function that models the number of beetles on your raspberry plants is $N = 17 + 4x$, where x is time in days starting at 0 on June 28. When were there 5 beetles in the berry patch? If the tolerance limit for producing marketable berries is 100 beetles per patch, when will the berries no longer be marketable?

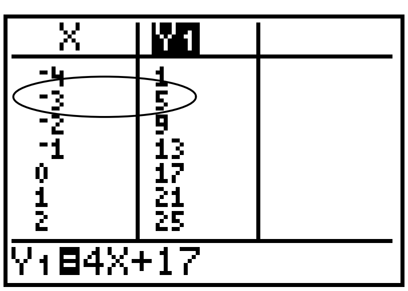


Figure 13

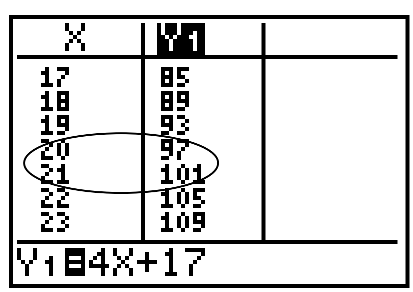


Figure 14

In this example, there is no need for exact answers. So, there were 5 beetles on the raspberries 3 days before June 28th. After about 21 days, the berries will no longer be marketable.

Comments: This is a good first method to introduce equation solving. The poorer students will usually continue to use this method even after simpler methods are introduced. When studying functions before solving equations and inequalities, solving both equations and inequalities are taught at the same time.

The Trace Method

A small search party is looking for a missing camper in a 150 acre wooded area. They can cover 2 acres in 6 minutes. The function that models the amount of area left to search at time *t* is $A = -\frac{2}{6}t + 150$ (Students must be able to explain why.). How much time will it take so that 75 acres remains to be searched? When will the search party have less than 75 acres to search? The problem is to solve the equation $-\frac{2}{6}t + 150 = 75$ and the inequality $-\frac{2}{6}t + 150 > 75$.

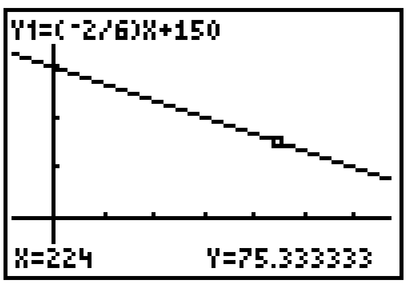


Figure 15

This is another technology-based method that is pedagogically sound when studying mathematics from a function approach. The answer to both questions is available in Figure 15. Zooming-in will yield a solution with a better accuracy than to the nearest hour.

Comments: The problem with this method is that if students are solving an equation taken out of the context of a problem, and the solution is a large number (like 224), they may have difficulty finding the solution.

The Zeros Method

If you throw a baseball straight up with an initial velocity of 60 feet per second and it leaves your hand when it is 5 feet from the floor. Ignoring air resistance, when will the ball hit the ground? The model physicists/mathematicians have developed is $h = -16t^2 + v_0 t + h_0$, where *h* is the height above ground in feet, t is time in flight measured in seconds, v_0 is the initial velocity, and h_0 is the initial height. Students must know that the equation to be solved is $-16t^2 + 60t + 5 = 0$

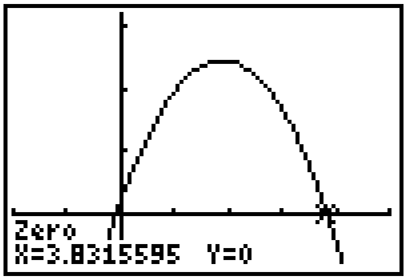


Figure 16

Figure 16 shows the use of the zero-finder on the TI-83. Zooming-in around the zero may also be used. This method is the most used method by students. They may prefer this method because they have found zeros of functions before they solved equations and therefore it is familiar. Or, they may like this method because they always know where to look for the solution to an equation – the *x*-axis.

Comments: The study of functions before studying equation solving is desirable because students should know where to look for zeros and how many they expect to find.

The Intersection Method

Suppose that taxi-company *A* has a fare schedule of $1.20 per mile plus a $3 usage fee. Taxi-company *B* charges $0.90 per mile plus a $5 usage fee. Students must know that the model of the taxi fare for company *A* is $F_A = 1.2m + 3$ and for company *B* it is $F_B = 0.9m + 5$. How far can you travel so that the fares are equal? When is fare $F_A > F_B$,? Exact answers are OK, but maybe not important.

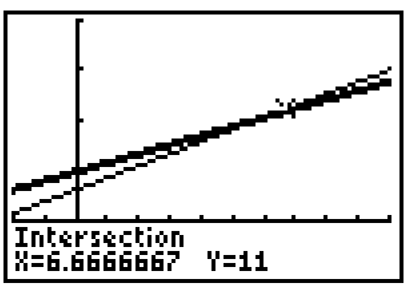


Figure 17

The answer to both questions can be found in Figure 17.

Comments: The graphing calculator can also be used as a teaching tool to help students to understand why the solution is at the intersection of the two graphs.

Factoring

To find the linear factors of a polynomial like $12x^2 + 11x - 91$, all students need is one zero.

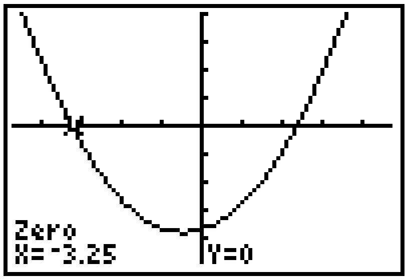


Figure 18

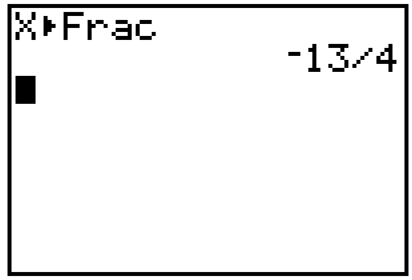


Figure 19

Since students know the connection between zeros and linear factors, they know a factor is (4*x* + 13). Because they know how to multiply binomials, they know the other factor must be (3*x* – 7)

Comments: One objection to factoring with technology is that it may be too time consuming; however, this method doesn't take any longer than would a traditional method. In addition, this method reinforces the concepts of the factor-zero connection.

Synthetic Division

Divide: $\dfrac{3x^4 - 2x^3 + 5x^2 - 4x + 1}{x - 1}$ using synthetic division and the calculator doing the arithmetic.

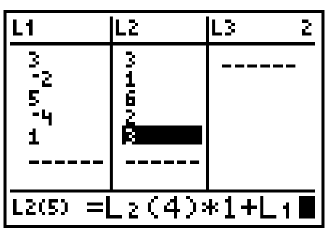


Figure 20

Algorithm:

1. Enter the first coefficient from L_1 to L_2
2. Multiply the number above the L_2 entry × 1 plus number across in L_1
3. Multiply the number above the L_2 entry × 1 plus number across in L_1
4. Etc.

The quotient is in L_2, with the last number being the remainder.

Comments: This technology-based method removes the arithmetic drudgery from the synthetic division process.

Confirming Pencil and Paper Symbol Manipulation

If a student thinks that the sum $\dfrac{3}{x-2} + \dfrac{5}{x^2-4}$ is $\dfrac{8}{x^2+x-6}$, (or anything else) the work can be checked with technology. Put the problem function in Y_1 and the proposed sum function in Y_2. If the two functions are the same, their numeric representations must be the same. Figure 21 shows that the two are not equivalent.

X	Y1	Y2
-4	-.0833	1.3333
-3	.4	ERROR
-2	ERROR	-2
-1	-2.667	-1.333
0	-2.75	-1.333
1	-4.667	-2
2	ERROR	ERROR

Y2=8/(X²+X−6)

Figure 21

Conclusion: Hand-held graphing calculator technology allows for many different methods of teaching and doing mathematics.