[1] G = group. (1) Center Z of G is $\{z\in G;\ zg=gz\ \forall g\in G\}$.

(2) Z is a subgroup: $e\in Z$ as $eg=ge\ \forall g\in G$; $z_1,z_2\in G\Rightarrow (z_1z_2)g=z_1(z_2g)=z_1(gz_2)=(z_1g)z_2$ $=(gz_1)z_2=g(z_1z_2)\Rightarrow z_1z_2\in Z$; $z\in Z\Rightarrow zg=gz\ \forall g\Rightarrow gz^{-1}=z^{-1}g\ \forall g\Rightarrow z^{-1}\in Z$ $\therefore$ Z is a subgroup.
$Z\trianglelefteq G$ since $\forall z\in Z$, $\forall g\in G$, $gzg^{-1}=z\in Z$.

(3) If $G/Z=\langle aZ\rangle$, $\forall g\in Z$ $\exists m=m(g)\in\mathbb{Z}_{\geq 0}$ with $gZ=(aZ)^m\overset{\text{as } Z\trianglelefteq G}{=}a^mZ$ $\therefore$ $g=a^mz$ for some $z=z(g)$ in Z.
For another $g'\in G$, $g'=a^{m'}z'$, and $gg'=a^mza^{m'}z'=a^{m+m'}zz'=a^{m'}z'a^mz=g'g$, so G is abelian.

[2] $f:G\to H$ hm, J subgroup of H; $f^{-1}(J)=\{x\in G;\ f(x)\in J\}$ is a subgroup: $e\in f^{-1}(J)$ since $f(e)\in J$; $x,y\in f^{-1}(J)\Rightarrow f(xy)=f(x)f(y)\in J\Rightarrow xy\in f^{-1}(J)$; $x\in f^{-1}(J)\Rightarrow f(x^{-1})\overset{f\text{ is hm}}{=}f(x)^{-1}\in J$ as $f(x)\in J$.
(2) $\ker(f)\subset f^{-1}(J)$ since $x\in\ker(f)\Rightarrow f(x)=e\in J\Rightarrow x\in f^{-1}(J)$ [you were not asked to show that $\ker(f)$ is sgp].

[3] Every $(a,b)\in\mathbb{R}\times\mathbb{R}$ can be expressed uniquely as $(a-b+b,b)=(a-b,c)+(b,b)$, so the cosets of $\mathbb{R}\times\mathbb{R}$ modulo $D=\{(x,x);\ x\in\mathbb{R}\}$ are $a+D$, $a\in\mathbb{R}$.

[4] If $\mathrm{ord}(a)=k$, then $a^k=e\Rightarrow(gag^{-1})^k=gag^{-1}\cdot gag^{-1}\cdot\ldots\cdot gag^{-1}=ga^kg^{-1}=e$, so $\mathrm{ord}(gag^{-1})$ divides k.
If $(gag^{-1})^m=e$, then $a^m=e$ (same argument: name $gag^{-1}=b$, then $b^m=e$ and so $a^m=(g^{-1}bg)^m=b^m=e$)
Hence $\mathrm{ord}(gag^{-1})=k$ (and not $m<k$), and the uniqueness implies $gag^{-1}=a\ \forall g\in G$ $\therefore$ a is central.

[5] (1) The centralizer $S(x)$ of $x\in G$ is $\{g\in G;\ gx=xg\}$. (2) $S(x)$ is a subgroup as $e\in S(x)$ (indeed, $ex=xe$); if $g,h\in S(x)$ then $(gh)x=g(hx)=g(xh)=(gx)h=g(xh)=g(hx)=(gh)x$ so $gh\in S(x)$; $g\in S(x)\Rightarrow gx=xg\Rightarrow xg^{-1}=g^{-1}x\Rightarrow g^{-1}\in S(x)$. (3) If $G=S_3$, $x=(12)$, $S((12))=\langle(12)\rangle\ntrianglelefteq S_3$.
(4) $C(x)=\{gxg^{-1};\ g\in G\}$ (conjugacy class of x in G). (5) $f:G/S(x)\to C(x)$, $gS(x)\mapsto gxg^{-1}$.
Since $G/S(x)$ and $C(x)$ are sets, not groups, don't attempt to show that f is a homomorphism!
f is well defined: if $gS(x)=hS(x)$, $g=hs$, $s\in S(x)\Rightarrow gxg^{-1}=hsx(hs)^{-1}=hsxs^{-1}h^{-1}=hxh^{-1}$.
f is onto: each element of $C(x)$ is gxg^{-1} ($\exists g\in G$), and $f(gS(x))=gxg^{-1}$. $\}$ f is isom
f is into: if $gxg^{-1}=hxh^{-1}$ then $x=h^{-1}gxg^{-1}h=(h^{-1}g)x(h^{-1}g)^{-1}\Rightarrow h^{-1}g=s\in S(x)\Rightarrow gS(x)=hS(x)$. $\}$ of sets!

[6] $G=\langle a,b;\ a^4=e=b^4,\ a^2=b^2,\ ba=a^3b\rangle=\{e,a,a^2,a^3,b,ab,a^2b,a^3b\}$
Conjugacy classes: $\{e\}$, $\{a^2\}$, $\{a,a^3=bab^{-1}\}$, $\{b,b^3=aba^{-1}\}$, $\{ab,a^3b=ba^3b^{-1}\}$
Centralizers: G, G, $\{e,a,a^2,a^3\}$ $\{e,b,b^2,b^3\}$ $\{e,a^2,ab,a^3b\}$